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A CRITERION FOR THE UNIFORM DISTRIBUTION OF SEQUENCES IN COMPACT METRIC SPACES

ROBERT F. TICHY

Let X be a compact metric space, le μ be a non-negative normalized Borel measure on X and let f be a measurable bounded real-valued function defined on X such that f is μ-almost everywhere continuous and different from zero. It is proved that a sequence (x_n), $n = 1, 2, \dots$ of points in X is μ-uniformly distributed if and only if for every Borel set $E \subseteq X$ with $\mu(bd(E)) = 0$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) 1_E(x_n) = \int_E f(x) d\mu(x),$$

where 1_E denotes the characteristic function of E and bdE the boundary of E. Furthermore some quantitative aspects and generalizations of this theorem are discussed.

1. Introduction.

In his classical paper [9] H. Weyl investigated uniformly distributed sequences of points in the unit interval $[0, 1)$.

A sequence (x_n), $n = 1, 2, \dots$ of points $x_n \in [0, 1)$ is said to be uniformly distributed (for short: u.d.) if the number of elements of $x_1, \dots, x_N$ contained in an arbitrary subinterval $I \subseteq [0, 1)$ is asymptotically N-times the length of I.E. Hlawka [5], [6] generalized this concept to sequences with elements in compact metric spaces.

Let X be a compact metric space and μ a non-negative normalized Borel measure on X. A sequence (x_n) with elements $x_n \in X$ is said to be u.d. with respect to μ (for short: μ-u.d.) if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_X d\mu \tag{1.3}$$

for all continuous real-valued functions f on X. In the case of the unit interval $X = [0,1)$ with the ordinary Lebesgue measure, (1.1) is equivalent to the above definition (cf. [8], Theorem 1.1, p. 2).

For the basic facts of the theory of uniform distribution we refer to the monographs of E. Hlawka [4] and of L. Kuipers and H. Niederreiter [8].

Remark 1.3. It is an easy consequence of the proof of [8], Theorem 1.2, p. 175 that a sequence (x_n) with elements in a compact metric space is μ-u.d. if and only if (1.1.) holds for all bounded measurable functions f that are continuous on a set of measure 1. For example, a characteristic function 1_E of a Borel set E ($1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ for $x \notin E$) has this property if $\mu(bdE) = 0$; such Borel sets are called μ-continuity sets.

By [8], Theorem 1.2, p. 175, (x_n) is μ-u.d. if and only if (1.1.) holds for all characteristic functions 1_E of μ-continuity sets E.

The object of this article is to give an extension of the following result of J. Horbowicz [7]: Let $f : [0,1) \to \mathbb{R}$ be a Riemann-integrable function which is almost everywhere non-zero and let (x_n) be a sequence with elements in $[0,1)$; then (x_n) is u.d. if and only if for every subinterval $[a,b)$ of $[0,1)$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n)1_{[a,b)}(x_n) = \int_a^b f(x)dx.$$

In section 2 we will prove

THEOREM 1. *Let X be a compact metric space and μ a non-negative normalized Borel measure on X and let $f : X \to \mathbb{R}$ be a bounded measurable function such that f is μ-almost everywhere continuous and different from zero. Then the sequence (x_n) with elements in X is μ-u.d. if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{n} f(x_n)1_E(x_n) = \int_E f(x)d\mu. \tag{1.3}$$

for every μ-continuity set $E \subseteq X$.

Remark 1.4. By Halmos [3], Sec. 50-52, any non-negative normalized Borel measure on a compact metric space is regular, i.e. for any Borel set $E \subseteq X$ we have

$$\mu(E) = \sup\{\mu(C) : C \subseteq E, C \text{ closed}\} = \inf\{\mu(D) : E \subseteq D, D \text{ open}\}.$$

We use the notations int E for the interior of E and $\bar{E}$ for the closure of a set $E \subseteq X$.

In section 3 a generalization of Theorem 1 to weighted means is established. Furthermore a concept of discrepancy is introduced for sequences with elements in an arbitrary compact metric space; quantitative aspects of Theorem 1 are investigated and extensions of several results of Fleischer [1], [2] are obtained.

2. Proof of Theorem 1.

The proof of Theorem 1 is based on the following well-known lemmas; see [8], p. 179, Exercises 1.5. and 1.6.

