FIRST EIGENVALUES AND COMPARISON OF GREEN'S FUNCTIONS FOR ELLIPTIC OPERATORS ON MANIFOLDS OR DOMAINS

By

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Abstract. Given a complete Riemannian manifold M (or a region U in $\mathbb{R}^N$) and two second-order elliptic operators L_1, L_2 in M (resp. U), conditions, mainly in terms of proximity near infinity (resp. near ∂U) between these operators, are found which imply that their Green's functions are equivalent in size. For the case of a complete manifold with a given reference point O the conditions are as follows: L_1 and L_2 are weakly coercive and locally well-behaved, there is an integrable and nonincreasing positive function Φ on $[0, \infty[$ such that the "distance" (to be defined) between L_1 and L_2 in each ball $B(x, 1) \subset M$ is less than $\Phi(d(x, O))$. At the same time a continuity property of the bottom of the spectrum of such elliptic operators is proved. Generalizations are discussed. Applications to the domain case lead to Dini-type criteria for Lipschitz domains (or, more generally, Hölder-type domains).

Introduction

In this paper, we mainly consider the following question. Given a complete Riemannian manifold M (or a region U in $\mathbb{R}^N$) and two second-order elliptic operators on M (resp. U), what condition of proximity near infinity (resp. near ∂U) between these operators insures that their Green's functions are equivalent in size? If each of these operators is connected to a diffusion, the last property essentially means that the related hitting probabilities are also uniformly comparable.

It turns out that the condition given in our main result (Theorems 1 and 1′, and the euclidean versions Theorems 9.1 and 9.1′) is a generalization of one part of a result by L. Carleson (see [C] Theorem, p. 1) which gives a sufficient (and in some sense necessary) condition for a second-order elliptic operator acting in the half-plane to have harmonic measures with bounded densities. The other part of the theorem in [C], namely a condition for the absolute continuity of these harmonic measures, has been deeply generalized in several papers starting with [FKJ] (see [FKP] and references there); in [FKP] a criterion for the mutual absolute continuity of the harmonic measures with respect to two elliptic operators in the unit ball of $\mathbb{R}^N$ is given. In contrast with these papers, the main results below do not rely on harmonic analysis techniques and require only a few structural assumptions on M. As a result, they may also be applied to domains which are far from Lipschitz (see Section 9). A crucial source of inspiration for us is the work of J. Serrin

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[Ser], where a result of our type for Poisson kernels of C^2 domains is proved. See Section 3.

Comparability (in the above sense) of Green's functions has been already studied in various situations involving regions in $\mathbb{R}^N$. [HS] is concerned with bounded $C^{1,1}$ domains—see also [Ser] and note that extensions to Dini–Liapounov-type regions follow from Widman [W1], [W2] (see [W1], p.523) — and [A1] with Lipschitz domains; in both papers the second-order coefficients are C^α, $0 < \alpha \leq 1$, up to the boundary. Results for global perturbations of the Laplace operator in $\mathbb{R}^N$ appear in [Pi4] (see also [Pi1]).

Other results deal with lower-order perturbations (mainly in domains in $\mathbb{R}^N$). Murata [Mu] shows among other things the stability of the classical Green's function in $\mathbb{R}^N$ under certain kinds of perturbations (see the final note there); [Pi3] considers more general operators and domains and introduces a notion of small perturbation (see also [Pi2]); and [Z1], [Z2] deal with Schrödinger operators satisfying a Kato class condition at infinity. See also [Pi3] and the references there. [CZ] studies Δ and $\Delta + B.\nabla$, $B \in L^p(D), p > N$, in a bounded domain D and shows in particular that when D is C^2 the corresponding Green's functions are equivalent.

For the manifold case, [SC2] exhibits a class of complete manifolds (e.g. complete manifolds of nonnegative Ricci curvature) for which all uniformly elliptic operators in divergence form and without lower-order terms have Green's functions equivalent in size (see [SC1], [SC2] for background and related references). We also note that independently of the present paper U. Hamenstädt ([Ham], Appendix) shows a stability property of the Martin kernels with respect to a class of elliptic operators with Hölder continuous coefficients for Cartan–Hadamard manifolds of pinched negative sectional curvatures, a result which is close to Theorem 1′ in Section 7 below.

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1. Notations and general assumptions. Statement of main results

We start with a Riemannian manifold M with bounded geometry and define as in [A3] classes of second-order elliptic operators in divergence form on M. Elliptic operators in nondivergence form will be considered later in Section 7. In Section 9, the results for domains in $\mathbb{R}^N$ are obtained as particular cases, using the same approach as [A3], §8. See also Section 6.

1.1. In what follows, M is a noncompact, connected, complete N-dimensional Riemannian manifold of class C^1 with the following property: there exists two positive numbers r_0 and c_0 and for each $a \in M$ a chart $\psi = \psi_a : B_a \to \mathbb{R}^N$ in the ball $B_a = B(a, r_0)$ of M such that $\psi(a) = 0$ and

$$c_0^{-1}\, d(x,y) \leq |\psi(x) - \psi(y)| \leq c_0\, d(x,y) \tag{1.1}$$

for $x,y \in B_a$; in particular, $U_a = \psi(B_a)$ contains the ball $B(0, r_0/c_0)$ of $\mathbb{R}^N$. For convenience, we may and will assume that $r_0 \leq \frac{1}{4}$. The obvious dependence on r_0 and c_0 of the various constants to appear below will be implicit, and as usual the letter c (or C) will refer to a positive constant whose value may change from line to line. The Riemannian volume in M is denoted by σ (or σ_M).

1.2. Examples. 1. The assumptions above are satisfied (with the exponential chart at $a \in M$) if M is C^3 with bounded sectional curvatures and injectivity radius bounded from below.

2. Another example (in fact, a special case of the previous one) is obtained by taking for M an open region Ω in $\mathbb{R}^N$, $\Omega \neq \mathbb{R}^N$, equipped with the metric $g_x(u,u) = \tilde{\delta}(x)^{-2}\,|u|^2$, where $\tilde{\delta}$ is a standard C^1-regularization of the distance function $\delta(x) = d(x, \partial\Omega)$; that is, $c^{-1}\delta(x) \leq \tilde{\delta}(x) \leq c\,\delta(x)$ and $|\nabla\tilde{\delta}(x)| \leq c$ on Ω, $c > 0$ (see [A3] §8).

1.3. Let θ and p be real numbers such that $p > N = \dim(M)$ and $\theta \geq 1$. We denote by $\mathcal{D}_M(\theta,p)$ the class of all elliptic operators $\mathcal{L}$ on M with a *given* representation in the following form:

$$\mathcal{L}u = \operatorname{div}(\mathcal{A}(\nabla u)) + D.\nabla u + \operatorname{div}(uD') + \gamma u. \tag{1.2}$$

Here $\mathcal{A} : x \mapsto \mathcal{A}_x \in \operatorname{End}(T_x(M))$ is a Borel section of the bundle $\operatorname{End}(T(M))$, D and D' are Borel vector fields on M, and γ is a real valued Borel function in M. It is further assumed that

$$\theta^{-1}\,|\xi|^2 \leq \langle \mathcal{A}_a(\xi), \xi\rangle \leq \theta\,|\xi|^2, \tag{1.3}$$

$$\|\mathcal{A}_a\|_{\operatorname{End}(T_a(M))} + \|D\|_{L^p(B_a)} + \|D'\|_{L^p(B_a)} + \|\gamma\|_{L^{p/2}(B_a)} \leq \theta, \tag{1.4}$$

when $a \in M$ and $\xi \in T_a(M)$. Recall that $B_a = B(a, r_0)$.

Some Sobolev spaces attached to a region U in M will be needed. Define $H^1(U)$ as the space of all functions $f \in L^2(U)$ with a weak gradient in $L^2(U)$—i.e. there is a L^2 vector field $V = \nabla f$ in U such that $\int V.W\,d\sigma = -\int f \operatorname{div}(W)\,d\sigma$ for all vector field W of class $C_0^1(U)$—equipped with the norm $\|f\|_{H^1(U)} = (\|f\|^2_{L^2(U)} + \|\nabla f\|^2_{L^2(U)})^{1/2}$. Let $H_0^1(U)$ denote the closure of $C_0^1(U)$ in $H^1(U)$. The dual $H^{-1}(U)$ of $H_0^1(U)$ is identified with the set of distributions S in U of the form $S = u + \operatorname{div}(V)$ where u (resp. V) is a function (resp. a vector field) in U

of class L^2. The spaces $H^1_{loc}(U)$ and $H^{-1}_{loc}(U)$ are defined in the obvious way and each operator $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ induces a map $\mathcal{L} : H^1_{loc}(U) \to H^{-1}_{loc}(U)$. (See [Sta] and Proposition 2.1 below.)

A function u in the region $U \subset M$ is an $\mathcal{L}$-solution if $u \in H^1_{loc}(U)$ and $\mathcal{L}(u) = 0$ on U. As is well-known, u is (after modification on a σ-null set) a continuous function in U and, if u is positive, the following Harnack inequalities hold:

$$(1.5) \qquad c^{-1}\, u(a) \le u(x) \le c\, u(a)$$

if $B(a,r) \subset U$, $r \le r_0$ and $d(x,a) \le r/2$, where $c = c_M(\theta,p) \ge 1$. Moreover, there are positive constants c' and β depending on θ, p and M such that

$$(1.6) \qquad (1 + c'(\rho/r)^\beta)^{-1}\, u(a) \le u(x) \le (1 + c'(\rho/r)^\beta)\, u(a)$$

if $d(x,a) \le \rho \le r/2$. A well-behaved local potential theory ([Her], [B]), whose harmonic functions are the $\mathcal{L}$-solutions is attached to $\mathcal{L}$ in M. (Using local charts we are left with the standard case $M = \mathbb{R}^N$, ref. [GT], [Sta], [HH].) Hence, we may speak of $\mathcal{L}$-superharmonic functions, $\mathcal{L}$ potentials, and so forth (ref. [B]).

1.4. Let $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ and let U be an open subset of M. Denote $\mathcal{S}_t(U)$ the set of all $\mathcal{L} + tI$-superharmonic functions in U and define the *critical level* of $\mathcal{L}$ in U as the number

$$(1.7) \qquad \lambda_1(\mathcal{L}, U) = \sup\{t \in \mathbb{R};\ \exists u \in \mathcal{S}_t(U),\ 0 < u < \infty \text{ in } U\}.$$

$\lambda_1(\mathcal{L}, U)$ is also the largest number t for which there exists a positive solution u on U to $\mathcal{L}(u) + tu = 0$. For $t < \lambda_1(U)$ the Green's function in U for $\mathcal{L} + tI$ exists. If $\mathcal{L}$ is formally self-adjoint, then $\lambda_1(\mathcal{L}, U)$ coincides with the usual bottom of the spectrum of $-\mathcal{L}$ seen as an unbounded operator on $L^2(U)$ with domain $\{u \in H^1_0(U)\,;\, \mathcal{L}(u) \in L^2(U)\}$ and $\lambda_1(\mathcal{L}, U) = \inf\{\langle -\mathcal{L}(\varphi), \varphi\rangle\,;\, \varphi \in C^1_0(U), \|\varphi\|_{L^2(U)} = 1\}$. Except the last equality, this interpretation of $\lambda_1(\mathcal{L}, U)$ holds also if the symmetry assumption on $\mathcal{L}$ is removed when U is relatively compact in M. We shall let $\lambda_1(\mathcal{L}) = \lambda_1(\mathcal{L}, M)$.

For $\varepsilon_0 > 0$, we denote $\mathcal{D}_M(\theta_0,p,\varepsilon_0)$ the class of all $\mathcal{L} \in \mathcal{D}_M(\theta_0,p)$ satisfying the following "weak coercivity" condition ([A3]):

(1.8) *There is a positive $\mathcal{L} + \varepsilon_0.I$-superharmonic function ($\not\equiv +\infty$) on M,*

i.e. $\lambda_1(\mathcal{L}) \ge \varepsilon_0$. This condition implies the existence of the Green's function G for $\mathcal{L}$, together with the estimate

$$(1.9) \qquad c^{-1} \le G(x,y) \le c$$

for some $c = c(\theta_0, p, \varepsilon_0) > 0$ and all x, y in M such that $d(x,y) = r_0$ (see [A3]). However, if $\mathcal{L}$ is not formally self-adjoint $G(x,y)$ need not be bounded when $d(x,y) \geq 1$. If $G = G^U$ is the Green's function in U, our convention is that $x \mapsto G(x,y)$ is $\mathcal{L}$-superharmonic in U (and harmonic in $U \setminus \{y\}$) whereas $y \mapsto G(x,y)$ is superharmonic in U (and harmonic in $U \setminus \{x\}$) with respect to the adjoint operator $\mathcal{L}^*$. Recall that for φ in $L^2(U,\sigma)$ and compactly supported $G(\varphi) = \int G(.,y)\,\varphi(y)\,d\sigma(y)$ solves $\mathcal{L}(G(\varphi)) = -\varphi$ in U with $G(\varphi) \in H^1_{loc}(U)$. Also, if we let $G(\varphi)(x) = 0$ on $M \setminus U$, then $G(\varphi) \in H^1_{loc}(M)$.

1.5. A reference point $O \in M$ is fixed and we set $d(x) = d(O,x)$ for $x \in M$. If $q \in [1,+\infty]$ and if $\mathcal{L}_1$ and $\mathcal{L}_2$ are members of some class $\mathcal{D}_M(\theta,p)$ with representations $\mathcal{L}_j(u) = \operatorname{div}(\mathcal{A}_j(\nabla u)) + D_j.\nabla u + \operatorname{div}(u D'_j) + \gamma_j u$, we define (recall that $B_a = B(a, r_0)$)

$$\begin{aligned}\operatorname{dist}_q[\mathcal{L}_1,\mathcal{L}_2](a) =&\|\mathcal{A}_1 - \mathcal{A}_2\|_{L^q(B_a)} + \|D_1 - D_2\|_{L^p(B_a)} \\ &+ \|D'_1 - D'_2\|_{L^p(B_a)} + \|\gamma_1 - \gamma_2\|_{L^{p/2}(B_a)},\end{aligned} \tag{1.10}$$

for $a \in M$; to avoid heavier notation, p is made implicit in the l.h.s. of (1.10). If $\Psi : [0,+\infty) \to \mathbb{R}_+$ is non-increasing, we shall write "$\operatorname{dist}_q[\mathcal{L}_1,\mathcal{L}_2] \prec \Psi$ in $\mathcal{D}_M(\theta,p)$" if $\operatorname{dist}_q[\mathcal{L}_1,\mathcal{L}_2](a) \leq \Psi(\rho)$ when $a \in M$ and $d(a) = \rho$. A similar notion appears in [FKP] for second-order elliptic operators in the unit ball of $\mathbb{R}^N$.

1.6. We may now state our main result. See Section 6 for generalizations to non-weakly-coercive operators.

Theorem 1 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be elements of $\mathcal{D}_M(\theta,p,\varepsilon_0)$ (with $p > N$ and $\varepsilon_0 > 0$) and let G^1 and G^2 be the corresponding Green's functions. If $\operatorname{dist}_\infty(\mathcal{L}_1,\mathcal{L}_2) \prec \Psi$ in $\mathcal{D}_M(\theta,p)$ for some nonincreasing function Ψ on $[0,\infty)$ such that $\int_0^{+\infty} \Psi(s)\,ds < +\infty$, there is a constant $c > 0$ such that*

$$c^{-1}\, G^2(x,y) \leq G^1(x,y) \leq c\, G^2(x,y) \tag{1.11}$$

for x, y in M such that $d(x,y) \geq r_0$. In fact, $\operatorname{dist}_\infty(\mathcal{L}_1,\mathcal{L}_2)$ may be replaced above by $\operatorname{dist}_{q_0}(\mathcal{L}_1,\mathcal{L}_2)$, for some $q_0 \in [1,+\infty[$ depending only on c_0, N, θ and p.

Moreover, for every $\delta > 0$ there is a number $\eta = \eta(M,\theta,p,\varepsilon_0,\delta) > 0$ such that if also $\int_0^{+\infty} \Psi(s)\,ds \leq \eta$, we may then let $c = 1 + \delta$ in (1.11).

The proof is given in Section 5. When M is a negatively curved Cartan–Hadamard manifold, Theorem 3 in Section 8 gives a version of Theorem 1 which is—roughly speaking—localised at one point on the sphere at infinity.

Remarks 1.1 (i) Let $\mathcal{L}_1, \mathcal{L}_2$ be members of $\mathcal{D}_M(\theta,p,\varepsilon)$, $\varepsilon > 0$. If $\operatorname{dist}_1(\mathcal{L}_1,\mathcal{L}_2) \prec \Psi$ in $\mathcal{D}_M(\theta,p)$ with $\Psi(t) = c\,\exp(-\alpha t)$, $\alpha > 0$, then $\operatorname{dist}_{q_0}(\mathcal{L}_1,\mathcal{L}_2) \prec \Phi$ with

$$\Phi(t) = (c + c^{1/q_0})\,(2\theta)^{(q_0-1)/q_0} \exp\Big(-\frac{\alpha}{q_0}t\Big)$$

since $\|\mathcal{A}_1 - \mathcal{A}_2\|_\infty \leq 2\theta$. Hence Theorem 1 applies and (1.11) holds.

