NOTES ON PÓLYA'S AND TURÁN'S HYPOTHESES CONCERNING LIOUVILLE'S FACTOR

by

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1. It was observed by Pólya [8] that if $\lambda(n)$ denotes Liouville's factor (i.e., the completely multiplicative function defined by $\lambda(p) = -1$ or, equivalently, by

(1) $$\sum_{n=1}^{\infty} \lambda(n)/n^s = \Pi(1+p^{-s})^{-1} \equiv \zeta(2s)/\zeta(s),$$

where $\sigma > 1$), then, on the one hand, there is a considerable numerical evidence for the truth of the hypothesis that

(2) $$\sum_{n=1}^{m} \lambda(n) \leqq 0 \quad \text{for} \quad m > 1$$

(in this regard, cf. [4]) and that, on the other hand, a general function-theoretical theorem of Landau [7] readily leads to the following conclusion: The hypothesis (2) (or its weakened form, requiring only

(3) $$\sum_{n=1}^{m} \lambda(n) \leqq 0 \quad \text{for} \quad m > m'$$

if m' is large enough) implies Riemann's hypothesis,

(4) $$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > \frac{1}{2}$$

[and, incidentally, the simplicity of all zeros of $\zeta(s)$ also; so that

(5) $$d\zeta(s)/ds \neq 0 \quad \text{at} \quad s = s_0 \quad \text{if} \quad \zeta(s_0) = 0,$$

where s_0 must be of the form $\frac{1}{2} + it_0$, by (4)].

A hypothesis similar to, but different from, (3) was proposed by Turán [9], p. 35, as follows:

$$\sum_{n=1}^{m} \lambda(n)/n \geqq 0 \quad \text{for} \quad m > m'', \tag{6}$$

if m'' is large enough. If (6) is true, then (4) and (5) can be concluded as before (in this regard, nothing would change if $\leqq 0$ and $\geqq 0$ in (3) were changed to $\geqq 0$ and $\leqq 0$ respectively). But what is alluring in this variant of (3) is that, owing to a general theorem of Bohr [2] (concerning the Diophantine irrelevance of « phase-shifts » for the zeros of absolutely convergent Dirichlet series on an open half-plane), the truth of (6) is implied by the following hypothesis (which, since

$$\zeta(s) \neq 0 \quad \text{for} \quad \sigma > 1 \tag{7}$$

is obvious from

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1} = \zeta(s), \tag{8}$$

where $\sigma > 1$, appears to be comparatively innocent):

$$\sum_{n=1}^{m} n^{-s} \neq 0 \quad \text{for} \quad \sigma > 1 \quad \text{if} \quad m > m''. \tag{9}$$

The deduction of (6) from the hypothesis (9) is contained in the more general argument on pp. 14-15 in [9]; concerning (9) ➤ (4), cf. p. 4 *loc. cit.* (Theorem 1, the simple wording of which a footnote attributes to Jessen).

2. It is worth mentioning that if (9) is true for $m'' = 0$, then not only (4) and (5) hold but there also follow the assertions, say (4 bis) and (5 bis), which result if $\zeta(s)$ is replaced by $L(s)$ in (4) and (5), where $L(s) = L(s; \chi)$ denotes the Dirichlet series belonging to an arbitrarily *real* character. This can be concluded as follows:

A recent result of Showalter (published in [1]) implies that if (6) is true for $m'' = 0$, then

$$\sum_{n=1}^{m} \chi(n)/n \geqq \sum_{n=1}^{m} \lambda(n)/n$$

is true for every m (and for every real χ). Hence, if (6 bis) denotes the assertion which results when λ is replaced by χ in (6), then, since (9) implies (6), it follows that the case $m'' = 0$ of (9) implies the case $m'' = 0$ of (6 bis). But

(6 bis) (even with some $m'' > 0$ only) leads to (4 bis) and (5 bis) if Landau's function-theoretical theorem is applied in the same way as before.

On the other hand, if (2 bis) denotes the hypothesis which results if λ is replaced by χ in (2), then (2 bis) is false (for certain χ), whether (2) be true or false. In fact, not even the hypothesis (4), which is weaker than (2), is involved in the failure of (2 bis). For it was shown by Heilbronn [5] that not only (2 bis) is false (without any hypothesis) but not even a weakened form of (2 bis), proposed by Chowla [3], can hold.

Accordingly, (4 bis) cannot be concluded from (2) in the way in which (4 bis) was concluded from the case $m'' = 0$ of (6). Hence the case $m'' = 0$ of the hypothesis (9) appears to contain substantially more than what is contained in the hypothesis (2), the latter being adjusted only to (7), i.e., to the *principal* character χ alone. This curious situation is pointed out here because of the comments of Turán [9], p. 35, which are to the effect that (6), or at any rate a relaxed form of (6), might be more amenable than (3) is.