LEMMA 1. *Let (X, d) be a metric space and μ a non-negative normalized Borel measure on X. Then for any $x \in X$ and any $\varepsilon > 0$ there exists a ball $B = B(x,r) = \{y \in X : d(x,y) < r\}$ with $0 < r < \varepsilon$ and $\mu(bdB) = 0$.*

Proof. Let us consider the family $\mathcal{F}$ of all balls $B(x,r)$ with center x and $0 < r < \varepsilon$. Since $bdB(x,r) \subseteq \{y \in x : d(x,y) = r\}$, the boundaries of two balls $B(x,r_1)$ and $B(x,r_2)$ with $r_1 \neq r_2$ are disjoint. Furthermore $\mathcal{F}$ contains more than countably many balls and so, because of $\mu(X) = 1$, there must exist a ball $B(x,r)$ with $\mu(bdB(x,r)) = 0$.

LEMMA 2. *A sequence (x_n) with elements in a compact metric space X is μ-u.d. in X if and only if*

$$\lim_{N\to\infty} \inf \sum_{n=1}^{N} 1_V(x_n) \geq \mu(V) \tag{2.1}$$

for all open sets $V \subseteq X$.

Proof. Let E be a μ-continuity set and put $C = \bar{E}, D =$int E. Then we have by (2.1)

$$\liminf_{N\to\infty} \frac{1}{N} 1_E(x_N) \geq \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_D(x_n) \geq \mu(D) = \mu(E),$$

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}1_E(x_n)\le 1-\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}1_{X\setminus C}(x_n)\le 1-\mu(X\setminus C)=\mu(C)=\mu(E).$$

By remark 1.2. it follows immediately that (x_n) is μ-u.d. Now let (x_n) be μ-u.d. Let V be an arbitrary open set, $C=X\setminus V$. Then, by Remark 1.4., for any $\varepsilon>0$ there exists an open set $U\supseteq C$ with $\mu(V\setminus C)\le\varepsilon$. Applying Lemma 1 we can find for all $x\in U$ a ball $B\subseteq U$ with $x\in B$ and $\mu(b\alpha\beta)=0$. Since C is compact, there exists a finite covering $F=\bigcup_{i=1}^{h}B_i$ of C consisting of such balls B_i. Since F is a μ-continuity set and (x_n) is μ-u.d., we obtain by Remark 1.2. and $V\supseteq F\supseteq C$, $\mu(F)-\mu(C)\le\varepsilon$ that

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}1_V(x_n)\ge 1-\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}1_C(x_n)\ge 1-\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}1_F(x_n)=$$

$$=1-\mu(F)\ge 1-\mu(C)-\varepsilon=\mu(V)-\varepsilon$$

for all $\varepsilon>0$. Hence we have (2.1) and the proof of Lemma 2 is complete.

LEMMA 3. *Let f be a bounded real-valued function on the metric space (X,d). We put $\limsup_{y\to x}f(y)=S$ if and only if for every $\varepsilon>0$ there is an open ball $B=B(x,\delta)=\{y:d(x,y)<\delta\}$ such that $f(y)\le S+\varepsilon$ for any $y\in B$ and a sequence (x_n) with $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}f(x_n)=S$. Furthermore we put $\liminf_{y\to x}f(y)=-\limsup_{y\to x}(-f(y))$ and*

$$\omega(f,x)=\limsup_{y\to x}f(y)-\liminf_{y\to x}f(y).$$

Then the set $D_0=\{x\in X:\omega(f,x)\ge y_0\}$ is closed.

We omit a proof of this Lemma since it is well-known and standard.

We begin the proof of Theorem 1 with the observation that (1.3) immediately follows from the μ-u.d. of the sequence (x_n) because of Remark 1.2. Now let $Z=Z(f)=\{x\in X:f(x)=0\}$ denote all zero-points of f and $D=D(f)$ the set of all discontinuity points of f. For every integer $m\ge 1$ we set

$$Z_m=\{x\in X:|f(x)|\le\frac{1}{m}\}$$

and

$$D_m=\{x\in X:\omega(f,x)\ge\frac{1}{m^2}\}.$$

Clearly, $Z = \bigcap_{m=1}^{\infty} Z_m$ and $D = \bigcup_{m=1}^{\infty} D_m$, hence D is measurable (D_m closed). Since $\mu(Z) = \mu(D) = 0$ and $Z_m \subseteq \bar{Z}_m \subseteq Z_m \cup D$ (Z_m measurable), we obtain

$$\lim_{m\to\infty} \mu(Z_m \cup D) = \lim_{m\to\infty} \mu(Z_m) = 0 \text{ because of } \limsup_{m\to\infty} \mu(Z_m \cup D) = 0. \tag{2.1}$$