(ii) By (1.9) and standard local estimates of Green's functions ([Sta]) (1.11) holds for $d(x,y) \leq r_0$ and another constant c.

We shall also prove (and use in the proof of Theorem 1) the following continuity property of $\lambda_1(\mathcal{L})$ with respect to $\mathcal{L}$ in $\mathcal{D}_M(\theta,p)$. See Section 4.

Theorem 2 *Let $\theta \geq 1$, $p > N$ be fixed. For every $\delta > 0$ there is number $\eta > 0$ such that if $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathcal{D}_M(\theta,p)$ and* $\text{dist}_1(\mathcal{L}_1, \mathcal{L}_2) \leq \eta$ *on M, then*

$$|\lambda_1(\mathcal{L}_1) - \lambda_1(\mathcal{L}_2)| \leq \delta. \tag{1.12}$$

In fact, λ_1 is Lipschitz continuous in $\mathcal{D}_M(\theta,p)$ with respect to the distance $d(\mathcal{L}, \mathcal{L}') = \|\text{dist}_{q_0}(\mathcal{L}, \mathcal{L}')\|_{\infty,M}$.

For $\mathcal{L}_j$ symmetric and without lower-order terms the statement is straightforward if dist_1 is replaced by dist_∞ (just use Rayleigh quotients). We also note that the Lipschitz continuity of λ_1 with respect to lower-order coefficients in L^∞ and for non-divergence-type elliptic operators is proved in [BNV] §5.

Remark 1.2 It follows from Theorem 2 that if $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p,\varepsilon_0)$, $\varepsilon_0 > 0$, and $\mathcal{L}_2 \in \mathcal{D}_M(\theta,p)$ are such that $\text{dist}_{q_0}(\mathcal{L}_1, \mathcal{L}_2) \prec \Psi$ in $\mathcal{D}_M(\theta,p)$ with Ψ decreasing on $(0,+\infty)$ and $\int_0^{+\infty} \Psi(s)\,ds$ small enough (depending on M, θ, p, and ε_0), then $\mathcal{L}_2 \in \mathcal{D}_M(\theta,p,\varepsilon_0/2)$ (see Lemma 2.6 and Remark 2.4). Thus, Theorem 1 applies.

Another criterion for $\lambda_1(\mathcal{L}) > 0$ follows from Theorem 2. See Sections 4.4 and 4.5.

Corollary 1.1 *Let $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p,\varepsilon_0)$ and $\mathcal{L}_2 \in \mathcal{D}_M(\theta,p)$ (with $p > N$ and $\varepsilon_0 > 0$). There is a number $\delta > 0$ depending only on $\mathcal{L}_1$, θ and p, such that if*

(i) *there is a Green's function in M for $\mathcal{L}_2$,*

(ii) $\text{dist}_1(\mathcal{L}_1, \mathcal{L}_2) \leq \delta$ *outside some compact subset K of M, (e.g.* $\text{dist}_1(\mathcal{L}_1, \mathcal{L}_2)(x)$ *tends to zero when $d(x) \to +\infty$),*

then $\lambda_1(\mathcal{L}_2) > 0$. Moreover, condition (i) *above can be dropped if $\mathcal{L}_j(1) = 0$ for $j = 1, 2$.*

Remark 1.3 In the case $\mathcal{L}_1(1) = 0$, $\mathcal{L}_2(1) \leq 0$, it will be seen that $\varepsilon > 0$ may be chosen depending only on M, K, θ, p and ε_0 so that $\lambda_1(\mathcal{L}_2) \geq \varepsilon$. This improves somehow the continuity property of Theorem 2.

Let us mention now two applications of Theorem 1 to elliptic operators in euclidean domains. More general results appear in Section 9 (Theorem 9.1, 9.1′). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $L = \sum_{1 \leq i,j \leq N} \partial_i(a_{ij}\partial_j(.)) +$

$\sum_{1\le j\le N} b_j\,\partial_j(.) + \gamma$ and $L' = \sum_{1\le i,j\le N} \partial_i(a'_{ij}\partial_j(.))$ be two uniformly elliptic operators in Ω with measurable coefficients such that $\gamma \le 0$ and $\sum_i \|b_i\|_{L^p(\Omega)} + \|\gamma\|_{L^{p/2}(\Omega)} \le \theta$ for some $p > N$, $\theta \ge 1$. Assume also that $\sum_{i,j} |a_{ij}|_\infty \le \theta$, $\sum_{ij} a_{ij}\xi_i\xi_j \ge \theta^{-1}|\xi|^2$ for all $\xi \in \mathbb{R}^N$ and similarly for the a'_{ij}. Let D_x denote the ball $D(x, \frac{1}{2}d(x,\partial\Omega)\,)$.

Corollary 1.2 *Assume that at least one of the following two conditions is satisfied:*

(i) $\sum_{i,j}\int_{D_x} |a_{ij}(x) - a'_{ij}(x)|\,dx \le |D_x|\,d(x,\partial\Omega)^\varepsilon$ *for some* $\varepsilon > 0$ *and all* $x \in \Omega$,

(ii) $\sum_{i,j} |a_{ij}(x) - a'_{ij}(x)| \le \varphi(d(x,\partial\Omega))$ *for* $x \in \Omega$ *and some nondecreasing function* φ *verifying* $\displaystyle\int_0^1 \frac{\varphi(s)}{s}\,ds < +\infty$.

Then G *and* G', *the Green's functions of* L *and* L' *respectively, are uniformly comparable, that is* $C^{-1}\,G' \le G \le C\,G'$ *with a constant* $C \ge 1$ *depending only on* Ω, θ, p *and* ε *(or* φ*).*

Corollary 1.2 extends a result of Cranston–Zhao ([CZ], Corollary 3.14) about first-order perturbations of the Laplacian in a $C^{1,1}$ domain. We are grateful to Zhen-Qing Chen for this reference and for raising the question of the Lipschitz domain case (i.e. if $B \in L^p(\Omega)$ then Δ and $\Delta + B.\nabla$ have comparable Green's functions in Ω) and later the question of the uniformity in this case of the constant C.

Another simple application is the absolute continuity of harmonic measures for nondivergence form elliptic operators in a Lipschitz domain Ω. Namely, if $L = \sum_{i,j} a_{ij}(x)\partial^2_{ij}$ is uniformly elliptic in Ω and with Hölder continuous coefficients a_{ij}, then the corresponding harmonic measures μ_x, $x \in \Omega$ are in the form $\mu_x = f_x.\lambda$ where λ is the area measure on $\partial\Omega$ and $f_x \in L^2(\lambda)$ (see Section 8). This is well-known when the a_{ij} are Lipschitz and (in fact) for wide classes of operators in divergence form (see [FKJ], [D]).

In Section 6 below, we relate Theorem 1 to some other earlier results and mention a generalization.

2. Auxiliary lemmas

Fix $p > N$ and let $\theta \ge 1$. The following proposition shows in particular that each $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ induces a map $\mathcal{L} : H^1(M) \to H^{-1}(M)$.

Proposition 2.1 *If* $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ *with* $\mathcal{L} = \mathrm{div}(\mathcal{A}\nabla.) + D\,\nabla. + \mathrm{div}(.\,D') + \gamma.$, *the bilinear map*

$$(\varphi,\psi) \mapsto a_{\mathcal{L}}(\varphi,\psi) = \int [\langle\mathcal{A}\nabla\varphi,\nabla\psi\rangle - \psi\,\langle D,\nabla\varphi\rangle + \varphi\,\langle D',\nabla\psi\rangle - \gamma\,\varphi\,\psi]\,d\sigma$$

is defined and continuous in $H^1(M) \times H^1(M)$.

Proof Let X be a maximal subset of M such that $d(m,m') \geq r_0/8$ whenever $m, m' \in X$, $m \neq m'$. The balls $B'_a = B(a, r_0/4)$, $a \in X$, cover M and if $B''_a = B(a, r_0/2)$, (1.1) implies that $\sum_{a\in X} 1_{B''_a} \leq C_M$ for some finite constant C_M.

For φ and ψ in $H^1(M)$, it follows from the Hölder inequality that

$$\int |D.\nabla\varphi|\,|\psi|\,d\sigma \leq \sum_{a\in X} \int_{B'_a} |D.\nabla\varphi|\,|\psi|\,d\sigma \leq \sum_{a\in X} \|D\|_{L^p(B'_a)} \|\nabla\varphi\|_{L^2(B'_a)} \|\psi\|_{L^{2^*}(B'_a)},$$

where $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{p} > \frac{1}{2} - \frac{1}{N}$. By Sobolev inequalities and (1.1),

$$\|\psi\|^2_{L^{2^*}(B'_a)} \leq C\,(\|\psi\|^2_{L^2(B''_a)} + \|\nabla\psi\|^2_{L^2(B''_a)}).$$

Thus,

$$\int_M |D.\nabla\varphi|\,|\psi|\,d\sigma \leq \frac{\theta}{2} \sum_{a\in X} (\|\nabla\varphi\|^2_{L^2(B'_a)} + \|\psi\|^2_{L^{2^*}(B'_a)}) \leq \frac{\theta}{2} C \sum_{a\in X} [\|\nabla\varphi\|^2_{L^2(B'_a)} + \|\psi\|^2_{L^2(B''_a)} + \|\nabla\psi\|^2_{L^2(B''_a)}].$$

Using the property of the cover $\{B''_a\}_{a\in X}$, we get

$$\int_M |D.\nabla\varphi|\,|\psi|\,d\sigma \leq \theta\,C\,\{\|\nabla\varphi\|^2_{L^2(M)} + \|\psi\|^2_{L^2(M)} + \|\nabla\psi\|^2_{L^2(M)}\} \leq \theta\,C\,\{\|\varphi\|^2_{H^1(M)} + \|\psi\|^2_{H^1(M)}\}.$$

Hence, $\int_M \psi\,D.\nabla\,\varphi\,d\sigma$ exists and $\int_M |\psi\,D.\nabla\,\varphi|\,d\sigma \leq 2\,\theta\,C\,\|\varphi\|_{H^1(M)}\,\|\psi\|_{H^1(M)}$ (it is sufficient to consider the case $\|\varphi\|_{H^1(M)} = \|\psi\|_{H^1(M)} = 1$).

Replacing D by D' and exchanging φ and ψ, we have the same bound for the integral $\int_M |\varphi\,D'.\nabla\,\psi|\,d\sigma$. Similarly,

$$\int_M |\gamma\,\varphi\,\psi|\,d\sigma \leq \sum_{a\in X} \int_{B'_a} |\gamma\,\varphi\,\psi|\,d\sigma \leq \sum_{a\in X} \|\gamma\|_{L^{p/2}(B'_a)} \|\varphi\|_{L^{2^*}(B'_a)} \|\psi\|_{L^{2^*}(B'_a)} \leq \frac{\theta}{2} \sum_{a\in X} [\|\varphi\|^2_{L^{2^*}(B'_a)} + \|\psi\|^2_{L^{2^*}(B'_a)}],$$

so that

$$\int_M |\gamma\,\varphi\,\psi|\,d\sigma \leq \theta\,C \sum_{a\in X}\{\|\varphi\|^2_{L^2(B''_a)} + \|\nabla\varphi\|^2_{L^2(B''_a)} + \|\psi\|^2_{L^2(B''_a)} + \|\nabla\psi\|^2_{L^2(B''_a)}\}$$
$$\leq \theta\,C\,[\|\varphi\|^2_{H^1(M)} + \|\psi\|^2_{H^1(M)}].$$

Since $(\varphi,\psi) \mapsto \int\langle\mathcal{A}(\nabla\varphi),\nabla\psi\rangle\,d\sigma$ is obviously defined and continuous on $H^1(M)\times H^1(M)$, the proposition follows.

Notice that the proof shows that $|a_{\mathcal{L}}(\varphi,\psi)| \leq c\,\|\varphi\|\,\|\psi\|$ for φ, ψ in $H^1(M)$ and $c = c_M(\theta,p)$.

Corollary 2.2 *There exists a positive real $\lambda = \lambda_M(\theta,p)$ such that $\mathcal{L}-\lambda I$ is coercive for all $\mathcal{L}\in\mathcal{D}_M(\theta,p)$, that is*

$$a_{\mathcal{L}}(u,u) + \lambda\int_M |u|^2\,d\sigma \geq c\,\{\|\nabla u\|^2_2 + \|u\|^2_2\} = c\,(\|u\|_{H^1(M)})^2$$

for $u\in H^1_0(M)$ and some constant $c>0$ (depending here only on θ, p and M).

Proof We adapt an argument from [Sta] (pp. 202–203). By the proof above, if V, V' are measurable vector fields on M, if γ_0 is a measurable function on M and if $\beta_p = \sup\{\|V\|_{L^p(B_a)} + \|V'\|_{L^p(B_a)} + \|\gamma_0\|_{L^{p/2}(B_a)}\,;\, a\in M\} < +\infty$, then

$$\int_M [|\psi\,V.\nabla\varphi| + |\varphi\,V'.\nabla\psi| + |\gamma_0\,\varphi\,\psi|]\,d\sigma \leq c_p\,\beta_p\,\|\varphi\|_{H^1(M)}\,\|\psi\|_{H^1(M)}$$

for φ and ψ in $H^1(M)$ with a constant c_p depending on p and M.

Fix p' with $N<p'<p$ and for $t>0$ write $D = D_1 + D_2$ where $D_2 = 1_{\{|D|>t\}}\,D$, and similarly $D' = D'_1 + D'_2$, $\gamma = \gamma_1 + \gamma_2$. Then, with $1/p' = 1/p + 1/q$,

$$\|D_2\|_{L^{p'}(B_a)} \leq \|D\|_{L^p(B_a)}\,[\sigma(\{D\geq t\}\cap B_a)]^{1/q} \leq t^{-p/q}(\|D\|_{L^p(B_a)})^{1+p/q}.$$

By the definition of D_1, D'_1 and γ_1, we have, for all $\eta>0$,

$$\int_M [|\varphi\,D_1.\nabla\varphi| + |\varphi\,D'_1.\nabla\varphi| + |\gamma_1|\varphi^2]\,d\sigma \leq t\,\{2\,\|\nabla\varphi\|_{L^2(M)}\,\|\varphi\|_{L^2(M)} + \|\varphi\|^2_{L^2(M)}\}$$
$$\leq t\,\{(1+\eta^{-1})\|\varphi\|^2_{L^2(M)} + \eta\|\nabla\varphi\|^2_{L^2(M)}\},$$

so that

$$\int_M [\,|\varphi\,D.\nabla\varphi| + |\varphi\,D'.\nabla\varphi| + |\gamma\,\varphi^2|]\,d\sigma \leq 3\,c_{p'}\,t^{-p/q}\,\beta_p^{1+p/q}\,[\,\|\varphi\|^2_2 + \|\nabla\varphi\|^2_2]$$
$$+\,t\,\{(1+\eta^{-1})\,\|\varphi\|^2_2 + \eta\,\|\nabla\varphi\|^2_2\}.$$

Thus, if we choose (and fix) t so large that $3\, c_{p'}\, t^{-p/q}\, \beta_p^{1+p/q} \leq 1/4\theta$ and then fix η such that $t\eta \leq 1/4\theta$,

$$a_{\mathcal{L}}(\varphi,\varphi) + \lambda \int \varphi^2\, d\sigma \geq \frac{1}{2\theta}\, \|\nabla\varphi\|_2^2 + \lambda\|\varphi\|_2^2 - C\,\|\varphi\|_2^2$$
$$\geq \frac{1}{2\theta}\,\{\, \|\nabla\varphi\|_2^2 + \|\varphi\|_2^2\,\},$$

provided λ is sufficiently large. The proof is complete.

Remark 2.1 It follows that for $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ we have a bound : $\lambda_1(\mathcal{L}) \geq -\lambda_0(M,\theta,p)$. (See e.g. [A3], Lemma 2). There is also a simpler bound $\lambda_1(\mathcal{L}) \leq \lambda_0'$ which will not be needed.

Lemma 2.3 *If $\mathcal{L} \in \mathcal{D}_M(\theta,p)$, if u is a bounded $\mathcal{L}$-solution (resp. a positive bounded $\mathcal{L}$-subsolution) in the open region $\omega \subset M$, we have for all $\varphi \in H_0^1(\omega)$*

$$\int_\omega \varphi^2\, |\nabla u|^2\, d\sigma\; \leq c\, \|u\|_\infty^2\; \|\varphi\|_{H_0^1(\omega)}^2$$

with $c = c(M,\theta,p)$. In particular, u is a multiplier for $H_0^1(\omega)$.

Remark 2.2 I learned from G. Mokobodzki that he has also proved a similar multiplier property in the framework of symmetric Dirichlet spaces [Mok].

Proof Fix a region $\omega' \subset\subset \omega$ and $\varphi \in H_0^1(\omega')$, with $\varphi \geq 0$ and bounded. Clearly, $u_{|\omega'} \in H^1(\omega')$, and the functions $u\varphi$, $u^2\varphi$, $u\varphi^2$ belong to $H_0^1(\omega')$. Write $\mathcal{L} = \mathrm{div}(\mathcal{A}(\nabla.)) + \mathrm{div}(D'.) + D.\nabla(.) + \gamma$ with (1.3)–(1.4) and consider the integral

$$I = \int_{\omega'} \varphi^2\, \langle \mathcal{A}(\nabla u), \nabla u\rangle\, d\sigma.$$

By several applications of the Leibnitz formula, and on using the assumptions on u, we shall derive an inequality of the form

$$I \leq -a_{\tilde{\mathcal{L}}}(\varphi, u^2\varphi) + \lambda \int_{\omega'} u^2\, \varphi^2\, d\sigma + \int_{\omega'} u^2\, \langle \mathcal{A}(\nabla\varphi), \nabla\varphi\rangle\, d\sigma,$$

for some coercive operator $\tilde{\mathcal{L}} \in \mathcal{D}_M(\theta',p)$, $\theta' > \theta$, and a large constant $\lambda = \lambda(\theta,p,M)$.