3. Turán's hypothesis (9) refers to the *partial sums* of $\zeta(s) = \sum n^{-s}$ for $\sigma > 1$. Can (4) [and (5)] be concluded if (9) is weakened to the corresponding hypothesis involving the Abel sums of $\zeta(s) = \sum n^{-s}$ for $\sigma > 1$? This question arises quite naturally. For, on the one hand, partial sums are generally more wobbly than Abel sums (the simplest instance of this remark being the content of Abel's own theorem, the converse of which is false) and, on the other hand, whether (3) or (6) is assumed, the deduction of (4) and (5) depends on an application of Landau's theorem to an integral representation of the function (1), a representation in which the partial sums of the numerical series of $\sum \lambda(n)$ or $\sum \lambda(n)/n$ form the integrand. There is, however, another integral representation of a Dirichlet series, a representation in which

$$(10) \qquad f(e^{-x}), \quad \text{where} \quad f(r) = \sum_{n=1}^{\infty} a_n r^n \quad \text{and} \quad 0 \leq r < 1,$$

takes the place of

$$(11) \qquad \sum_{n \leq x} a_n,$$

if $\sum a_n/n^s$ is the function to be represented by the integral (in both cases, the range of integration is the half-line $0 < x < \infty$, which can be reduced to $1 \leq x < \infty$ in the second case). In particular, Riemann's proofs of the functional equation depend (with $f(r) = \sum r^n$ or $f(r) = \sum r^{n^2}$) on the integral representation of $\sum a_n/n^s = \zeta(s)$ in terms of (10), rather than of (11); and it was shown in [11] that even such Tauberian theorems as that of Ikehara (cf. [10], § 19) can be generalized so as to put the (Tauberian) assumption of *monotony* on the Abelian generator, rather than on the partial sum (as a function of x).

On the other hand, Landau's general theorem is applicable whether (11) or (10) is supposed to be of constant sign for large x (this, of course, is less than the corresponding assumption of monotony, just mentioned). Since Turán's way of applying Bohr's general theorem (cf. § 1 above) can also be adjusted to the replacement of (11) by (10), the answer to the question, raised at the beginning of this § 3, easily proves to be affirmative.

4. In order to see this, replace the partial sums of $\zeta(s) = \sum n^{-s}$, where $\sigma > 1$, by its Abelian generator *

$$\zeta_r(s) = \sum_{n=1}^{\infty} r^n/n^s, \tag{12}$$

where $0 \leqq r < 1$. Then what corresponds to the assumption of a sufficiently large m'' satisfying (9) is the assumption that there exists a sufficiently small ε having the property that

$$\zeta_r(s) \neq 0 \quad \text{for} \quad \sigma > 1 \quad \text{if} \quad 1 - \varepsilon < r < 1 \tag{13}$$

$(0 < \varepsilon < 1)$; a hypothesis which, in view of (12), goes over into the trivial fact (7) if $r \to 1 - 0$. Clearly, the refinement announced in § 3 for Turán's result can be formulated as follows:

If there exists a positive constant $\varepsilon(< 1)$ *for which the functions* (12) *satisfy* (13), *then* (4) *is true [as is, incidentally,* (5) *as well].*

REMARK. There are two comments to be made on this inference.

First, the converse inference can hardly be made; we know of nothing to prevent that (4) [even with (5)] be possibly true under the assumption that the « Abelian hypothesis » (13) is false. But we shall refer in § 8 to *another* « Abelian hypothesis », one which is *equivalent* to (4) [without (5)].

Secondly, the Abelian relaxation of (9) to (13) is an *actual* relaxation; for, lacking any conceivable Tauberian restriction (such as an appropriate condition of unilateral boundedness), we cannot infer (9) from (13). We shall mention in

* Actually, if r has a fixed value within the unit circle of the complex r-plane, then (12) is an entire function of s, since the series is uniformly convergent on every fixed half-plane $\sigma > -$ const.

§ 8 a striking parallel instance, one concerning (4) itself, in which precisely the Abelian relaxation is certainly not vacuous (unless (4) in false).

In view of § 3, the proof of the italicized implication requires no new ideas. But since the details are not lengthy, and since a little formal trick (the use of the parameter u in § 6 below) with the *subsequent* identification $s+u$ with s is involved, for the sake of completeness we shall give the details.