Hence we have for any $\varepsilon > 0$

$$\mu(\bar{Z}_m \cup D) < \varepsilon \tag{2.2}$$

for all sufficiently large $m \geq m_0(\varepsilon)$. By Lemma 3, D_m is closed and so $\bar{Z}_m \cup D_m$ is a compact set. Let $V \subseteq X$ be an open set and choose an arbitrary $\varepsilon > 0$. Consider a fixed integer $m \geq \max(m_0, \frac{1}{\varepsilon})$.

By (2.2) and Remark 1.4 there exists an open set $U \supseteq (\bar{Z}_m \cup D_m)$ and a compact set $C \subseteq V$ such that

$$\mu(\bar{Z}_m \cup D_m) < \mu(U) < 2\varepsilon, \quad \mu(V \setminus C) < \varepsilon. \tag{2.3}$$

Furthermore for every point $x \in C \setminus U$, by Lemma 1, there exists an open ball $B = B(x, r_m(x))$ with $\mu(bdB) = C$ and

$$\sup_{y\in B} f(y) - \inf_{y\in B} f(y) < \frac{1}{m^2}. \tag{2.4}$$

Since $C \setminus U$ is compact there is a finite covering $B_1, \ldots, B_t$ (t minimal) of $C \setminus U$ with open balls $B_j (j = 1, \ldots t)$ having property (2.4). Next we define a finite family $A_j (j = 1, \ldots, t)$ of disjoint, non-empty, measurable sets recursively by

$$A_1 = (C \setminus U) \cap B_1 \tag{2.5}$$

$$A_j = ((C \setminus U) \cap B_j) \setminus (\bigcup_{i=1}^{i-1} A_i).$$

We set $A_0 = (V \setminus C) \cup (C \cap U)$ and obtain

$$\mu(A_0) < 3\varepsilon, \quad \bigcup_{J=0}^{t} A_j = V. \tag{2.6}$$

For fixed A_j put $m_j = \inf_{y\in A_j} f(y)$ and $M_j = \sup_{y\in A_j} f(y)$. We obtain by the definition of U

$$|M_j|, \ |m_j| \geq \frac{1}{m} \quad \text{and} \quad M_j - m_j < \frac{1}{m_2} \quad (j = 1, \ldots, t). \tag{2.7}$$

Hence f is of constant sign on A_j. First we suppose that f is positive on A_j. Since $M_j \geq \frac{1}{m}$ and $m \geq \frac{1}{\varepsilon}$ we get

$$\frac{m_j}{M_j} > 1 - \frac{1}{m_j m^2} \geq 1 - \frac{1}{m} \geq 1 - \varepsilon \quad (j = 1, \ldots, t). \tag{2.8}$$

Furthermore by the hypothesis of the theorem we obtain

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_{A_j}(x_n) \geq \liminf_{N\to\infty} \frac{1}{NM_j} \sum_{n=1}^{N} f(x_n) 1_{A_j}(x_n) = \tag{2.9}$$

$$= \frac{1}{M_j} \int_{A_j} f(x) d\mu \geq \frac{m_j}{M_j} \mu(A_j) \geq (1-\varepsilon)\mu(A_j).$$

This estimate can be shown in the same way if f is negative on A_j.

Summing up for $j = 0, 1, \ldots t$ and using (2.6) we derive from (2.9)

(2.10)

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_V(x_n) \geq \sum_{j=0}^{t} \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_{A_j}(x_n) \geq \sum_{j=1}^{t} (1-\varepsilon)\mu(A_j) \geq \mu(V) - 4\varepsilon.$$

Since $\varepsilon > 0$ can be taken arbitrary small, the proof of Theorem 1 is completed by Lemma 2.