Observe that $I = \int_{\omega'} \langle \mathcal{A}(\nabla u), \nabla(u\varphi^2)\rangle\, d\sigma - 2\int_{\omega'} u\,\varphi\, \langle \mathcal{A}(\nabla u), \nabla\varphi\rangle\, d\sigma$ and, since $\mathcal{L}(u) = 0$ (resp. $u \geq 0$ and $\mathcal{L}(u) \geq 0$),

$$I \leq \int_{\omega'} \{u\,\varphi^2\, D.\nabla u\; - u D'.\nabla(u\varphi^2) + \gamma\, u^2\varphi^2\,\}\, d\sigma - \int_{\omega'} \langle \nabla(u^2\varphi), \mathcal{A}^*(\nabla\varphi)\rangle\, d\sigma$$
$$+ \int_{\omega'} u^2\, \langle \nabla\varphi, \mathcal{A}^*(\nabla\varphi)\rangle\, d\sigma.$$

But $u\varphi^2\nabla u = \frac{1}{2}\{\varphi\,\nabla(u^2\varphi) - u^2\varphi\,\nabla(\varphi)\}$, $u\,\nabla(u\varphi^2) = \frac{1}{2}\{\varphi\,\nabla(u^2\varphi) + 3\,u^2\varphi\,\nabla\varphi\}$, and

$$I \le \int_{\omega'} \Big\{ - \langle \mathcal{A}^*(\nabla\varphi), \nabla(u^2\varphi)\rangle - \frac{1}{2} u^2\varphi\,\{D + 3D'\}.\nabla\varphi + \frac{1}{2}\varphi\,\{D - D'\}.\nabla(u^2\varphi) + (\gamma - \lambda)\,u^2\varphi^2\Big\}\,d\sigma + \lambda \int_{\omega'} u^2\varphi^2\,d\sigma + \int_{\omega'} u^2\,\langle \mathcal{A}(\nabla\varphi), \nabla\varphi\rangle\,d\sigma,$$

or

$$I \le -a_{\tilde{\mathcal{L}}}(\varphi, u^2\varphi) + \lambda \int_{\omega'} u^2\varphi^2\,d\sigma + \int_{\omega'} u^2\,\langle \mathcal{A}(\nabla\varphi), \nabla\varphi\rangle\,d\sigma$$

where $\tilde{\mathcal{L}}(\varphi) = \operatorname{div}(\mathcal{A}^*(\nabla\varphi)) - \frac{1}{2}(D + 3\,D').\nabla\varphi - \frac{1}{2}\operatorname{div}(\varphi(D - D')) + (\gamma - \lambda)\varphi$. If λ is chosen (and fixed) large enough (depending on θ and p) then, by Corollary 2.2 above, $\tilde{\mathcal{L}}$ is coercive in M and belongs to some class $\mathcal{D}_M(\theta', p)$.

If φ is a $\tilde{\mathcal{L}}$-supersolution in ω', the function φ is nonnegative since $\varphi \in H_0^1(\omega')$ and $\tilde{\mathcal{L}}$ is coercive. Thus, $a_{\tilde{\mathcal{L}}}(\varphi, u^2\varphi) \ge 0$ and by the above

$$I \le \lambda\,\|u\|_{\infty,\omega'}^2\,\|\varphi\|_{L^2(\omega')}^2 + \theta\,\|u\|_{\infty,\omega'}^2\,\|\nabla\varphi\|_{L^2(\omega')}^2.$$

Using the uniform ellipticity of $\mathcal{A}$, we see that

$$\int_{\omega'} \varphi^2\,|\nabla u|^2\,d\sigma \le \theta\,(\theta + \lambda)\,\|u\|_\infty^2\,\|\varphi\|_{H_0^1(\omega')}^2. \tag{2.1}$$

This inequality can be extended to all $\tilde{\mathcal{L}}$ supersolutions $\varphi \in H_0^1(\omega')$ (not necessarily bounded) as follows. Since $\tilde{\mathcal{L}}$ is coercive and ω' is bounded, there exists a bounded and > 0 supersolution $s_0 \in H_0^1(\omega')$. Applying (2.1) to $\varphi_n = \inf\{\varphi, n\,s_0\}$ and letting n go to infinity, we obtain (2.1) for such φ.

Finally, if φ is arbitrary in $H_0^1(\omega')$, it is well-known that there is a $\tilde{\mathcal{L}}$-supersolution $\psi \in H_0^1(\omega')$ such that $|\varphi| \le \psi$ and $\|\psi\|_{H_0^1(\omega')} \le C\,\|\varphi\|_{H_0^1(\omega')}$ for some $C = C(\theta, p)$. Just take for ψ the projection (in the Stampacchia sense and with respect to the form $a_{\tilde{\mathcal{L}}}$, cf. [Sta]) of the origin in $H_0^1(\omega')$ onto the convex set $\Gamma = \{f \in H_0^1(\omega')\,; f \ge |\varphi|\,\}$. The continuity and the coercivity of $a_{\tilde{\mathcal{L}}}$ provide the constant C. Thus,

$$\int_{\omega'} \varphi^2\,|\nabla u|^2\,d\sigma \le \int_{\omega'} \psi^2\,|\nabla u|^2\,d\sigma \le C_1\,\|u\|_{\infty,\omega'}^2\,\|\psi\|_{H^1(\omega')}^2 \le C\,C_1\,\|u\|_{\infty,\omega'}^2\,\|\varphi\|_{H^1(\omega')}^2.$$

Since $C_2 = C\,C_1$ is independent of the choice of $\omega' \subset\subset \omega$, an obvious argument yields the estimate (2.1) in general. The proof is complete.

The following lemma says that after being suitably normalized a positive $\mathcal{L}$-solution, $\mathcal{L} \in \mathcal{D}_M(\theta, p)$, has few critical points.

Lemma 2.4 *Let $\mathcal{L} \in \mathcal{D}_M(\theta, p, \varepsilon)$, $\varepsilon > 0$. If u is a positive $\mathcal{L}$-solution on the ball $B = B(a, \rho)$ and if h is a positive $(\mathcal{L} + \varepsilon I)$-solution on B, then*

$$\int_B h(x)^2 \left|\nabla\left(\frac{u}{h}\right)(x)\right|^2 d\sigma(x) \geq C\varepsilon\, |u(a)|^2$$

with $C = C_M(\theta, p, \rho) > 0$.

Proof Note that since u and h^{-1} are locally bounded in B the function u/h is locally of class H^1. Also we may assume from the start that $h(a) = u(a) = 1$ so that by the Harnack inequalities u and h are in between two positive constants on $B' = B(a, \rho/2)$. Let $v = u/h$ and let φ be a Lipschitz cutoff function on M with $\varphi = 1$ on $B(a, \rho/4)$, $\mathrm{supp}(\varphi) \subset \overline{B}(a, \rho/2)$, $0 \leq \varphi \leq 1$ and $\|\nabla\varphi\|_\infty \leq 4\,\rho^{-1}$. Then,

$$\begin{aligned} 0 &= a_{\mathcal{L}}(vh, vh\varphi) \\ &= \int \{\langle \mathcal{A}\nabla(vh), \nabla(hv\varphi)\rangle - hv\varphi\, D.\nabla(hv) + hv\, D'.\nabla(hv\varphi) - \gamma h^2 v^2 \varphi\}\, d\sigma \end{aligned}$$

since $u = vh$ is a $\mathcal{L}$-solution. Using a few simple transformations, we find

$$0 = a_{\mathcal{L}}(h, hv^2\varphi) + A = \varepsilon \int h^2 v^2 \varphi\, d\sigma + A \tag{2.2}$$

where

$$\begin{aligned} A = \int h\,\langle \mathcal{A}(\nabla v), \nabla(hv\varphi)\rangle d\sigma - \int h\,v\,\varphi\,\langle \mathcal{A}(\nabla h), \nabla v\rangle\, d\sigma - \int h^2\, v\,\varphi\, D.\nabla v\, d\sigma \\ - \int h^2 v\,\varphi\, D'.\nabla v\, d\sigma. \end{aligned}$$

From this equality, the uniform estimates for $\|v\|_{\infty,B'}$, $\|\varphi\|_\infty$, $\|\nabla\varphi\|_\infty$, and $\|\nabla h\|_{L^2(B')}$ (using Caccioppoli's inequality) lead to

$$|A| \leq C\,\|\nabla v\|_{L^2(B')}\,\{1 + \|\nabla v\|_{L^2(B')}\}.$$

Thus by (2.2) there is a constant $c' > 0$ such that

$$c'\,\varepsilon \leq \varepsilon \int h^2\, v^2\, \varphi\, d\sigma \leq C\{\|\nabla v\|^2_{L^2(B')} + \|\nabla v\|_{L^2(B')}\},$$

whence

$$\|\nabla v\|_{L^2(B')} \geq \sqrt{\frac{c'}{C}\varepsilon + \frac{1}{4}} - \frac{1}{2}$$

and the lemma is proven.

We require the following formula. For $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ in the form (1.2)–(1.4), we set $\alpha_{\mathcal{L}}(\nabla v, \nabla v) = \langle \mathcal{A}(\nabla v), \nabla v\rangle$ where $\mathcal{A}$ is the section of $\mathrm{End}(T(M))$ related to $\mathcal{L}$ by (1.2).

Lemma 2.5 *Let u and h be (strictly) positive continuous functions in the region Ω of M such that $u\varphi \in H^1_{loc}(\Omega)$ and $h\varphi \in H^1_{loc}(\Omega)$ for all $\varphi \in H^1_{loc}(\Omega)$. Let also $f : (0,+\infty) \to (0,+\infty)$ be of class C^2 and set $v = u/h$. Then, for each $\mathcal{L} \in \mathcal{D}_M(\theta,p)$, we have the following identity in $H^{-1}_{loc}(\Omega)$:*

$$\mathcal{L}(hf(v)) = hf''(v)\,\alpha_{\mathcal{L}}(\nabla v, \nabla v) + f'(v)\,[\mathcal{L}(u) - v\,\mathcal{L}(h)] + f(v)\,\mathcal{L}(h).$$

Remark 2.3 1. Observe that by the assumptions on u and h, each term in the r.h.s. is a well-defined element of $H^{-1}_{loc}(\Omega)$. Clearly, $hf(v) \in H^1_{loc}(\Omega)$ so that the l.h.s. is also a well-defined element in $H^{-1}_{loc}(\Omega)$

2. By Lemma 2.3, we may take for u (resp. h) a positive solution of $\mathcal{L}_1(u) = 0$ (resp. $\mathcal{L}_1(h) + \varepsilon h = 0$) in Ω for some $\mathcal{L}_1 \in \mathcal{D}_M(\theta',p)$.

The proof, which is (at least formally) a straightforward computation, is left to the reader.

Finally, the following simple technical remark will also be needed.

Lemma 2.6 *There is a constant $C > 1$ depending only on M and such that*

$$\mathrm{dist}_q(\mathcal{L}_1, \mathcal{L}_2)(a) \le C\ \sup\{\mathrm{dist}_q(\mathcal{L}_1, \mathcal{L}_2)(x)\,;\, d(x) = r_0/4\,\}$$

when $a \in M$, $d(a) \le r_0/8$, $\mathcal{L}_j \in \mathcal{D}_M(\theta,p)$, $j = 1, 2$ and $q \in [1,+\infty]$.

Proof Consider a maximal set $E \subset \partial B(O, r_0/4)$ such that $d(x,x') \ge \frac{1}{8} r_0$ when x, x' are distinct points in E. By (1.1) the cardinality of E is bounded by a constant $C' = C'_M$ and, if $z \in \overline{B}(O, \frac{9}{8} r_0)$, there is a point $z' \in \partial B(O, r_0/4)$ such that $d(z,z') \le \frac{7}{8} r_0$, and thus also a point $z'' \in E$ with $d(z,z'') \le r_0$.

Hence $B(a, r_0) \subset \bigcup_{b\in E} \overline{B}(b, r_0)$. Thus, with obvious notation for the coefficients of $\mathcal{L}_j$,

$$\begin{aligned} \|\mathcal{A}_1 - \mathcal{A}_2\|_{L^q(B_a)} &\le \|\sum_{b\in E} 1_{\overline{B}_b}\, |\mathcal{A}_1 - \mathcal{A}_2\,|\|_{L^q(M)} \\ &\le \sum_{b\in E} \|\mathcal{A}_1 - \mathcal{A}_2\,\|_{L^q(\overline{B}_b)} \\ &\le C' \sup\{\|\mathcal{A}_1 - \mathcal{A}_2\|_{L^q(B_b)}\,;\, d(b) = r_0/4\,\}. \end{aligned}$$

Similar inequalities hold for the three other terms in the expression of $\mathrm{dist}_q(\mathcal{L}_1, \mathcal{L}_2)(a)$ and the lemma follows with $C = 4\,C'$.

Remark 2.4 The lemma shows that whenever we have a relation $\mathrm{dist}_q(\mathcal{L}_1, \mathcal{L}_2) \prec \psi$ in $\mathcal{D}_M(\theta,p)$, with ψ nonincreasing on $[0,+\infty)$, we may replace the function $\psi(t)$, $t \in [0,\infty)$, by $\psi_1(t) = \inf\{\psi(t), C\,\psi(r_0/8)\,\}$ where C is the constant in Lemma 2.6.

3. The main construction

Let $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p,\varepsilon_0)$, $p > N$, $\varepsilon_0 > 0$ and let ψ_1 be a continuous nonincreasing and integrable function on $[0,+\infty)$. Starting from some positive and $\mathcal{L}_1$-harmonic function u in the ball $B_R = B(O,R)$ in M, with $R > 2$ and $u(O) = 1$, we shall construct a function w which is in some sense close to u and (uniformly) "almost" superharmonic with respect to all $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ such that $\mathrm{dist}_\infty(\mathcal{L}_1,\mathcal{L}) \prec \psi_1$ in $\mathcal{D}_M(\theta,p)$ (see 1.5) provided that $\|\psi_1\|_1 = \int_0^{+\infty} \psi_1(s)\,ds$ is small enough. A key idea in the construction goes back to the work [Ser] of J. Serrin and was already used by the author in other contexts (e.g. [A1]). Serrin used functions $f_\pm(P_\zeta)$ of the standard Poisson kernel P_ζ in the unit ball B of $\mathbb{R}^N$, $\zeta \in \partial B$, to get bounds for the Poisson kernel of a sufficiently regular second order elliptic operator L in $\overline{B}$ whose principal part at ζ is the Laplacian (more general principal parts at ζ are treated similarly). The bounds follow from the L-superharmonicity (resp. L-subharmonicity) of $f_+(P_\zeta)$ (resp. $f_-(P_\zeta)$) which is checked by explicit computations (see [Ser]). Here, we combine this construction with a relativization procedure which is allowed by Lemma 2.3. Relativization methods are familiar in potential (and probability) theory and they have proved useful in a number of problems.

Let $\tilde{\psi}_1(t) = \psi_1(t + r_0')$ where $r_0' = r_0/32$, and let

$$Q(t) = \frac{\kappa}{\sqrt{\|\psi_1\|_1}}\,\frac{1}{t}\,\tilde{\psi}_1\Big(\frac{|\log(t)|}{\kappa}\Big) \tag{3.1}$$

for $t > 0$. Here κ is some large positive constant which will be chosen later and which will depend *only* on M, θ, p (and r_0). Observe that Q is positive, continuous, and integrable on $(0,+\infty)$. Also, $\int_0^{+\infty} Q(s)\,ds \le 2\,\kappa^2 \sqrt{\int_0^{+\infty} \psi_1(s)\,ds}$.

Let f be the solution of the differential equation $y''(t) + Q(t)\,y'(t) = 0$ with initial conditions $y(0) = 0, y'(0) = 1$. In fact, we just set (using the integrability of Q)

$$f(t) = \int_0^t \exp(-\int_0^s Q(\tau)\,d\tau)\,ds. \tag{3.2}$$

The function f is concave and C^1 on $[0,+\infty)$, and $C_0\,t \le f(t) \le t$ with $C_0 = \exp(-\int_0^{+\infty} Q(\tau)\,d\tau)$.

Finally, fix a positive $(\mathcal{L}_1 + \varepsilon_0.I)$-solution h on M with $h(O) = 1$ and let $w = hf(u/h)$. It is well-known and easily seen that w is $\mathcal{L}_1$-superharmonic (see e.g. [GK] or the end of Remark 3.2.2 below). Clearly, $w \in H^1_{loc}(B_R)$ and $C_0\,u \le w \le u$.