5. For a fixed positive $r<1$, write the Dirichlet series $\zeta_r(s)$ in the form

$$(14) \qquad \zeta_r(s)=\sum_{n=1}^{\infty} r^n \exp(-s\log n), \qquad (\exp = n^{-s}),$$

confine s to the half-plane $\sigma>1$ (even though any half-plane $\sigma>c$ would be allowed), and apply to (14) Bohr's general theorem in [2], by using for the sequence $\log 1, \ldots, \log n, \ldots$ the basis $\log 2, \ldots, \log p, \ldots$, and for Bohr's « phase-shifting » of (14) into

$$(15) \qquad \sum_{n=1}^{\infty} r^n \exp(i\varphi_n) n^{-s}$$

the angles defined (mod 2π) as follows: $\varphi_n=0$ or $\varphi_n=\pi$ according as the number of all (not necessarily distinct) prime divisors p of n is even or odd. In view of (1), this means that $\exp(i\varphi_n)$ is chosen to be $\lambda(n)$ for every n (also for $n=1$). Hence (15) goes over into

$$(16) \qquad \zeta_r(s)=\sum_{n=1}^{\infty} r^n \lambda(n)/n^s.$$

Accordingly, Bohr's general theorem is applicable to the two functions (14), (16); its assertion reduces to the following: For a fixed r, the function $\zeta_r(s)$ attains on the half-plane $\sigma>1$ those and only those values which the function $\zeta_r(s)$, belonging to the same r as $\zeta_r(s)$, attains on the half-plane $\sigma>1$. In particular, the hypothesis (13) (for a fixed ε) is equivalent to

$$(17) \qquad \zeta_r(s)\neq 0 \quad \text{for} \quad \sigma>1 \quad \text{if} \quad 1-\varepsilon<r<1$$

(with the same ε).

Choose $s=\sigma$ in (17) and note that, according to (16), the function $\zeta_r(\sigma)$ is real-valued on the half-line $1<\sigma<\infty$ and tends to the limit $r^1\lambda(1)/1^s=r\lambda(1)=r>0$ as $\sigma\to\infty$. Consequently, the case $s=\sigma$ of (17) means that

$$(18) \qquad f(r;\, u)\equiv\sum_{n=1}^{\infty} r^n \lambda(n)/n^u>0 \quad \text{for} \quad 1<u<\infty \quad \text{if} \quad 1-\varepsilon<r<1,$$

where, for later use, $s = \sigma$ is replaced by u (and the first of the relations (18) is the definition of f).

6. In order to pass from (18) to an appropriate application of Riemann's general identity (referred to in § 3), it will be sufficient to use that identity only in its following particular form: If an ordinary Dirichlet series $\sum a_n/n^s$ is absolutely convergent for $\sigma > 1$, then

$$\Gamma(s) \sum_{n=1}^{\infty} a_n/n^s = \int_0^{\infty} x^{s-1} f(e^{-x})\, dx \tag{19}$$

holds for $\sigma > 1$, if $f(r)$ is defined by (10). Choose $a_n = \lambda(n)/n^u$, where u is fixed on the region $1 < u < \infty$. Then $\sum a_n/n^s$ is absolutely convergent for $\sigma > 1$ (and even for $\sigma > 0$) and (19) reduces to

$$\Gamma(s)\zeta(2s + 2u)/\zeta(s + u) = \int_0^{\infty} x^{s-1} f(e^{-x};\ u)\, dx, \tag{20}$$

if use is made of the definitions, (1) and (18), of $\lambda(n)$ and f.

It is now possible to apply Landau's theorem, his extension [7] (to Laplace or Mellin integrals) of the Vivanti-Pringsheim theorem (on power series). In fact, let Landau's theorem be applied to the function (20) of s, *while u is kept fixed* (*but* $u > 1$). Then it follows from (18) that, when the real part of s decreases (from $\sigma = \infty$), the function (of s) on the left of (20) will *first* become singular at a real s (unless it is an entire function-which it is not).

Finally, $\Gamma(s)$ has no zero, and is regular, for $\sigma > 0$, and it is well-known (from the functional equation of ζ) that $\zeta(z)$ has no zero on the half-line $0 < z < \infty$. Since $\zeta(2z)$ is regular on the half-line $\frac{1}{2} < z < \infty$, it follows that the function $\zeta(2s + 2u)/\zeta(s + u)$ of s is regular on the s-domain $\sigma + u > \frac{1}{2}$. Hence, the function $\zeta(2z)/\zeta(z)$ of the complex variable z is regular so long as the real part of z exceeds $\frac{1}{2}$. Thus (4) follows.

By keeping t fixed but letting $\sigma \to \frac{1}{2} + 0$ in $s = \sigma + it$, the additional information (5) follows from (20) and (18) in *precisely* the same way as when (9) is applied to the corresponding integral representation of $\zeta(2s)/\zeta(s)$ (that is to say, in *about* the same way in which (5) is usually concluded from the Mertens hypothesis).