Since the metric d of X is only used to construct μ-continuity sets, it is clear that Theorem 1 can be generalized to arbitrary compact Hausdorff spaces; the μ-continuity sets can be constructed by means of Urysohn functions (cf. [8], p. 174). Furthermore the assumption on f can be slightly relaxed:

THEOREM 1*. *Let X be a compact Hausdorff space and μ a non-negative normalized regular Borel meausre on X. Let $f : X \to \mathbb{R}$ be a bounded, measurable and μ-almost everywhere continuous function and let $Z(f) = \{x \in X : f(x) = 0\}$. If*

$$\mu(Z(f)) = 0 \tag{2.11}$$

or

$$\text{card } Z(f) = 1, \tag{2.12}$$

then any sequence (x_n) of points in X is μ-u.d. if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) 1_E(x_n) = \int_E f(x) d\mu \tag{2.13}$$

for every μ-continuity set $E \subseteq X$.

Proof. If (2.11) holds, the assertion follows in the same way as in the proof of Theorem 1 (using however open sets insetad of balls). Now suppose that (2.12) holds and $\mu(Z(f)) > 0$. Let z be the only element of $Z(f)$ and let E be an arbitrary μ-continuity set. First we assume that $z \in \text{int}\,(X \setminus E)$. Then $f(x) \neq 0$ for every $x \in \bar{E}$. As in the proof of Theorem 1 we obtain that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n} 1_E(x_n) \geq \mu(\text{int}E),$$

and similarly

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n} 1_{\bar{E}}(x_n) \leq \mu(\bar{E}).$$

If $z \in \text{int } E$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n} 1_{X\setminus E}(x_n) = \mu(X \setminus E). \tag{2.14}$$

Since $\mu(E) + \mu(X \setminus E) = 1$, it follows that also the set E satisfies (2.14), and the proof can be completed easily.

3. A generalization and some quantitative aspects

Let $A = (a_{Nn})$ be a positive Toeplitz matrix (cf. [8]); i.e.

$$a_{Nn} \geq 0 \quad \lim_{N\to\infty} \sum_{n=1}^{\infty} a_{Nn} = 1, \quad \sup a_{Nn} < \infty.$$

A sequence (x_n) is said to be (A, μ)-u.d. if and only if

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} a_{Nn} g(x_n) = \int_X g(x) d\mu \tag{3.1}$$

for all continuous real-valued functions g on X. In the case $a_{Nn} = \frac{1}{N}(1 \leq n \leq N)$ and $a_{N'n} = 0\,(n > N)$ the classical concept of μ-u.d. is obtained. The proof of the following result runs along the same lines as the proof of Theorem 1.

THEOREM 2. *Let X be a compact metric space and μ a non-negative normalized Borel measure on X and let f be a bounded, measurable, real-valued and μ-almost everywhere non-zero and continuous function on X; $A = (a_{Nn})$ denotes a positive Toeplitz matrix. Then the sequence (x_n), $n = 1, 2\ldots$ is (A, μ)-u.d. if and only if*

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} a_{Nn} f(x_n) 1_E(x_n) = \int_E f(x) d\mu \tag{3.2}$$

for every μ-continuity set $E \subseteq X$.

Remark 3.3. It is easy to show (compare to Remark 1.2) that (3.2) is equivalent to

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} a_{Nn} f(x_n) g(x_n) = \int_X f(x) g(x) d\mu \tag{3.4}$$

for all bounded, measurable, real-valued functions g the disconties of which are contained in a null set.

Furthermore, similar to Theórem 1 a stronger version of Theorem 2 can be shown.

Suggested by [1], [2], we consider an arbitrary normed space B and a bounded linear operator $T : B \to C(X)$, where $C(X)$ denotes the Banach space of all real-valued continuous functions (with uniform convergence).

By $\langle TB, 1\rangle$ we denote the linear subspace of $C(X)$ spanned by the T-image of B and the identity: $\overline{\langle TB, 1\rangle}$ denotes the closure of $\langle TB, 1\rangle$ (in $C(X)$). Let T be as above and f a given bounded real-valued and μ-almost everywhere continuous and non-zero function. Then we introduce for any sequence (x_n), $x_n \in X$ the linear functionals $L_N : B \to \mathbb{R}$

$$L_n(g) = \sum_{n=1}^{\infty} a_{Nn} f(x_n)(Tg)(x_n) - \int_X f(x)(Tg)(x) d\mu. \tag{3.5}$$

In the followng $||L_N||$ denotes the norm of L_N.