Proposition 3.1 *Let* $I_1 = \int_0^\infty \psi_1(s)\,ds$. *Let* $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ *and* $R > 10$ *be such that*

(i) $\mathcal{L} = \mathcal{L}_1$ on the "annulus" $\omega_R = \{x \in M\,;\, R-1 < d(O,x) < R\}$, and

(ii) $\mathrm{dist}_q(\mathcal{L}_1,\mathcal{L}) \prec \psi_1$ in $\mathcal{D}_M(\theta,p)$, *where $q \in [q_0, +\infty]$ and q_0 is sufficiently large depending on M, θ and p. Then for each $\delta > 0$, there is a number $\eta(\delta) = \eta_M(\theta, p, \varepsilon_0, \delta) > 0$ independent of R and such that, if $I_1 \leq \eta(\delta)$, we may write*

$$\mathcal{L}(w) = S - \mu \quad \text{in } B(O,R) \tag{3.3}$$

where μ is a positive measure on $B(O,R)$, $\mu \in H^{-1}_{loc}(B_R)$, $S \in H^{-1}(B_R)$, supp$(S) \subset \overline{B}(O, R-1)$ *and*

$$\|S\|_{H^{-1}(B_a(r_0/2))} \leq \delta \ \mu(B(a, r_0/4)) \tag{3.4}$$

for every $a \in B(O, R-1/2)$.

Proof of Proposition 3.1 We may assume from the start that $\|\psi\|_1$ is so small that

$$t/2 \leq f(t) \leq t \quad \text{on } [0,+\infty) \quad \text{and} \quad \tilde{\psi}_1(0) = \psi_1(r_0') \leq 1.$$

(Recall that κ is to be chosen below independently of δ.)

1. By Lemma 2.5,

$$\mathcal{L}(w) = \mathcal{L}(hf(v)) = hf''(v)\,\alpha_{\mathcal{L}}(\nabla v, \nabla v) + f'(v)\,[\mathcal{L}(u) - v\,\mathcal{L}(h)] + f(v)\,\mathcal{L}(h),$$

where $v = u/h$, so that by (1.2), (3.2) and the assumptions on u and h (namely $\mathcal{L}_1(u) = 0$ and $\mathcal{L}_1(h) = -\varepsilon_0 h$)

$$\begin{aligned}\mathcal{L}(w) = f'(v)\,[-Q(v)\,h\,\langle \mathcal{A}\nabla v, \nabla v\rangle + (\mathcal{L} - \mathcal{L}_1)(u)] + [f(v) - vf'(v)][\mathcal{L}(h) - \mathcal{L}_1(h)] \\ - \varepsilon_0 h[f(v) - vf'(v)].\end{aligned}$$

From the concavity of f we have $f(v) - vf'(v) \geq 0$. We thus define a positive and absolutely continuous measure $\mu = \ell.\sigma_M$ on B_R on setting

$$\ell = Q(v)\,h\,\langle \mathcal{A}\nabla v, \nabla v\rangle f'(v) + \varepsilon_0\,h\,(f(v) - vf'(v)). \tag{3.5}$$

Clearly, by Lemma 2.3, $\mu \in H^{-1}_{loc}(B_R)$. We also set

$$S = f'(v)\,[\mathcal{L} - \mathcal{L}_1](u) + (f(v) - vf'(v))\,[\mathcal{L}(h) - \mathcal{L}_1(h)].$$

2. By the Harnack inequalities, we have that for some constant $A = A_M(\theta, p, r_0)$,

$$\exp(-A\,(d(a) + r_0/2)) \leq u(x)/h(x) \leq \exp(A\,(d(a) + r_0/2))$$

for $x \in B_a \subset B(O, R-\frac{1}{4})$. Thus, choosing $\kappa = 18A$, using the definition of Q and setting $C = (\|\psi_1\|_1)^{-\frac{1}{2}}$, we have for $2r_0' \le d(a) \le R - \frac{1}{2}$

$$Q(v(x)) = C\kappa \frac{h(x)}{u(x)}\,\tilde{\psi}_1\left[\kappa^{-1}\left|\log\left(\frac{u}{h}\right)\right|\right] \ge C\kappa\,\frac{h(x)}{u(x)}\,\tilde{\psi}_1\left[\frac{A}{\kappa}\left(\frac{r_0}{2}+d(a)\right)\right]$$
$$\ge C\kappa\,\frac{h(x)}{u(x)}\,\psi_1(d(a)).$$

(Recall that ψ_1 is nonincreasing and that $r_0' = r_0/32$.) Taking into account Lemma 2.4, we see from (3.5) that for all $a \in M$ with $r_0/8 \le d(a) < R - \frac{3}{4}$,

$$\mu(B(a, r_0')) \ge c\,\varepsilon_0\,C\,\psi_1(d(a))\,u(a). \tag{3.6}$$

Here, c is a positive constant which depends only on M, θ, p and r_0.

3. It is easy to see that for each ball $B_a = B(a, r_0) \subset B(O, R-\frac{1}{4})$, the function $f'(v)$ is a multiplier for $H_0^1(B_a)$ with a multiplier norm estimated by some constant $c = c_M(\theta, p, r_0) > 0$. Observe that by (3.1), $|f''(t)| \le Q(t) \le [\kappa\,\|\psi_1\|_1^{-\frac{1}{2}}\psi_1(r_0')]\,t^{-1} \le c/t$ since

$$\psi_1(r_0')/\sqrt{\|\psi_1\|_1} \le \frac{1}{\sqrt{r_0'}}\,(\psi_1(r_0'))^{\frac{1}{2}}$$

(using the monotonicity of ψ_1). Thus,

$$|f''(v)\,\nabla v| \le c\,v^{-1}\,|\nabla v|,$$

and the claim follows from Lemma 2.3 and Harnack inequalities for u and h.

This also shows that $f(v) - vf'(v)$ is a multiplier of $H_0^1(B_a)$ with a multiplier norm less than $c\,v(a) = c\,u(a)\,(h(a))^{-1}$.

4. The next step is to bound the norm of $[\mathcal{L} - \mathcal{L}_1](u)$ in $H^{-1}(B_a)$ (if $d(a) \le R - \frac{1}{2}$). We have

$$\|(\mathcal{L} - \mathcal{L}_1)(u)\|_{H^{-1}(B_a)} \le c\,u(a)\,\psi_1(d(a)) \tag{3.7}$$

(if q is large enough) as the following computation shows. Recall that by a theorem of Meyers [Mey], there exists $\varepsilon = \varepsilon(c_0, \theta, p, N) > 0$ such that $\|\nabla u\|_{2+\varepsilon, B_a} \le c\,u(a)$

(see also [Gia], Chap. 5). It follows that for $\varphi \in H_0^1(B_a)$, we have (using obvious notation and Harnack, Caccioppoli's and Sobolev inequalities)

$$\begin{aligned}
|\langle(\mathcal{L}-\mathcal{L}_1)u,\varphi\rangle| \le \int \{ &|\langle(\mathcal{A}-\mathcal{A}_1)\nabla u,\nabla\varphi\rangle| + |\langle D-D_1,\nabla u\rangle\,\varphi| + |\langle D'-D_1',\nabla\varphi\rangle\,u| \\
&\cdots + |(\gamma-\gamma_1)\,u\,\varphi|\,\}\,d\sigma \\
\le & c\,\|\nabla u\|_{2+\varepsilon,B_a}\|\nabla\varphi\|_{2,B_a}\|\mathcal{A}-\mathcal{A}_1\|_{q,B_a} \\
&+ \|D-D_1\|_{p,B_a}\|\nabla u\|_{2,B_a}\|\varphi\|_{2^*,B_a} \\
&+ \|\tilde{D}-\tilde{D}_1\|_{p,B_a}\|\nabla\varphi\|_{2,B_a}\|u\|_{2^*,B_a} \\
&+ \|\gamma-\gamma_1\|_{p/2,B_a}\|u\|_{2^*,B_a}\|\varphi\|_{2^*,B_a} \\
\le & c'\,\|\varphi\|_{H_0^1(B_a)}\,\mathrm{dist}_q(\mathcal{L},\mathcal{L}_1)(a)\,u(a),
\end{aligned}$$

if $2^* = 2p/(p-2)$ and if $q \ge q_0$ where q_0 is such that $\frac{1}{2} + 1/(2+\varepsilon) + 1/q_0 = 1$, whence (3.7).

It also follows from (3.7) that $\|(\mathcal{L}-\mathcal{L}_1)(h)\|_{H^{-1}(B_a)} \le c\,h(a)\,\psi_1(d(a))$. Thus, by the previous paragraph and the definition of S,

$$\|S\|_{H^{-1}(B_a)} \le c\,u(a)\,\psi_1(d(a)). \tag{3.8}$$

5. At least if $d(a) \ge r_0/16 = 2r_0'$, (3.4) follows at once from (3.6) and (3.8) since C increases to $+\infty$ as $\|\psi_1\|_1$ tends to 0. If $d(a) \le 2\,r_0'$, observe that $B(a,r_0/2) \subset B(b,r_0)$, and $B(a,r_0/4) \supset B(b,r_0/8)$ for any b taken on the sphere $\partial B(O,2\,r_0')$ and (3.6)–(3.8) at b imply (3.4) at a. The proof of Proposition 3.1 is complete.

Remarks 3.2 1. (*Added in final version*) The proof above is made simpler if one uses the second term in the r.h.s. of (3.5) to bound ℓ from below. Observe that by the Taylor formula $f(t) - tf'(t) = t^2 \int_0^1 s\,Q(ts)f'(ts)\,ds$ is larger than $\frac{1}{8}t^2 \inf\{Q(s)\,;\,t/2 \le s \le t\}$. This argument makes it possible to get rid of Lemma 2.4.

2. We also need a slightly different version of Proposition 3.1 which follows easily from the proof above. Here, u is positive $\mathcal{L}_1$ superharmonic in $B(O,R)$, continuous and $\mathcal{L}_1$ harmonic outside a ball $B(a_0,r_0) \subset B(O,R)$ with $d(a_0,O) \ge 2\,r_0$ and $u(O) = 1$. Besides (i) and (ii) in Proposition 1.3, it is also assumed that $\mathcal{L} = \mathcal{L}_1$ on $B(a,2r_0)$. Then, the conclusions in Proposition 3.1 hold—except that we now only assert that μ is locally of class H^{-1} in $B(O,R) \setminus \overline{B}(a_0,r_0)$—and $\mathrm{supp}(S) \subset \overline{B}(O,R-1) \setminus B(a,2\,r_0)$.

Also, it is easily seen that $C^{-1}\,\mu \ge -\mathcal{L}_1(u)$ in $B(a,2r_0)$. Just observe that since f is concave, $w = \inf\{d_j\,u + d_j'\,h\,;\,j \ge 1\}$ where d_j and d_j' are positive and $1 \ge d_j \ge C_0 = \inf\{f'(t)\,;\,t \ge 0\}$. Thus, $-\mathcal{L}_1(w) \ge \inf\{-d_j\,\mathcal{L}_1(u)\,;\,j \ge 1\} = -C_0\,\mathcal{L}_1(u)$ in $B(a,2r_0)$ (since if $s = \inf_{j\ge1} s_j$, with $s_j \ge 0$ and $\mathcal{L}(s_j) \le 0$, then $\mathcal{L}(s) \le 0$).

While Proposition 3.1 is the main ingredient in the proof of Theorem 1, our proof of Theorem 2 is based upon a (simpler) variant where f is replaced by the concave function $x \mapsto \sqrt{x}$ (so that now $f(t) \sim t$ does not hold).

Proposition 3.3 *Let* $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p,\varepsilon_0)$, *u, h,* $v = u/h$ *and* ω_R *be as in Proposition* 3.1 *and set now* $w = h\sqrt{u/h} = \sqrt{u\,h}$. *Then for every given* $\delta > 0$ *there is a number* $\varepsilon = \varepsilon(\theta, p, \delta) > 0$ *such that if* $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ *verifies* $\mathrm{dist}_1(\mathcal{L}_1, \mathcal{L}) \le \varepsilon$ *uniformly in* M *and* $\mathcal{L} = \mathcal{L}_1$ *on* ω_R, *we have*

$$\mathcal{L}(w) = S - \mu \quad \text{in } B(O,R)$$

where μ *is a positive measure on* $B(O,R)$, $\mu \in H^{-1}_{loc}(B_R)$, $S \in H^{-1}(B_R)$, $\mathrm{supp}(S) \subset \overline{B}(O, R-1)$ *and*

$$\|S\|_{H^{-1}(B_a(r_0/2))} \le \delta\; \mu(B(a, r_0/4))$$

when $a \in B(O, R-1/2)$.

Proof We have now $Q(t) = 1/2t$. Computing $\mathcal{L}(w)$ as in the preceding proof, we find

$$\mathcal{L}(w) = -\frac{h}{2}\sqrt{v}\left[\varepsilon_0 + v^{-2}\langle \mathcal{A}\nabla v, \nabla v\rangle\right] + \frac{h}{2}\sqrt{v}\left[h^{-1}(\mathcal{L}-\mathcal{L}_1)(h)) + u^{-1}(\mathcal{L}-\mathcal{L}_1)(u))\right].$$

Let μ be the positive measure on $B(O,R)$ defined by the density

$$g = \frac{h}{2}\sqrt{v}\left[\varepsilon_0 + v^{-2}\langle \mathcal{A}\nabla v, \nabla v\rangle\right],$$

and set

$$S = \frac{h}{2}\sqrt{v}\left[h^{-1}(\mathcal{L}-\mathcal{L}_1)(h)) + u^{-1}(\mathcal{L}-\mathcal{L}_1)(u))\right].$$

As in the end of the proof of Proposition 3.1 (parts 3 and 4), it is easily seen that

$$\|S\|_{H^{-1}(B(a,r_0/2))} \le c\,h(a)\sqrt{v(a)}\left\{\mathrm{dist}_1(\mathcal{L},\mathcal{L}_1)(a)\right\}^{1/q},$$

if one uses also the inequality $\|\mathcal{A} - \mathcal{A}_1\|_{L^q(B_a)} \le (2\theta)^{1-1/q}\left[\|\mathcal{A}-\mathcal{A}_1\|_{L^1(B_a)}\right]^{1/q}$. Proposition 3.3 follows.

Remark 3.4 The normalization conditions $u(O) = h(O) = 1$ are now superfluous.

Remark 3.5 The proof shows that $\delta \le C(M,\theta,p)\,\varepsilon_0^{-1}\|\mathrm{dist}_{q_0}(\mathcal{L},\mathcal{L}_1)\|_{\infty,M}$.

4. Proof of Theorem 2 and Corollary 1.1

4.1. It is enough to show that if $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p,\varepsilon_0)$, $\varepsilon_0 > 0$, and if $\alpha > 0$ is sufficiently small depending on M, θ, p, ε_0, then $\lambda_1(\mathcal{L}) \geq 0$ holds for $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ with $\operatorname{dist}_1(\mathcal{L}_1, \mathcal{L}) \leq \alpha$ in M. To this end, we shall show that under these conditions the first eigenvalue $\lambda_1[\mathcal{L}, B(O,r)]$ of $\mathcal{L}$ for the Dirichlet problem in $B(O,r)$ is positive for $r \geq 1$. Observe that from the Harnack convergence theorem (for $\mathcal{L}$) and the second definition of $\lambda_1(\mathcal{L})$ in paragraph 1.4, it follows that $\lim_{r\to\infty} \lambda_1(\mathcal{L},\Omega_r) = \lambda_1(\mathcal{L})$.

Let $\Omega_R = B(O,R+2)$. Because $\lambda_1(\mathcal{L}, B(0,r))$ is a nonincreasing function of r, it even suffices to show that $\lambda_1(\mathcal{L},\Omega_R) \geq 0$ under the additional assumption that $\mathcal{L}_1 = \mathcal{L}$ on $\omega'_R = \{x \in M;\, R < d(x) < R+2\}$, the number $R \geq 1$ being now fixed.

Let $\mathcal{L}^*$ denote the formal adjoint of $\mathcal{L}$ and let $s^* \in H^1_0(\Omega_R)$ be a positive eigenfunction for $\mathcal{L}^*$ associated to the first eigenvalue $\lambda_0 = \lambda_1(\mathcal{L},\Omega_R) = \lambda_1(\mathcal{L}^*,\Omega_R)$.

4.2. Since $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p,\varepsilon_0)$, we may choose a continuous positive $\mathcal{L}_1$-superharmonic function $u \in H^1_0(\Omega_R)$, the function u being $\mathcal{L}_1$-harmonic on $\Omega_R \setminus B(a_0, 2r_0)$ for some a_0 in M with $d(a_0) = R + \frac{3}{2}$. We may take for u the solution of the problem $\mathcal{L}_1(u) = -1_{B(a,2r_0)}$ in Ω_R, $u = 0$ on $\partial\Omega_R$.

Fix a positive $[\mathcal{L}_1 + \varepsilon_0.I]$-solution h on M and let $f : [0,+\infty[\to \mathbb{R}$ be a smooth concave function such that $f(t) = \sqrt{t}$ if $t > 2\varepsilon_1$ and $f(t) = \varepsilon_1^{-1/2}\, t$ if $0 \leq t \leq \varepsilon_1$, where ε_1 is positive and small. Set now $w = hf(u/h)$ and choose ε_1 so small that $w = \sqrt{hu}$ in $B(O,R+1)$.