7. No implicative connection between (3) and (6) or between (3) and (13) is known. It is therefore worth mentioning that, so far as (4) and (5) are concerned, Pólya's hypothesis can be relaxed in the same way in which the hypothesis (13) on the Abelian generator, (12), of (1) relaxes Turán's hypothesis.

In fact, let $f(r)$ now denote the case $a_n = \lambda(n)$ of (10) and, instead of (3), suppose merely that there exists a sufficiently small ε^* having the property that

$$(18^*) \qquad f(r) \equiv \sum_{n=1}^{\infty} \lambda(n) r^n \leqq 0 \quad \text{for} \quad 1 - \varepsilon^* < r < 1$$

$(0 < \varepsilon^* < 1)$. Then, since (19) reduces for $a_n = \lambda(n)$ to

$$(20^*) \qquad \Gamma(s)\zeta(2s)/\zeta(s) = \int_0^{\infty} x^{s-1} f(e^{-x})\, dx,$$

(4) [and (5)] can be concluded as before (and, in one respect, more conveniently, since there is now no parameter u to deal with).

It is similarly seen that (4) [and (5)] can be concluded not only from the Mertens hypothesis, $M(x) = O(x^{\frac{1}{2}})$, but also from its Abelian relaxation,

$$(21) \qquad g(r) = O(1-r)^{-\frac{1}{2}},$$

where

$$M(x) = \sum_{n \leqq x} \mu(n) \quad \text{and} \quad g(r) = \sum_{n=1}^{\infty} \mu(n) r^n;$$

that either of the unilateral hypotheses $g(r) = O_L(1-r)^{-\frac{1}{2}}$, $g(r) = O_R(1-r)^{-\frac{1}{2}}$ would also suffice; finally, that not only the relation $M(x) = o(x^{\frac{1}{2}})$, claimed by Stieltjes, is false (Landau) but (for the same reason) not even its Abelian relaxation, $g(r) = o(1-r)^{-\frac{1}{2}}$, can possibly be true.

8. The Abelian reductions considered above, the reductions of Mertens', Pólya's and Turán's hypotheses to (21), (18) and (13) respectively, are more than mere *analytical* subtleties, since there is a *number-theoretical* parallel in which Abelian reduction is basic (under, and for, the truth of Riemann's hypothesis). This instance is the case $a_n = \Lambda(n) - 1$ of (10)-(11); so that

$$\sum_{n=1}^{\infty} a_n/n^s = -\zeta'/\zeta(s) - \zeta(s).$$

Let $h(r)$, $H(x)$ denote the corresponding functions (10), (11), i.e., put

$$h(r) = \sum_{n=1}^{\infty} \Lambda(n) r^n - (1-r)^{-1} \quad \text{and} \quad H(x) = \sum_{n \leqq x} \Lambda(n) - [x].$$

Then the situation is as follows:

(i) *Whether Riemann's hypothesis* (4) *be true or false,* it is certainly true that

$$H(x) \neq O(x^{\frac{1}{2}}), \tag{22}$$

and even

$$H(x) \neq O_R(x^{\frac{1}{2}}) \quad \text{and} \quad H(x) \neq O_L(x^{\frac{1}{2}}). \tag{22 bis}$$

(ii) Nevertheless, *under Riemann's hypothesis* (4), the Abelian relaxation,

$$h(r) = O(1-r)^{\frac{1}{2}}, \tag{23}$$

of the (certainly false) estimate $H(x) = O(x^{\frac{1}{2}})$ is true; in fact, (23) is equivalent to (4), since even

$$h(r) = O_R(1-r)^{\frac{1}{2}} \quad \text{or} \quad h(r) = O_L(1-r)^{\frac{1}{2}} \tag{23 bis}$$

implies (4).

In order to conclude (4) from (23), or just from either of the assumptions (23 bis), it is sufficient to apply (19) to $a_n = \Lambda(n) - 1$, and then to use Landau's theorem [7] in the same way as above. The corresponding (but unconditional) assertion, (22), of (i), and even the unilateral refinement, (22 bis), of (22) goes back to Phragmén (1891-1901; for later references, cf. [6], p. 139). On the other hand, Hardy and Littlewood computed the « Abelian relaxation » of the explicit formula of Riemann-Mangoldt (a relaxation which, incidentally, is much easier to prove than that formula itself), and concluded from it that (23), the Abelian relaxation of the (unconditionally) wrong relaxation $H(x) = O(x^{\frac{1}{2}})$ of (23), is implied by (4); cf. [6], pp. 134-138. Hence (23) and (4) are equivalent (and the whole of (ii) is now clear, since (23) is equivalent to the two relations (23 bis) together).

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