Example 3.6. Let $A = (a_{Nn})$ be the arthmetic mean: $a_{Nn} = \frac{1}{N}$ for $1 \leq n \leq N$, $a_{Nn} = 0$ for $n > N$; let $X = [0, 1)$, $B = L^1[0, 1]$ and T is defined by

$$(Tg)(x) = \int_0^1 1_{[0,y)}(x) g(y) dy \quad \text{for} \quad g \in L^1[0, 1].$$

Let (x_n) bean arbitrary sequence with $x_n \in [0,1)$. Obviously T is a bounded linear operator. For the linear functional $L_N \in (L^1)^*$ we have

$$L_N(g) = \frac{1}{N}\sum_{n=1}^{N} f(x_n)(Tg)(x_n) - \int_0^1 f(x)(Tg)(x)dx =$$

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n)\int_0^1 1_{[0,y)}(x_n)g(y)dy - \int_0^1 f(x)\int_0^1 1_{[0,y)}(x)g(y)dydx =$$

$$\int_0^1 \left(\frac{1}{N}\sum_{n=1}^{N} f(x_n)1_{[0,y)}(x_n) - \int_0^y f(x)dx\right) g(y)dy.$$

From this it follows that

$$||L_N|| = \sup_{0\leq y\leq 1} \left|\frac{1}{N}\sum_{n=1}^{N} f(x_n)1_{[0,y)}(x_n) - \int_0^y f(x)dx\right|.$$

Hence $||L_N||$ is a generalization of the classical concept of discrepancy (cf. [4] and [8]). The following theorem is a generalization of the well-known fact that a sequence is u.d. if and only if its discrepancy tends to 0.

THEOREM 3. *Let T be a compact operator given as above and $\overline{\langle TB, 1\rangle} = C(X)$. Then the sequence (x_n) with elements in a compact metric space X is (A,μ)-u.d. if and only if* $\lim_{N\to\infty} ||L_N|| = 0$. *($A = (a_{Nn})$ denotes a positive Toepliz matrix, μ a non-negative Borel measure and L_N is defined as in (3.5)).*

Proof. If $\lim_{N\to\infty} ||L_N|| = 0$ then (3.4) holds for all $g \in TB$. Since $\lim_{N\to\infty}\sum_{n=1}^{\infty} a_{Nn} = 1$, (3.4) holds also for $g \equiv 1$. Furthermore $\sup_N ||L_N|| < \infty$ and so (3.4) is valid for all $g \in C(X)$. By Remark 3.3, (x_n) is (A,μ)-u.d. Now we assume that (3.4) holds for all $g \in C(X)$. Since T is compact it follows that the image TS of the unit sphere S of B is relatively compact. Hence for any $\varepsilon > 0$ there exists a finite ε-net $g_1, \dots, g_r$ in $\overline{TS}$. Let $g \in S$ and $\varepsilon > 0$; then there exist positive integers $N(\varepsilon)$ and j with $1 \leq j \leq r$ such that for all $n \geq N(\varepsilon)$

$$\left|\sum_{n=1}^{\infty} a_{Nn} f(x_n)(Tg)(x_n) - \int_X f(x)(Tg)(x)d\mu\right| \leq$$

$$|\sum_{n=1}^{\infty} a_{Nn} f(x_n)(Tg)(x_n) - \sum_{n=1}^{\infty} a_{Nn} f(x_n) g_j(x_n)| +$$

$$+|\sum_{n=1}^{\infty} a_{Nn} f(x_n) g_j(x_n) - \int_X f(x) g_j(x) d\mu| + |\int_X f(x) g_j(x) d\mu - \int_X f(x)(T_g)(x) d\mu|$$

$$< MF\varepsilon + \varepsilon + F\varepsilon,$$

where $M = \sup_N \sum_{n=1}^{\infty} a_{Nn}$, $F = \sup_{x \in X} |f(x)|$. Hence $\lim_{N\to\infty} ||L_N|| = 0$ and the proof of Theorem 3 is complete.

Remark 3.7. In [1] it is shown that for any normed space B and any compact metric space X there exists a compact operator T with the property $\overline{\langle TB, 1\rangle} = C(X)$. Furthermore we want to remark that the following converse of Theorem 3 can be proved:

Assume that (x_n) *is* (A,μ)*-u.d. if and only if* $\lim_{N\to\infty} ||L_N|| = 0$. *Then* T *is compact and* $\langle TB, 1\rangle = C(X)$.

We do not work out the proof in detail since it is rather technical; for a special case see [2], Satz 2.

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Robert F. Tichy
Abt. f. Techn. Math.
Technical Univ. Vienna
Wiedner Hauptstrasse 8-10
1040 Vienna, AUSTRIA