Since f is Lipschitz on $[0,\infty)$ with $f(0) = 0$ and $(u/h) \in H^1_0(\Omega_R)$, the function w is in $H^1_0(\Omega_R)$. Moreover, by Proposition 3.3 and the concavity of f, if α is small enough depending on $\delta > 0$, the continuous function w has the following properties:

$$\mathcal{L}(w) = -\nu + S, \qquad S \in H^{-1}(\Omega_R), \qquad \operatorname{supp}(S) \subset \overline{B}(O,R),$$

ν being a positive measure in $H^{-1}(\Omega_R)$. Also, for all $a \in B(O,R+1)$

$$\|S\|_{H^{-1}(B(a,r'_0))} \leq \delta\, \nu(B(a, r'_0/2)),$$

if $r'_0 = r_0/2$.

4.3. Now, arguing by contradiction and assuming that $\lambda_0 < 0$, we have

$$\langle -\mathcal{L}(w), s^* \rangle = -\langle w, \mathcal{L}^* s^* \rangle = \lambda_0 \langle w, s^* \rangle < 0.$$

On the other hand, on using a Whitney partition $\{\varphi_j\}$ corresponding to the radius r'_0 (see the definition below), we find also that because $\operatorname{supp}(S) \subset \overline{B}(O,R)$ and ν is a positive measure,

$$\langle -\mathcal{L}(w), s^* \rangle = \langle \nu, s^* \rangle - \langle S, s^* \rangle \geq \sum_{d(x_j) \leq R + \frac{1}{2}} \left[\int s^* \varphi_j \, d\nu - \langle S, \varphi_j s^* \rangle \right].$$

Here $\{\varphi_j\}_{j\geq 1}$ is a smooth partition of unity in M with $F_j = \operatorname{supp}(\varphi_j) \subset B(x_j, r'_0)$, $x_j \in M$, $\varphi_j \geq c^{-1}$ on $B(x_j, r'_0/2)$ and $|\nabla \varphi_j| \leq c$ where $c = c_M(r'_0)$. Such a partition is easily constructed starting with a maximal subset $\{x_j \,; j \geq 1\}$ in M with $d(x_j, x_k) \geq r'_0/4$ when $j \neq k$ and with smooth nonnegative functions g_j in M such that $g_j = 1$ on $B(x_j, r'_0/2)$, and $\operatorname{supp}(g_j) \subset B(x_j, r'_0)$. Clearly, $n(x) = |\{j \geq 1 \,; x \in \overline{B}(x_j, r'_0)\}|$, $x \in M$, is bounded by a constant $c = c_M(r'_0)$ and we may let $\varphi_j = g_j/(\sum_{k\geq 1} g_k)$.

It now follows from the Harnack and Caccioppoli inequalities that

$$|\langle S, \varphi_j s^* \rangle| \leq \; c\, s^*(x_j)\; \|S\|_{H^{-1}(B(x_j, r'_0))}$$

(recall that by Corollary 2.2 there is a bound $|\lambda_0| \leq c'_M(\theta, p)$) and

$$\int s^* \varphi_j \, d\nu \geq c^{-1} s^*(x_j)\, \nu(B(x_j, r'_0/2))$$

for some constant $c = c_M(\theta, p, r_0) > 0$. Taking δ so small that $c^2 \delta \leq 1$ we get $\langle -\mathcal{L}(w), s^* \rangle \geq 0$, a contradiction. This proves the first claim in Theorem 1. Since $\delta \leq c^{-2}$ was what we wanted above, Remark 3.5 shows that $\sup_M \operatorname{dist}_{q_0}(\mathcal{L}, \mathcal{L}_1) \leq c(M, \theta, p)\, \varepsilon_0$ insures $\lambda_1(\mathcal{L}) \geq 0$, and the last claim follows.

4.4. Proof of Corollary 1.1 Choose $\delta > 0$ such that the condition $\operatorname{dist}_1(\mathcal{L}_1, \mathcal{L}) \leq \delta$ in M, $\mathcal{L} \in \mathcal{D}_M(\theta, p)$, implies that $\mathcal{L} \in \mathcal{D}_M(\theta, p, \varepsilon_0/2)$.

If $\mathcal{L}_2$ verifies (ii) (in Corollary 1.1), the operator $\mathcal{L} \in \mathcal{D}_M(\theta, p)$ whose coefficients coincide with those of $\mathcal{L}_1$ on K and with those of $\mathcal{L}_2$ on $M \setminus K$ is in $\mathcal{D}_M(\theta, p, \varepsilon_0/2)$. Thus $\lambda_1(\mathcal{L}_2, M \setminus K) \geq \varepsilon_0/2$. By assumption (i) the Green's function in M with respect to $\mathcal{L}_2$ exists and it follows from Lemma 21 in [A3] that $\lambda_1(\mathcal{L}_2, M) > 0$.

Assume now, instead of (i), that constant functions are $\mathcal{L}_j$-harmonic in M for $j = 1, 2$. Set $\mathcal{L} = \varphi \mathcal{L}_1 + (1 - \varphi)\mathcal{L}_2$ where φ is a cutoff function with $0 \leq \varphi \leq 1$, $\varphi = 1$ in a neighborhood of K, $\varphi(x) = 0$ if $d(x, K) \geq 2$ and $|\nabla \varphi| \leq 1$. It is easily checked that $\mathcal{L}$ may be represented in the form (1.2) such that with respect to this representation $\mathcal{L} \in \mathcal{D}_M(\theta', p)$ for some θ' depending only on M, θ and p (not on K) and $\operatorname{dist}_1(\mathcal{L}, \mathcal{L}_1) \leq c \operatorname{dist}_1(\mathcal{L}_2, \mathcal{L}_1)$. Thus by Theorem 2, $\lambda_1(\mathcal{L}) \geq 3\varepsilon_0/4$ if δ is small. Also, $\mathcal{L}(1) = 0$. If U is an open neighborhood of $\operatorname{supp}(\varphi)$ the réduite function (ref. [B] p. 36, [Her]) $v = R_1^U$ (in M and with respect to $\mathcal{L}$) is a $\mathcal{L}$-potential because Green's function for $\mathcal{L}$ exists. Hence v being nonconstant is $\mathcal{L}_2$ superharmonic and nonharmonic. Thus (i) holds and $\lambda_1(\mathcal{L}_2) > 0$.

4.5. Proof of Remark 1.3 Constant will mean a constant depending only on M, K, θ, p, ε_0. Assume first that $\mathcal{L}_2(1) = 0$ and consider $\mathcal{L} \in \mathcal{D}_M(\theta', p, \frac{3}{4}\varepsilon_0)$ as above. Let G_2, G denote the Green's functions in M of $\mathcal{L}_2$, $\mathcal{L}$ respectively, let G_2^j resp. G^j denote the corresponding Green's functions in smooth domains U_j chosen such that $\overline{B}(O,j) \subset U_j \subset B(O,j+1)$. Fix $P_0 \in M$ with $d(O,P_0) = 1$ and $R \geq 3$ such that $K \subset B(O,R-1)$. By Harnack inequalities and by (1.9) for $\mathcal{L}^*$, there exists a constant $C \geq 1$ such that

$$C^{-1}\, G^j(O,P)) \leq G_2^j(O,P)/G_2^j(O,P_0) \leq CG^j(O,P)$$

for $P \in \partial B(0,R)$ and j large, and hence by the maximum principle, for $P \in U_j \setminus B(O,R)$. It follows (using the Stokes formula and regularizations of $\mathcal{L}_2$ near ∂U_j) that the harmonic measures μ_j^2 (resp. μ_j) of O in U_j with respect to $\mathcal{L}_2$ (resp. $\mathcal{L}$) verify $C^{-1}\,\mu_j \leq [G_2^j(O,P_0)]^{-1}\,\mu_j^2$, whence $G_2^j(O,P_0) \leq C$. Letting $j \to \infty$ and using Harnack inequalities, this yields $G_2(P,Q) \leq C$ for P, Q in $B(O,R+2)$, $d(P,Q) = 1$, and another C. By the argument in Lemma 5.2 below, it follows that $C_1^{-1}\, G(P,Q) \leq G_2(P,Q) \leq C_1\, G(P,Q)$, for all P, Q in M and a constant $C_1 \geq 1$.

Fix a positive solution s of $\mathcal{L}(s) + \frac{1}{2}\varepsilon_0\, s = 0$ in M. Then $s = \frac{1}{2}\varepsilon_0\, G(s)$ in M (since $G(s) \geq C^{ste}\, s$ by (1.9), s is a potential). Hence $w = G_2(s)$ verifies

$$\mathcal{L}_2(w) + \frac{\varepsilon_0}{2\,C_1}\, w \leq 0.$$

This means that $\mathcal{L}_2 \in \mathcal{D}_M(\theta, p, \varepsilon_0/2\,C_1)$.

For general $\mathcal{L}_2$, denote $\mathcal{A}_j, D_j, \ldots$ the coefficients of $\mathcal{L}_j$ in the representations (1.2) of $\mathcal{L}_1$, $\mathcal{L}_2$. Write $\mathcal{L}_2 = \mathcal{L}' + \mathcal{L}''$ where $\mathcal{L}'(s) = \operatorname{div}(\mathcal{A}_2 \nabla s) + (D_2 + D_2' - D_1').\nabla s + \operatorname{div}(sD_1') + \gamma_1 s$ and $\mathcal{L}''(s) = s\left[\operatorname{div}(D_2' - D_1') + \gamma_2 - \gamma_1\right]$. By the case already treated, there is a positive solution s to

$$\mathcal{L}'(s) + \frac{\varepsilon_0}{2\,C_1} s = 0$$

and by the assumption $\mathcal{L}''(s) \leq 0$, whence $\lambda_1(\mathcal{L}_2) \geq \varepsilon_0/2\,C_1$ and the proof is complete.

5. Proof of Theorem 1

Fix a class $\mathcal{D}_M(\theta,p)$ and a positive number $r_1 \in (0, r_0/100)$ which is a coercivity radius for $\mathcal{D}_M(\theta,p)$. This means that for some constant $c = c_M(\theta,p,r_1) > 0$,

$$a_{\mathcal{L}}(\varphi,\varphi) \geq c\left(\|\varphi\|_{H_0^1(B(a,r_1))}\right)^2$$

when $a \in M$, $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ and $\varphi \in H_0^1(B(a,r_1))$. Fix also $\varepsilon_0 > 0$.

5.1. We shall need the following simple fact.

Lemma 5.1 *Let $S \in H^{-1}(B(a,r_1))$ with* $\mathrm{supp}(S) \subset B(a,r_1')$ *where $a \in M$ and $0 < r_1' < r_1$. Let $\mathcal{L} \in \mathcal{D}_M(\theta,p,\varepsilon_0)$, and let G_U denote the Green's kernel of $\mathcal{L}$ with respect to some region $U \supset \overline{B}(a,r_1)$. Then, for all $A > 0$*

$$\sigma\big[\{x \in B\,;\, |G_U(S)(x)| \geq A\}\big] \leq \frac{C}{A^2}\, \|S\|^2_{H^{-1}(B)}, \tag{5.1}$$

where $B = B(a,r_1)$ and $C = C_M(\theta,p,\varepsilon_0,r_1,r_1')$ is a positive constant.

Proof Set $G = G_U$, $\eta = \|S\|_{H^{-1}(B)}$. There is a positive constant c (depending on θ, p, ε_0, r_1, r_1' and M) such that for $P \in \partial B(a,r_1)$

$$|G(S)(P)| = |\langle S,\, G_P^*\rangle| = |\langle S, \varphi\, G_P^*\rangle| \leq \eta\, \|\varphi\, G_P^*\|_{H_0^1(B)} \leq c\,\eta$$

if φ is a C^1 cut-off function with $\varphi = 1$ on $\overline{B}(a,r_1')$, $\mathrm{supp}(\varphi) \subset B(a,r_1'')$, $r_1'' = (r_1' + r_1)/2$ and $\|\nabla\varphi\|_\infty \leq 3\,(r_1 - r_1')^{-1}$. We have used (1.9), Caccioppoli's and Harnack inequalities to estimate $\|\varphi\, G_P^*\|_{H_0^1(B)}$.

Write $G(S) = u + G_B(S)$ in $B = B(a,r_1)$, where u is the $\mathcal{L}$-harmonic function in B with boundary value $u = G(S)$ on ∂B. Since $|u| \leq c\,\eta$ on ∂B, it is easy to see that $|u| \leq c'\,\eta$ on $\overline{B}$ for another constant $c' > 0$. (Compare with a positive $\mathcal{L}$ solution in M using the maximum principle and the local Harnack inequalities.)

By uniform coerciveness of $\mathcal{L}$ in B, we have $\|G_B(S)\|_{H_0^1(B)} \leq c''\,\eta$. Hence, if $A' \geq 2\,c'$,

$$\begin{aligned}\sigma[\{x \in B\,;\, |G(S)(x)| \geq A'\,\eta\}] &\leq \sigma[\{x \in B\,;\, |G_B(S)(x)| \geq \tfrac{1}{2}A'\,\eta\}]\\ &\leq 4\,A'^{-2}\,\eta^{-2}\,\|G_B(S)\|^2_{L^2(B)}\\ &\leq 4\,(c'')^2\,A'^{-2}.\end{aligned}$$

If $c_1 > 0$ is such that $(c_1/c')^2 \geq \sigma[B(x,r_1)]$ for all $x \in M$, and if $c_2 = \sup(c'', c_1)$,

$$\sigma[\{x \in B\,;\, |G(S)| \geq A'\,\eta\}] \leq 4[c_2]^2\,A'^{-2}$$

for all $A' > 0$. The proof is complete.

5.2. With the notations and assumptions of Theorem 1, *and under the extra assumption* that $\mathcal{L}_1 = \mathcal{L}_2$ on some ball $B(a,r_1)$ in M with $d(a) \geq r_1$, we show the following: *if $\int_0^{+\infty} \psi(s)\,ds$ is small enough depending on $\delta > 0$ (θ, p, ε_0 and M are regarded as fixed),*

$$G^2(a,b) \leq (1+\delta)\,G^1(a,b) \tag{5.2}$$

for all $b \in M$ such that $d(a,b) \geq 3\, r_1$.

Proof Let $\tilde{G}_R$ denote the Green's function in $\Omega_R = B(O, R+1)$ with respect to the operator $\tilde{\mathcal{L}}$ which coincides with $\mathcal{L}_2$ on $B(O, R-1)$ (i.e. the coefficients of $\tilde{\mathcal{L}}$ coincide with those of $\mathcal{L}_2$ on $B(O, R-1)$) and with $\mathcal{L}_1$ on $M \setminus B(O, R-1)$. Observe that $\mathrm{dist}_1(\mathcal{L}_1, \tilde{\mathcal{L}}) \prec c\,\psi$ in $\mathcal{D}_M(\theta, p)$, so that by Theorem 2 and Remark 1.2, $\tilde{\mathcal{L}} \in \mathcal{D}_M(\theta, p, \varepsilon_0/2)$ if $\int_0^{+\infty} \psi(s)\, ds$ is small; thus, we may assume that $\tilde{\mathcal{L}} \in \mathcal{D}_M(\theta, p, \varepsilon_0/2)$ and consider $\tilde{G}_R$.

It suffices to show that if $\int_0^{\infty} \psi(s)\, ds$ is small, then for every large ball Ω_R,

$$\tilde{G}_R(a,b) \leq (1+\delta)\, G_R^1(a,b) \tag{5.3}$$

when $b \in \Omega_{R-1}$ and $d(a,b) \geq 3\, r_1$. In fact, it follows from (5.3) that $G_{R-1}^2(a,b) \leq \tilde{G}_R(a,b) \leq (1+\delta)\, G_R^1(a,b)$ (for R large), whence (5.2) with R tending to infinity. (G_R^j denotes the Green's function of $\mathcal{L}_j$ in Ω_R.)

To derive (5.3), we use the construction of Section 3. We take for u the Martin kernel (with respect to $\mathcal{L}_1$ in Ω_R) $u = K_a^1 = G_R^1(.,a)/G_R^1(O,a)$ and construct $w = hf(u/h)$ as in Proposition 3.1, the function h being any fixed positive solution of $\mathcal{L}_1 h + \varepsilon_0 h = 0$ with $h(O) = 1$. If $\delta_1 > 0$ is given and if $\int_0^{+\infty} \psi(s)\, ds$ is small enough,

$$(1-\delta_1) \leq f'(t) \leq 1 \tag{5.4}$$

on $(0, +\infty)$ by (3.1)–(3.2), and thus $(1-\delta_1)\, t \leq f(t) \leq t$ for $t \geq 0$.

The function w is $\mathcal{L}_1$-superharmonic on Ω_R and is in $H^1(\Omega_R \setminus B(a,\rho))$ with vanishing boundary value on $\partial\Omega_R$, for all $\rho > 0$. Moreover, it follows from Proposition 3.1 (see Remark 3.2) that $\tilde{\mathcal{L}}(w) = -\nu + S$ where ν is a positive measure on Ω_R, $S \in H^{-1}(\Omega_R)$, $\mathrm{supp}(S) \subset \overline{B}(O, R-1) \setminus B(a, r_1)$ and

$$\|S\|_{H^{-1}(B(x, r_1/2))} \leq \delta_1\, \nu(B(x, r_1/4)) \tag{5.5}$$

for all $x \in \Omega_{R-1}$ (provided that $\int_0^{+\infty} \psi(s)\, ds$ is small enough). The measure ν is such that $\nu_{|(\Omega_R \setminus B(a,\rho))} \in H^{-1}(\Omega_R)$ for $\rho > 0$, and by (5.4), the concavity of f (see Remark 3.2)

$$\nu \geq (1-\delta_1)\, [G_{\Omega_R}^1(O,a)]^{-1}\, \varepsilon_a \tag{5.6}$$

where ε_a is the Dirac measure at a.

Now, we may consider $\tilde{G}_R(S)$ as well as $\tilde{G}_R(\nu)$ and we have $w = -\tilde{G}_R(S) + \tilde{G}_R(\nu)$. If $\{\varphi_j\}$ is a Whitney partition associated to $r'_1 = r_1/4$ (see Section 4.3) and $J = \{j \geq 1\,;\, d(x_j) \leq R\}$,

$$\begin{aligned}\tilde{G}_R(S)(b) &= \sum_{j\in J} \tilde{G}_R(\varphi_j S)(b)\\ &= \sum_{j\in J, d(x_j,b)>r_1/3} \langle \tilde{G}_R(b,.),\, \varphi_j S\rangle + \sum_{j\in J, d(x_j,b)\leq r_1/3} \tilde{G}_R(\varphi_j S)(b)\,.\end{aligned}$$

By Harnack and Caccioppoli's inequalities and by (5.5) each term $|\langle \tilde{G}_R(b,.),\, \varphi_j S\rangle$ with $d(x_j, b) \geq r_1/3$ is less than $c_2\,\delta_1 \int_{B(x_j,r'_1)} \tilde{G}_R(b,.)\, d\nu$. The sum of these terms is hence less than $c_2\,\delta_1\, \tilde{G}_R(\nu)(b)$.

If $d(x_j, b) \leq r_1/3$, the set of points $b' \in B(b, r_1)$ with

$$|\tilde{G}_R(\varphi_j S)(b')| \geq \sqrt{\delta}_1\, \tilde{G}_R(\varphi_j\, \nu)(b)$$

has measure less than $c\,\delta_1$ by Lemma 5.1 (since by (1.9),

$$\tilde{G}_R(\varphi_j\, \nu)(b) \geq c\, \nu(B(x_j, r'_1/2)) \geq c'\, \delta^{-1} \|\varphi_j S\|_{H^{-1}(B(b,r_1))}\ ).$$

It follows that for δ_1 small and $s = \tilde{G}_R(\nu)$ there is a b' with $d(b, b') \leq \delta_1^{1/(N+1)}$ such that $|\tilde{G}_R(S)(b')| \leq \sqrt{\delta_1}\, \tilde{G}_R(\nu)(b)$ and

$$(1 - \sqrt{\delta}_1)\, w(b) \leq s(b') \leq (1 + \sqrt{\delta}_1)\, w(b).$$

The function s is $\tilde{\mathcal{L}}$ superharmonic and positive. Thus, by (5.6) and the Riesz decomposition, we find that $s \geq (1 - \delta_1)\, [G^1_R(O, a)]^{-1}\; \tilde{G}_R(., a)$ on Ω_R (globally). Hence

$$(1 - \delta_1)\, \tilde{G}_R(b', a) \leq G^1_R(O, a)\, s(b') \leq G^1_R(O, a)\, (1 + \sqrt{\delta}_1)\, w(b).$$

Since

$$w \leq h\, \frac{G^1_{R,a}}{G^1_{R,a}(O)\, h} = [G^1_R(O, a)]^{-1}\, G^1_{a,R}$$

because $f(t) \leq t$, we have

$$(1 - \delta_1)\, \tilde{G}_R(b', a) \leq (1 + \sqrt{\delta}_1)\; G^1_R(b, a).$$

Finally, by Harnack inequalities (1.6)

$$(1 - \delta_1)(1 - \kappa(\delta_1^{1/(N+1)}))\; \tilde{G}_R(b, a) \leq (1 + \sqrt{\delta}_1)\; G^1_R(b, a)$$

where $\kappa(t)$ tends to zero when $t \to 0$. This proves (5.3) and hence (5.2).

Interchanging $\mathcal{L}_1$ and $\mathcal{L}_2$, we also have under the same assumptions on $\mathcal{L}_j$, a, b that

$$(1+\delta)^{-1} G^1(b,a) \leq G^2(b,a) \leq (1+\delta) G^1(b,a). \tag{5.7}$$

5.3. We check now that the restriction $d(a) \geq r_1$ may (of course) be dropped in (5.7). To this end fix $O' \in M$ with $d(O,O') = r_0' = r_0/16$ and take O' as a new origin. If $\varphi(a) = \text{dist}_q'(\mathcal{L}_1, \mathcal{L}_2)(a)$, we have $\varphi(a) \leq \psi(r_0' - d(a,O'))$ if $d(a,O') \leq r_0'$ and $\varphi(a) \leq \psi(d(a,O') - r_0')$ otherwise. If $\psi_1(t) = C\psi(r_0')\, 1_{[0,2r_0']}(t) + C\psi(t-r_0')\, 1_{(2r_0',\infty)}(t)$, Lemma 2.6 shows that $\varphi(a) \leq \psi_1(d(a,O'))$. Clearly, ψ_1 is nonincreasing and $\int_0^\infty \psi_1(s)\,ds \leq 2C\|\psi\|_1$. Applying the previous step—for $a \in M$ such that $d(a,O) \leq r_1$—we obtain (5.7) for all $a \in M$, $b \in M$ with $d(a,b) \geq 3r_1$, $\mathcal{L}_1 = \mathcal{L}_2$ on $B(a,r_1)$, if $\|\psi\|_1$ is small enough.

It is also quite easy to remove in (5.7) the assumption that $\mathcal{L}_1 = \mathcal{L}_2$ on $B(a,r_1)$. Consider the operator $\mathcal{L}_3 \in \mathcal{D}_M(\theta,p)$ whose coefficients are equal to those of $\mathcal{L}_1$ outside $B(a,r_1)$ and equal to those of $\mathcal{L}_2$ on $B(a,r_1)$. Clearly, $\mathcal{L}_3 \in \mathcal{D}_M(\theta,p)$ and $\text{dist}(\mathcal{L}_j,\mathcal{L}_3) \prec \psi$ for $j = 1,2$ if $\text{dist}(\mathcal{L}_1,\mathcal{L}_2) \prec \psi$ in $\mathcal{D}_M(\theta,p)$. Also, if $\int_0^\infty \psi(s)\,ds$ is small $\mathcal{L}_3 \in \mathcal{D}_M(\theta,p,\varepsilon_0/2)$ by Theorem 2 and Remark 1.2. From what has already been proved,

$$(1+\delta)^{-1} G^1(b,a) \leq G^3(b,a) \leq (1+\delta) G^1(b,a), \tag{5.8}$$

$$(1+\delta)^{-1} G^2(b,a) \leq G^3(b,a) \leq (1+\delta) G^2(b,a), \tag{5.8$'$}$$

if $d(a,b) \geq 3r_1$ and if $\int_0^\infty \psi(s)\,ds$ is small. (In (5.8) we have applied (5.7) to the adjoint operators and in (5.8′) we have interchanged a and b in (5.7).) It follows that $(1+\delta)^{-2} G^1(b,a) \leq G^2(b,a) \leq (1+\delta)^2 G^1(b,a)$ and the last claim of Theorem 1 is established.

5.4. Finally, the first claim in Theorem 1 will be proved by combining the above and the following simple lemma.

Lemma 5.2 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ in $\mathcal{D}_M(\theta,p,\varepsilon_0)$ be such that $\mathcal{L}_1 = \mathcal{L}_2$ on $M \setminus B(O,R)$ for some finite $R > 0$. The corresponding Green's functions in M verify*

$$c^{-1} G^2(x,y) \leq G^1(x,y) \leq c\, G^2(x,y) \tag{5.9}$$

for all x, y in M with $d(x,y) \geq r_0$ and a constant $c = c_M(\theta,p,\varepsilon_0,R) > 0$.

Proof Consider x in the ball $B(O,R)$. From the Harnack inequalities and the local estimates (1.9), we see that if $y \in B(O,R+1)$, $d(x,y) \geq r_0$,

$$G_x^1(y) \geq c\, G_x^2(y) \tag{5.10}$$

with $c = c_M(\theta,p,\varepsilon_0,R) \geq 1$. Using the maximum principle and the equality of $\mathcal{L}_1$ and $\mathcal{L}_2$ on $M \backslash B(O,R)$ we thus have $G_x^1(y) \geq c\, G_x^2(y)$ for all $x \in B(O,R), y \in M$ with $d(x,y) \geq r_0$. Exchanging $\mathcal{L}_1$ and $\mathcal{L}_2$ and considering then the adjoint operators it follows that $c^{-1}\, G_y^2 \leq G_y^1 \leq c\, G_y^2$ on $B(O,R)$ if $d(y) \geq R+1$. By a variant of the maximum principle ([B], p. 39)

$$G_y^1 \geq c^{-1}\, G_y^2 \tag{5.11}$$

on M. Here, we have observed that $u = G_y^1 - c^{-1}\, G_y^2$ is $\mathcal{L}_1$ superharmonic on $M \setminus B(O,R)$ (since $c \geq 1$), positive and continuous on $\partial B(O,R)$ and bounded from below by $-c^{-1}\, G_y^1$.

The lemma follows then from (5.10) and (5.11).

End of proof of Theorem 1. Let $\mathcal{L}'$ be the operator in $\mathcal{D}_M(\theta,p)$ which coincides with $\mathcal{L}_1$ on $B(O,R)$ and with $\mathcal{L}_2$ on $M \backslash \overline{B}(O,R)$. If R is chosen sufficiently large, then $\mathcal{L}' \in \mathcal{D}_M(\theta,p,\varepsilon_0/2)$ (Theorem 2), and moreover $\mathrm{dist}(\mathcal{L}_1,\mathcal{L}') \prec \psi_R = \inf(\psi,\psi(R))$ in $\mathcal{D}_M(\theta,p)$ so that $\int_0^{+\infty} \psi_R(s)\, ds$ can be made arbitrarily small. The second part of Theorem 1 which has already been proved shows that if R is fixed large enough (depending only on θ, p, ε_0, $\|\psi\|_1$), and if G' denotes the Green's function for $\mathcal{L}'$, then $\frac{1}{2}\, G^1(x,y) \leq G'(x,y) \leq 2G^1(x,y)$ for all x and y in M with $d(x,y) \geq r_0$. Also, by the previous lemma, $c^{-1}\, G^2(x,y) \leq G'(x,y) \leq c\, G^2(x,y)$ for some constant $c = c_M(\theta,p,\varepsilon_0,\|\psi\|_1) > 0$. (1.10) follows and the proof of Theorem 1 is complete.

6. Comments and first examples. Generalizations to the case $\lambda_1(\mathcal{L}) = 0$

6.1. We first relate Theorem 1 to a known criteria for comparability of Green's functions which is specific to zero-order perturbations. Assume that $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p)$, $\theta \geq 1, p > N$, admits a Green's function $G_1 = G_{\mathcal{L}_1}$. Following Definition 2.1 of [Pi3], a measurable function $W : M \to \mathbb{R}$ is called a small perturbation for $\mathcal{L}_1$ if for $R \to \infty$,

$$\sup\{\int_{d(z)\geq R} G_1(x,z)\, W(z)\, G_1(z,y)\, d\sigma(z)\,;\, d(x) \geq R,\, d(y) \geq R\}/G_1(x,y) \to 0.$$

A simple adaptation of the argument in [Pi3] shows that if W is a small perturbation for $\mathcal{L}_1$ and if $\mathcal{L} = \mathcal{L}_1 + W$ admits a Green's function G, then G_1 and G are comparable. A weak converse of this is observed in Remark 2.6 of [Pi3].

Corollary 6.1 *Let* $\psi : [0,\infty[\to \mathbb{R}$ *be nonincreasing and integrable and assume that* $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p,\varepsilon)$ *for some* $\varepsilon > 0$. *Then* $V(x) = \psi(d(x))$, $x \in M$, *is a small perturbation for* $\mathcal{L}_1$.

Proof Choose $\Phi : [0,\infty[\to \mathbb{R}$ positive, nonincreasing integrable and such that $\lim_{t\to\infty} \Phi(t)/\psi(t) = +\infty$. By Theorems 1 and 2, $\mathcal{L}_1$ and $\mathcal{L}_1 + \Phi(d(.))\, 1_{M\setminus B(O,R)}(\cdot)$ have, for R large enough, comparable Green's functions. By the resolvant argument in Remark 2.6 of [Pi3], it follows that $\int G_1(x,z)\,\Phi(d(z))\,G_1(z,y)\,d\sigma(z) \le C\,G_1(x,y)$ for some $C > 0$, whence the corollary. The same argument shows that any zero order perturbation $\mathcal{L} = \mathcal{L}_1 + W$ of $\mathcal{L}_1$ allowed by Theorem 1 is a small perturbation for $\mathcal{L}_1$.

Note that for the case at hand the proof of Theorem 1 reduces considerably. However, I do not know of a "direct" proof of Corollary 6.1 (except e.g. when M is hyperbolic and on using the Harnack principle at infinity of [A3]). Note also that using the methods in Section 9 the corollary implies a version for domains. Details are left to the reader.

6.2. Let us now consider the case of the Laplacian Δ in $\mathbb{R}^N$, $N \ge 3$, and of its perturbations, a case which is treated in [Pi1], [Pi4]. It may seem that Theorem 1 is useless here since $\lambda_1(\Delta) = 0$; moreover, the other requirements in Theorem 1 (with $p = \infty$ say) look different from the sharp conditions in [Pi4]. However, after a simple change of metric, Theorem 1 leads to similar conditions.

Fix a uniformly coercive $N \times N$-matrix $\mathcal{A}_x$ which is a bounded measurable function of $x \in \mathbb{R}^N$ and an elliptic operator in $\mathbb{R}^N$ in the form

$$L = \mathrm{div}(\mathcal{A}\nabla.) + B.\nabla. + \mathrm{div}(B'.) + b$$

with B, B' locally L^p, b locally $L^{p/2}$ in $\mathbb{R}^N$ for some $p > N$. Note $L_1 = \mathrm{div}(\mathcal{A}\nabla.)$. Let M denote $\mathbb{R}^N$ equipped with the metric $g(dx) = \varphi(r)\,|dx|^2$ where $r = |x|$, φ is positive and smooth with $\varphi(r) = r^{-2}$ when $r \ge 1$. It is easily seen that M satisfies our assumptions in Section 1.

Let $\mathcal{L}_1 = \varphi^{-1} L_1$, $\mathcal{L} = \varphi^{-1} L$. Simple computations show that $\mathcal{L} = \mathrm{div}_M(\mathcal{A}\nabla_M.) + D. + \mathrm{div}(D'.) + \gamma$, with $D = \varphi^{-1} B - (\frac{1}{2}n - 1)\,\varphi^{-2}\mathcal{A}^*(\nabla\varphi)$, $D' = \varphi^{-1} B'$, $\gamma = \varphi^{-1} b - \frac{1}{2}(n-2)\,\varphi^{-2}\,\nabla\varphi.B'$ where in the last three equations the gradients and the scalar products are the standard ones in $\mathbb{R}^N$.

Clearly $\mathcal{L}_1 \in \mathcal{D}_M(\theta,\infty)$ for θ large enough. Also, $\mathcal{L}_1$ is weakly coercive in M: this means that for small $\varepsilon > 0$, the operator $L_1 + \varepsilon/(1+r^2)$ admits a positive supersolution. This is clear for $L_1 = \Delta$ (just take $s(x) = 1/|x|^\alpha$, $0 < \alpha < N-2$) and amounts to the inequality

$$\int (1+r^2)^{-1}\,u^2 dx \le \varepsilon^{-1} \int |\nabla u|^2\,dx \qquad (6.1)$$

for $u \in C_0^\infty(\mathbb{R}^N)$. This inequality implies in turn the desired property for general L_1.

It is easily checked that $\mathcal{L} \in \mathcal{D}_M(\theta,\infty)$ for some $\theta \geq 1$ if $(1+r)(|B|+|B'|) + (1+r)^2|b| \leq C$. Moreover if ψ is nonincreasing on $\mathbb{R}_+$ and such that $\int_0^\infty \psi(t)\,dt < \infty$ then $\mathrm{dist}_\infty(\mathcal{L}_1,\mathcal{L}) \prec \psi$ in $\mathcal{D}_M(\theta,\infty)$ if $(1+r)(|B|+|B'|)+(1+r)^2|b| \leq \psi(\log(r))$ (note that $d_M(0,x) \sim \log(|x|)$ for $|x| \geq 2$).

In agreement with the results of [Pi4], it follows that if L is Greenian and if $(1+r)(|B|+|B'|)+(1+r)^2|b| \leq f(r)$ with f nonincreasing and such that $\int_1^\infty (f(t)/t)\,dt < \infty$, then the Green's functions G and G_1 of L and L_1 (acting in $\mathbb{R}^N$ with its usual metric) are comparable. It is well-known that G_1 is comparable to G_Δ [Sta]. Note that (for $p=\infty$) our conditions on B, B' and b are slightly stronger than the Kato conditions of [Pi4] (see Lemma 2.3 there) and that an extra uniform $C^{1,\alpha}$ regularity condition is made there on $\mathcal{A}$. Also, by Theorem 1, if for some $p > N \geq 3$, and all $\rho \geq 1$,

$$(1+\rho)\left(\rho^{-N}\int_{\rho\leq|x|\leq 2\rho}[\|B\|+\|B'\|]^p\,dx\right)^{1/p} + (1+\rho)^2\left(\rho^{-N}\int_{\rho\leq|x|\leq 2\rho}|b|^{p/2}\,dx\right)^{2/p} \leq f(\rho)$$

with f as before, then $G_L \sim G_\Delta$.

For nondivergence-type elliptic operators, using now the results of Section 7, the argument above yields (again in agreement with [Pi4]) the following. Let $L = \sum a_{ij}(x)\,\partial_i\partial_j + \sum B_i\,\partial_i + b$ be uniformly elliptic in $\mathbb{R}^N$ with bounded measurable coefficients. Assume that $\|a_{ij}\|_{\alpha,\mathbb{R}^N} < \infty$, $|a_{ij}(x) - a_{ij}^0| \leq C|x|^{-\delta}$ for some $\delta > 0$, $\alpha > 0$ and constants a_{ij}^0. Suppose further that $\sum_{1\leq i\leq N}|x|\,|B_i(x)| + |x|^2\,|b(x)| \leq f(|x|)$ with f satisfying the same Dini condition as above. Then the Green's function of L, if it exists, is comparable to G_Δ.

6.3. We now mention generalizations of Theorem 1 and Theorem 2 for general M as in Section 1. Let $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive nonincreasing function such that $\pi(r+1) \geq c\,\pi(r)$ for all $r \geq 0$ and some constant $c > 0$. We denote by the same letter π the function $m \mapsto \pi(d(0,m))$, $m \in M$, and for $\theta \geq 1$, $p > N$, denote $\mathcal{D}_M(\theta,p,\pi)$ the set of $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ such that there exists a $\mathcal{L}+\pi$ positive superharmonic function in M. Let now

$$\lambda_1^\pi(\mathcal{L}) = \sup\{t \in \mathbb{R};\ \mathcal{L}+t\pi \text{ has a Green's function}\} \in [-\infty,\infty).$$

It is quite straightforward to generalize Theorem 2 and its proof in Section 4 in the following way.

Theorem 6.2 *Let $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p)$ with $\lambda_1^\pi(\mathcal{L}_1) \geq -A$, $A < \infty$. There is a constant $c = c_{\pi,M}(\theta,p,A) > 0$ such that $|\lambda_1^\pi(\mathcal{L}_1) - \lambda_1^\pi(\mathcal{L}_2)| \leq \delta$ for $\mathcal{L}_2 \in \mathcal{D}_M(\theta,p)$, verifying*

$\mathrm{dist}_{q_0}(\mathcal{L}_1, \mathcal{L}_2)(m) \leq c\,\pi(d(m))\,\delta$ for $m \in M$. *Here $q_0 \geq 1$ is some large constant depending only on c_0, θ, and p, in fact the same q_0 as in the proof of Proposition 3.1.*

Let now $\mathcal{L}_1 \in \mathcal{D}_M(\theta,p,\pi)$, $\mathcal{L}_2 \in \mathcal{D}_M(\theta,p,\pi)$, $\theta > 1, p > N$. Let u (resp. h) be a positive $\mathcal{L}_1$-harmonic (resp. $\mathcal{L}_1 + \pi$-harmonic) function in B_R (resp. M) such that $u(O) = h(O) = 1$. The proof of Proposition 3.1 shows the following more general statement. Let $\psi : [0,\infty) \to \mathbb{R}_+$ be a positive continuous nonincreasing and integrable function and let f be as in Section 3. Let $w = hf(u/h)$.

Proposition 6.3 *We have $\mathcal{L}_2(w) = S - \mu$ where $S \in H^{-1}_{\mathrm{loc}}(B_R)$, and μ is a positive measure in B_R such that for each $a \in B_{R-1}$*

$$(6.2) \qquad \|S\|_{H^{-1}(B(a,r_0/2))} \leq c\,\sqrt{\|\psi\|_1}\;\frac{\mathrm{dist}_{q_0}(\mathcal{L}_1,\mathcal{L}_2)(a)}{\pi(d(a))\,\psi(d(a))}\;\mu(B(a,r_0/4))$$

where $c = c_M(\pi,\theta,p)$ is a positive constant.

Proposition 6.3 leads to the following extension of Theorem 1.

Theorem 6.4 *If $\mathrm{dist}_{q_0}(\mathcal{L}_1,\mathcal{L}_2)(m) \leq \pi(m)\,\psi_1(d(m))$, $m \in M$, with ψ_1 nonincreasing in $[0,\infty[$ and $\int_0^\infty \psi_1(t)\,dt < \infty$, then for some constant $c \geq 1$, we have*

$$c^{-1}\,G_1(x,y) \leq G_2(x,y) \leq c\,G_1(x,y)$$

for all $(x,y) \in M \times M$ such that $d(x,y) \geq r_0$ and some constant $c \geq 1$. In fact, for a given $\delta > 0$ we may even take $c = 1 + \delta$ if $\int_0^\infty \psi_1(t)\,dt$ is sufficiently small depending on δ.

The extension of the proof in Section 5 requires the following remarks. Firstly, if in the statement of Lemma 5.1 it is only assumed that $\mathcal{L} \in \mathcal{D}_M(\theta,p)$ admits a Green's function, then the conclusion holds if one replaces A in the l.h.s. of (5.1) by $\gamma(a)A$ where $\gamma(a) = \sup\{G(a,P)\,;\, P \in \partial B(a,r_1')\}$. The estimates of the terms with $d(x_j,b) < r_1/3$ in the paragraph after (5.6) are then easily extended.

As shown by the examples in 6.1 the above result is far from sharp in the case of $M = \mathbb{R}^N$, $N \geq 3$, equipped with the standard euclidean metric, and $\mathcal{L}_1 = \Delta_M$, $\mathcal{L}_2 = \Delta + D.\nabla$ say. One of the reasons behind this is that we have used in the proof of Theorem 1 the Harnack inequalities in their weakest form whereas in the above example with the Laplacian in $\mathbb{R}^N$ much better Harnack inequalities are available. More generally, the next paragraph shows that Theorem 6.4 can be seriously improved when $\mathcal{L}_1$ has no lower-order terms and if M has nonnegative Ricci curvature, or if M is a Lie group with polynomial growth endowed with a left invariant metric (such M verifies conditions (PI) and (DV) below).

6.4. Assume now that the complete manifold M verifies: (PI) Uniform Poincaré inequalities hold for all balls, (DV) the volume doubling condition for balls holds (for precise definitions see [SC2]), and that moreover with respect to the fixed point $O \in M$ we have $\mathrm{Vol}(B(O,s))/\mathrm{Vol}(B(O,r)) \geq c(s/r)^\sigma$ for $s \geq r \geq 1$ and some $\sigma > 2$. Here we may drop the assumptions (1.1).

Manifolds M verifying the first two conditions above have been extensively studied ([SC2], see also account and references there). In particular, uniform Harnack inequalities in (all) balls for operators in the form $\mathcal{L}_1 = \mathrm{div}(\mathcal{A}\nabla)$ with $\mathcal{A}$ bounded verifying (1.3) are known as well as the fact that our third assumption implies that the Green's function $G_{\mathcal{L}_1}$ for $\mathcal{L}_1$ exists and is comparable with G_Δ.

Fix $\nu > 2$ such that $\mathrm{vol}(B(x,r))/\mathrm{Vol}(B(x,s)) \leq C(r/s)^\nu$ for some $C \geq 1$ and all $0 < s < r, x \in M$. Put $r_m = 2^{m-1}$ if $m \geq 1$ and $r_0 = 0$. Let $\mathcal{L} = \mathrm{div}(\mathcal{A}\nabla) + D.\nabla. + \mathrm{div}(D'.) + \gamma$ be such that

$$
\begin{aligned}
a_m =&(1 + r_m)[\oint_{r_m \leq d(x) \leq r_{m+1}} (|D| + |D'|)^p \, d\sigma]^{1/p} \\
&+ (1 + r_m)^2[\oint_{r_m \leq d(x) \leq r_{m+1}} |\gamma|^{p/2} \, d\sigma]^{2/p} < \infty,
\end{aligned}
$$

for all $m \geq 1$ and some $p > \nu$. Here $\oint_A$ means $\frac{1}{\sigma(A)} \int_A$. We then have the following.

Theorem 6.5 *If the Green's function $G_\mathcal{L}$ for $\mathcal{L}$ exists and if $a_m \leq b_m$, $m \geq 0$, for some nonincreasing and summable sequence $\{b_m\}$, then $G_\mathcal{L}$ and $G_{\mathcal{L}_1}$ are comparable.*

The proof follows to some extent the same lines as before. Details will appear elsewhere.

7. The case of second-order elliptic operators in nondivergence form

7.1. In this section, in addition to (1.1) it is also assumed that in every chart $\psi = \psi_a$, $a \in M$, there is a bound

$$
|\partial_{x_k} g_{ij}| \leq c_0 \tag{7.1}
$$

on $B_a = B(a, r_0)$, $1 \leq i, j, k \leq N$, for the coefficients g_{ij} of the metric of M, and that a (global) orthonormal moving frame $\{X_1, \ldots, X_N\}$ verifying

$$
|\nabla_{X_k}(X_j)| \leq c_0 \tag{7.2}
$$

for j and k in $\{1, \ldots, N\}$, is given in M. For $\theta \geq 1$ and $0 < \alpha \leq 1$, we denote by $\Lambda_M(\theta, \alpha)$ the set of all second-order elliptic operator $\mathcal{L}$ on M with a given representation of the form

$$\mathcal{L}(u) = \sum_{i,j=1}^{N} a_{ij} X_j X_i(u) + \sum_{k=1}^{N} b_k X_k(u) + \gamma u, \tag{7.3}$$

where the coefficients a_{ij}, b_k, γ are bounded (borel) functions on M satisfying

$$\theta^{-1} \sum_{k=1}^{N} \xi_k^2 \leq \sum_{i,j} a_{ij}(x)\, \xi_i\, \xi_j \leq \theta \sum_{k=1}^{N} \xi_k^2, \tag{7.4}$$

$$\sum_{i,j=1}^{N} |a_{ij}(x) - a_{ij}(x')| \leq \theta\, d(x,x')^{\alpha}, \tag{7.5}$$

$$\sum_{1 \leq i,j \leq N} |a_{ij}(x)| + \sum_{k=1}^{N} |b_k(x)| + |\gamma(x)| \leq \theta, \tag{7.6}$$

when $x \in M$, $x' \in M$ are such that $d(x,x') \leq 1$ and $\xi \in \mathbb{R}^N$. The global existence of the frame $\{X_1, \ldots, X_N\}$ is assumed for the sake of notational simplicity and what follows may easily be extended to the class considered in [A3], pp. 512–514.

Let $\mathcal{L} \in \Lambda_M(\theta, \alpha)$. A $\mathcal{L}$-solution (or a $\mathcal{L}$ harmonic function) on a region $U \subset M$ is a function of class $W^{2,p}$ for some (or for all) finite $p > N$ satisfying $\mathcal{L}(u)(x) = 0$ a.e. It is well-known that Harnack inequalities (1.5), (1.6) hold for positive $\mathcal{L}$-solutions (with $\beta = 1$ in (1.6)) ([Ser]) and that a well-behaved local potential theory may be attached to $\mathcal{L}$ ([Her]). (Using a local chart, one is left with the standard case where $M = \mathbb{R}^N$ and $\{X_1, \ldots, X_N\}$ is the (constant) standard frame of $\mathbb{R}^N$.) On each transient region U ([A4]), there is a well-defined Green's function $G^U_{\mathcal{L}}(x,y)$ which is continuous in $U \times U$, $\mathcal{L}$-harmonic with respect to x in $U \setminus \{y\}$ and such that for each compactly supported $\varphi \in L^p(U)$, $G(\varphi) \in W^{2,p}_{loc}(U)$ and $\mathcal{L}G(\varphi) = -\varphi$; moreover, $G(\varphi)$ admits no positive $\mathcal{L}$ harmonic minorant in U. Finally, *an adjoint* potential theory ([Her]) may be defined: by definition, each function $y \mapsto G^U_{\mathcal{L}}(x,y)$ is $\mathcal{L}^*$-harmonic in $U \setminus \{x\}$ and adjoint potentials in U are the functions of the form $s = G^U_{\mathcal{L}}(\mu) = \int G^U_{\mathcal{L}}(x,.)\, d\mu(x)$ where μ is a positive measure in U such that $G^U_{\mathcal{L}}(\mu) \not\equiv +\infty$. Harnack inequalities (1.5) hold for the adjoint theory with a constant $c = c(\theta, \alpha, r_0)$ (by the local estimate of the Green's functions in the case $M = \mathbb{R}^N$). By the invariance of the class $\Lambda_{\mathbb{R}^N}(\theta, \alpha)$ under dilations, one also gets (1.6) (for adjoints) in the case $M = \mathbb{R}^N$ and hence also in the general case.

If $\mathcal{L} \in \Lambda_M(\theta, \alpha)$ and if $U \subset M$ is open, $\lambda_1(\mathcal{L}, U)$ is defined as before by (1.7) and we set $\lambda_1(\mathcal{L}) = \lambda_1(\mathcal{L}, M)$, $\Lambda_M(\theta, \alpha, \varepsilon_0) = \{\mathcal{L} \in \Lambda_M(\theta, \alpha)\,;\, \lambda_1(\mathcal{L}) \geq \varepsilon_0\}$. Moreover, for $\mathcal{L} \in \Lambda_M(\theta, \alpha, \varepsilon_0)$, $\varepsilon_0 > 0$, estimates (1.9) still hold (see [A3]).

For $\mathcal{L}_j \in \Lambda_M(\theta, \alpha), j = 1, 2$, $q \geq 1$ and $a \in M$ we now set

$$\mathrm{dist}'_q(\mathcal{L}_1, \mathcal{L}_2)(a) = \sum_{i,j} \|a^1_{ij} - a^2_{ij}\|_{L^q(B_a)} + \sum_i \|b^1_i - b^2_i\|_{L^1(B_a)} + \|\gamma^1 - \gamma^2\|_{L^1(B_a)},$$

where the a^k_{ij}, b^k_i, γ^k are the (given) coefficients of $\mathcal{L}_k$. The choice $q = +\infty$ is certainly the most natural, but the method works as well for $q > 1$. For $\psi : (0, +\infty) \to \mathbb{R}_+$ and $\mathcal{L}_j \in \Lambda_M(\theta, \alpha), j = 1, 2$, the notation $\mathrm{dist}'_q(\mathcal{L}_1, \mathcal{L}_2) \prec \psi$ in $\Lambda_m(\theta, \alpha)$ means that $\mathrm{dist}'_q(\mathcal{L}_1, \mathcal{L}_2)(a) \leq \psi(\rho)$ when $a \in M$ and $\rho = d(a)$.

7.2. To establish the analogues of Theorems 1 and 2 in this setting we follow the same lines as above for divergence-form operators and in some respect the proof is much simpler now. We start with the following (obvious) version of 2.5.

Lemma 7.2 *Let u and h be two continuous positive functions of class $W^{2,p}_{loc}$ in the region Ω of M and let $v = u/h$. Let also $f : (0, +\infty) \to (0, +\infty)$ be of class C^2 on $(0, +\infty)$. Then, for each $\mathcal{L} \in \Lambda_M(\theta, \alpha)$,*

$$\mathcal{L}(hf(v)) = hf''(v)\, a_{\mathcal{L}}(\nabla v, \nabla v) + f'(v)\, [\mathcal{L}(u) - v\, \mathcal{L}(h)] + f(v)\, \mathcal{L}(h)$$

holds a.e. in Ω. Here $a_{\mathcal{L}}(\nabla v, \nabla v) = \sum_{i,j} a_{ij} X_i(v) X_j(v)$ if $\mathcal{L}$ is in the form (7.3).

Proof Straightforward computation.

We also have the following obvious substitute to Corollary 2.2. *If $\mathcal{L} \in \Lambda_M(\theta, \alpha)$ then $\mathcal{L} - (\varepsilon + \theta)I \in \Lambda_M(\theta, \alpha, \varepsilon)$.* Finally, we have to replace the last argument is §5.3. This is the content of the next lemma.

Lemma 7.3 *Let $\mathcal{L}_k$, $k = 1, 2$ be two elements in $\Lambda_M(\theta, \alpha, \varepsilon_0)$, $\varepsilon_0 > 0$, such that $\mathcal{L}_1 = \mathcal{L}_2$ in $M \setminus B(a, r_0)$ for some $a \in M$, and denote by G_j the corresponding Green's functions. For each given $\delta > 0$, there is a positive ε such that when $\mathrm{dist}'_1(\mathcal{L}_1, \mathcal{L}_2)(a) \leq \varepsilon$*

$$(1 + \delta)^{-1}\, G_2(b, a) \leq G_1(b, a) \leq (1 + \delta)\, G_2(b, a) \tag{7.8}$$

for all $b \in M$ such that $d(a, b) \geq 2r_0$. (See also Remark 7.4 below.)

Proof We may as well assume that $\mathrm{dist}'_\infty(\mathcal{L}_1, \mathcal{L}_2)(a) \leq \varepsilon$. By Harnack inequalities (1.6) for adjoint harmonic functions with respect to $\mathcal{L}_j$,

$$(1 + \delta/4)^{-1}\, G_j(., a) \leq G_j(\varphi) \leq (1 + \delta/4) G_j(., a) \tag{7.9}$$

in $M \setminus B(a, r_0)$ if $\varphi = c_\rho 1_{B(a,\rho)}$, $c_\rho = [\sigma(B(a,\rho)]^{-1}$ and if ρ is sufficiently small (depending on M, θ, α and δ, but not on ε). Next we note the formula

$$G_2(\varphi) = G_1(\varphi) + G_2((\mathcal{L}_2 - \mathcal{L}_1)[G_1(\varphi)]) \tag{7.10}$$

where $\psi = (\mathcal{L}_2 - \mathcal{L}_1)[G_1(\varphi)]$ is in $L^p(M)$, $p < \infty$, with $\operatorname{supp}(\psi) \subset B(a, r_0)$. To prove the formula observe that by the basic properties of the Green's functions, the r.h.s. is a $W^{2,p}_{loc}(M)$ function w such that $\mathcal{L}_2 w = -\varphi$ and, in particular, w is $\mathcal{L}_2$-superharmonic. Also, because $\mathcal{L}_1 = \mathcal{L}_2$ on $M \setminus B(a, r_0)$ the function $|w|$ is dominated by $C\,G^2(.,a)$ where C is a large constant (by the maximum principle, [B] p. 39). It follows that w is a potential ([B]) and hence that $w \equiv G_2(\varphi)$, which proves the formula.

Observe now that $\|\psi\|_{L^p(M)} \to 0$ when $\varepsilon \to 0$. In fact, by the interior $W^{2,p}$ estimates, $\|G_1(\varphi)\|_{W^{2,p}(B_a)} \le c\,[\|\varphi\|_{L^p(M)} + \|G_1(\varphi)\|_{\infty, B(a,2r_0)}] \le c'$ since by (1.9) $G_1(\varphi)$ is bounded in $B(a, 2r_0)$ by a constant which depends on δ. Thus

$$\begin{aligned} \|\psi\|_{L^p} &\le c\,\varepsilon \sum_{i,j} \|X_i X_j G_1(\varphi))\|_{L^p(B)} + \sum_i \|b_i^2 - b_i^1\|_{L^p(B)} \|\nabla G_1(\varphi)\|_{\infty,B} \\ &\qquad + \|\gamma^2 - \gamma^1\|_{L^p(B)} \|G_1(\varphi)\|_{\infty,B} \\ &\le c'\{\varepsilon \|\varphi\|_{L^p(B)} + \varepsilon^{1/p}\}. \end{aligned}$$

By Harnack's inequalities for $\mathcal{L}^*$-harmonic functions,

$$G_2(|\psi|)(b) \le c\,G_2(b,a)\,\|\psi\|_{L^1}$$

for $b \in M$ such that $d(b,a) \ge 2r_0$. Thus, if ε is sufficiently small $G_2(|\psi|) \le (\delta/4)\,G_2(\varphi)$ on $M \setminus B(a, 2r_0)$ and formula (7.10) yields $G_1(\varphi) \le G_2(\varphi)(1 + \delta/4)$. Combining this with (7.9) we obtain (7.8).

Remark 7.4 The restriction $d(a,b) \ge 2r_0$ may be removed. Note that the proof above extends to the case where this condition is replaced by $d(b,a) \ge r_1$, for any fixed r_1 in $(0, r_0)$. On the other hand, by the known local behavior of Green's function, for each given $\delta > 0$ there is a number r_1 such that (7.8) hold for all $b \in B(a, r_1)$ provided r_1 is small enough.

7.3. It is easy to adapt the key construction in Section 3 and Proposition 3.5. Fix $\mathcal{L}_1 \in \Lambda_M(\theta, \alpha, \varepsilon_0)$, $\varepsilon_0 > 0$. Let u be positive $\mathcal{L}_1$ harmonic in $B(0,R)$ and let h be a positive $\mathcal{L}_1 + \varepsilon_0 I$ solution in M with $u(0) = h(0) = 1$. As in Section 3, we may construct a function w in the form $w = hf(u/h)$ in $B(0,R)$ where f is given by (3.1)–(3.2) and depends on the choice of the auxiliary nonincreasing function $\psi_1 : [0, +\infty) \to \mathbb{R}_+$.

Proposition 7.4 *Fix $q > 1$ and let $\mathcal{L} \in \Lambda_M(\theta, \alpha)$ be such that*
(i) $\mathcal{L} = \mathcal{L}_1$ *in* $\omega_R = \{x \in M;\, R - 1 < d(0,x) < R\}$ *and*
(ii) $\operatorname{dist}'_q(\mathcal{L}_1, \mathcal{L}) \prec \psi_1$ *with* $\int_0^{+\infty} \psi_1(s)\, ds \leq \eta$.
Then, for each given $\delta > 0$, there is a number $\eta(\delta) = \eta_M(\theta, \alpha, q, \varepsilon_0, \delta) > 0$, such that if $\eta \leq \eta(\delta)$ we may write $\mathcal{L}(w) = S - \mu$ in $B(0,R)$, where μ is positive and in $L^1_{loc}(B_R)$, $S \in L^1(B_R)$, $\operatorname{supp}(S) \subset \overline{B}(0, R-1)$ *and for every $a \in B(0, R - 1/2)$*

$$\|S\|_{L^1(B_a(r_0/2))} \leq \delta \int_{B(a,r_0/4)} \mu(x)\, d\sigma(x). \tag{7.8}$$

Observe that $\eta(\delta)$ is independent of R and can be taken in the form $\eta(\delta) = C(M, c_0, \theta, \alpha, q)\varepsilon_0^{-1}\delta$. The proof is similar to the proof of Proposition 3.1, using Remark 3.2.1 and the norms $\|.\|_{H^{-1}}$ being replaced by L^1 norms. The required bounds on $\|[\mathcal{L} - \mathcal{L}_1](u)\|_{L^1(B_a)}$ and $\|[\mathcal{L} - \mathcal{L}_1](h)\|_{L^1(B_a)}$ (compare (3.7)–(3.8)) are now straightforward (and are the reason for the assumption $q \neq 1$). The content of Remark 3.2 and Proposition 3.3 extend in the obvious way to the present setting. We omit further details and rather state now the analogues of Theorem 1 and Theorem 2.

Theorem 1′ *Fix $q > 1$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two element in $\Lambda_M(\theta, \alpha, \varepsilon_0)$ (with $0 < \alpha \leq 1$ and $\varepsilon_0 > 0$) and denote G^1 and G^2 the corresponding Green's functions in M. If ψ is nonincreasing in $[0, \infty)$ with $\int_0^{+\infty} \psi(s)\, ds < +\infty$ and if* $\operatorname{dist}'_q(\mathcal{L}_1, \mathcal{L}_2) \prec \psi$ *in $\Lambda_M(\theta, \alpha)$,*

$$c^{-1}\, G^2(x,y) \leq G^1(x,y) \leq c\, G^2(x,y) \tag{7.9}$$

for all $x, y \in M$ and some constant $c > 0$. Moreover, for every $\delta > 0$ there is a number $\eta = \eta(M, \theta, \alpha, \varepsilon_0, \delta) > 0$ such that if $\int_0^{+\infty} \psi(s)\, ds \leq \eta$ we may let $c = 1 + \delta$ in (7.9).

Theorem 2′ *Let $\theta > 0$, $\alpha \in (0, 1]$ be fixed. For each $\delta > 0$ there is a positive real η such that when $\mathcal{L}_1$, $\mathcal{L}_2 \in \Lambda_M(\theta, \alpha)$ and* $\operatorname{dist}_1'(\mathcal{L}_1, \mathcal{L}_2) \leq \eta$ on M,

$$|\lambda_1(\mathcal{L}_1) - \lambda_1(\mathcal{L}_2)| \leq \delta. \tag{7.10}$$

In fact, λ_1 is Lipschitz continuous in $\Lambda_M(\theta, \alpha)$ with respect to the distance $d(\mathcal{L}, \mathcal{L}') =$ $\sup_M \operatorname{dist}_q(\mathcal{L}, \mathcal{L}')$ *for each $q > 1$.*

7.4. Proof of Theorem 2′ To extend the proof in Section 4, we use the following fact which holds for every $\mathcal{L} \in \Lambda_M(\theta, \alpha)$ and every bounded region Ω in M such that $\lambda_0 = \lambda_1(\mathcal{L}, \Omega) > 0$: if G is the Green's function for $\mathcal{L}$ in Ω and if $G^*(x,y) = G(y,x)$ for x and y in Ω, there is a positive continuous function σ^* on Ω

such that $\sigma^* = \lambda_0 G^*(\sigma^*)$. This is well-known at least under some extra smoothness assumptions. See 7.5 below.

The proof in Section 4 may then be repeated, with $\Omega_R = \Omega$. Now, S is a negative measure with compact support $\subset B(0, R-1)$, and the integration-by- parts formula $\langle \nu - S, \sigma^* \rangle = \langle w, \lambda_0 \sigma^* \rangle$ holds since by Fubini's theorem

$$\langle \nu - S, \sigma^* \rangle = \langle \nu - S, \lambda_0 G^*(\sigma^*) \rangle = \langle G(\nu - S), \lambda_0 \sigma^* \rangle.$$

The other parts of the proof are unchanged.

7.5. *The existence of σ^*.* We sketch a proof for the existence of σ^* in 7.4. Replacing $\mathcal{L}$ by $\mathcal{L} - \lambda_2 I$, $\lambda_2 = \|\mathcal{L}(1)\|_\infty$, we may assume that $\mathcal{L}(1) \leq 0$ thanks to the resolvant equation. In this case, and since Ω is bounded, there is a bound $G(x,y) \leq C k_N(d(x,y))$ where $k_N(r) = r^{2-N}$ if $N \geq 3$ and $k_2(r) = 1 + \log_+(1/r)$. By Harnack property (1.6) it follows that G defines a compact operator in $L^2(\Omega)$. Fredholm's theorem shows then that Green's function for $\mathcal{L} + \lambda_0 I$ fails to exist, so that up to scalar multiples there is a unique $\mathcal{L}^* + \lambda_0 I$ positive supersolution σ^* in Ω and σ^* is $\mathcal{L}^* + \lambda_0 I$ harmonic (see e.g. [A4], Chap. 1 and 3). Finally, in the Riesz decomposition $\sigma^* = \lambda_0 G^*(\sigma^*) + h$ with $h \geq 0$, it is easily checked that $u = G^*(\sigma^*)$ is $\mathcal{L}^* + \lambda_0 I$ superharmonic and thus $\sigma^* = \lambda_0 G^*(\sigma^*)$.

7.6. Proof of Theorem 2 Using now Proposition 7.3 instead of Proposition 3.1, the proof in Section 5 may be repeated, the only changes being as follows.

(i) Given $\mathcal{L}_1$, $\mathcal{L}_2$ in $\Lambda_M(\theta, \alpha)$ and a closed set $F \subset M$, in general there is no $\mathcal{L} \in \Lambda_M(\theta, \alpha)$ which agree with $\mathcal{L}_1$ on F and with $\mathcal{L}_2$ on $M \setminus F$. However, using a smooth cutoff function φ we may define in the obvious way $\mathcal{L} \in \Lambda_M(4\theta, \alpha)$ equal to $\mathcal{L}_1$ on F and to $\mathcal{L}_2$ on $\{x \in M;\ d(x,F) \geq 1\}$.

(ii) In the formula after (5.6), S is in the form $S = f\sigma$ with $f_{|B_a} \in L^1(B_a), f \leq 0$ and (7.8). It follows that the terms corresponding to $j \in J$ with $d(x_j, b) \leq r_1/3$ in the l.h.s. may now be estimated using the Hölder inequality and the standard local estimate of $\tilde{G}$ by $d(x,y)^{2-N}$ (resp. $-\log|x-y|$ if $N = 2$):

$$\|\tilde{G}_R(\varphi_j S)\|_{L^1(B(b,r_1))} \leq c\,\|S\|_{L^1(B(x_j,r_1))}, \tag{7.11}$$

so that $\sigma\{x \in B(b, r_1)\,;\ |\tilde{G}_R(\varphi_j S)(x)| \geq t\,\|\varphi_j S\|_{L^1(B(x_j,r_1))}\} \leq c^{-1} t^{-1}$.

(iii) At the end of the proof (see Section 5.3) the argument is replaced by Lemma 7.3.

8. Some applications of Theorem 1 to manifolds

8.1. *A version of Theorem 1 localized at one point at infinity.* In this paragraph we assume that M, besides the assumptions in Section 1, is also hyperbolic in the sense of Gromov (e.g. M is a Cartan–Hadamard manifold with pinched negative sectional curvatures). We refer to [A4], [A3] for definitions, notations and potential theoretic results. Let $\Gamma = 0\zeta$ be a geodesic (minimizing) ray in M, $\zeta \in S_\infty(M)$, and let Φ be a positive function on $[0, +\infty)$. Set

$$U_\Phi(\zeta) = \{x \in M\,;\, d(x,\Gamma) < \Phi(d(0,x))\,\}.$$

Theorem 3 *Assume that* $\log(t) = o(\Phi(t))$ *when* $t \to +\infty$. *Let* $\mathcal{L}_j \in \mathcal{D}_M(\theta,p,\varepsilon)$, $j = 1,2$ *and* $\varepsilon > 0$ *be such that* $\mathrm{dist}_{q_0}(\mathcal{L}_1,\mathcal{L}_2)(x) \le \psi(d(x))$ *for* $x \in U_\Phi(\zeta)$ *where* ψ *is a nonincreasing and integrable function on* $(0,+\infty)$. *Then the corresponding Green's functions verify*

$$C^{-1}\,G^1(x,y) \le G^2(x,y) \le C\,G^1(x,y)$$

for x *and* y *on the ray* $\Gamma = 0\zeta$ *and some* $C = C_M(\Phi,\theta,p,\psi,\varepsilon) > 0$. *Moreover, the ratio* $G^1(x,0)/G^2(x,0)$ *has a limit when* $x \to \zeta$, $x \in \Gamma$.

Here $q_0 = q_0(M,\theta,p)$ is as in Theorem 1. Simple changes in the proof show that the similar statement with $\mathcal{L}_j \in \Lambda_M(\theta,\alpha,\varepsilon)$ and $\mathrm{dist}'_\infty(\mathcal{L}_1,\mathcal{L}_2) \le \psi(d(x))$ for $x \in U_\Phi(\zeta)$ holds as well.

Proof It suffices to show that the conclusions of the theorem hold if we keep the assumptions on the operators $\mathcal{L}_j$ but take $\Phi(t) = \Phi_A(t) = A\log(2+t)$ with a constant $A > 0$ sufficiently large (depending on M, θ, p and ε). This will follow from Theorem 1 and the following properties. Fix $\mathcal{L} \in \mathcal{D}_M(\theta,p,\varepsilon)$, set $U = U_{A,\rho} = \{x \in M\,;\, d(x,\Gamma_\rho) < \Phi_A(d(a_\rho,x))\,\}$ where a_ρ is the origin of the ray $\Gamma_\rho = \Gamma \setminus B(0,\rho)$. Let G (resp. g) denote the Green's function of $\mathcal{L}$ in M (resp. in U). We then have

(i) $g(x,y) \le G(x,y) \le C g(x,y)$ for x and y on Γ,

(ii) the limit $\ell = \lim_{x\in\Gamma, x\to\zeta} g(x,0)/G(x,0)$ exists and $\ell > 0$.

Assuming for the moment that (i) and (ii) hold, let us see how Theorem 3 follows, using Theorem 1. Introduce the operator $\mathcal{L}$ having the same coefficients as $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) in U (resp. in $F = M\setminus U$). By Theorem 2, if ρ is chosen sufficiently large $\mathcal{L} \in \mathcal{D}_M(\theta,p,\varepsilon/2)$, so that $\mathcal{L}$ and $\mathcal{L}_2$ satisfy the assumptions of Theorem 1 and thus have Green's functions of similar size. By (i) above, it follows that G^2, G, g and G^1 are also equivalent in size on Γ (because g is also the Green's function for $\mathcal{L}_1$ in U). The first claim in the theorem follows. By Proposition 8.4 below the second claim follows similarly from (ii).

Let us now prove (i) and (ii) following closely the method in [A5], §VII (see also [A4]). We assume as we may that $\rho = 0$ and $a_\rho = O$. Denote R_s^B the réduite of s over $B \subset M$ with respect to $\mathcal{L}$ (ref. [B]).

Lemma 8.1 *For $x \in M$, $R \geq 1$ and $V_R = M \setminus B(x,R)$,*

$$R^{V_R}_{G_x}(x) \leq \beta^{-1} e^{-\beta R} \tag{8.1}$$

with $\beta = \beta_M(\theta, p, \varepsilon) > 0$.

Proof Fix t with $0 < t < \varepsilon$ and let G^t be the Green's function for $\mathcal{L}+tI$. By [A4], Prop. 10, there is a constant $\beta = \beta_M(\theta,p,\varepsilon) > 0$ such that $G_x(y) \leq \beta^{-1} e^{-\beta R} G^t_x(y)$ for $d(x,y) \geq R$. Hence, setting $w = R^{V_R}_{G^t_x}$ (the réduite is taken with respect to $\mathcal{L}$),

$$R^{V_R}_{G_x}(x) \leq \beta^{-1} e^{-\beta R} w(x) \leq C e^{-\beta R},$$

since $w(z) \leq G^t_x(z)$, by the $\mathcal{L}$-superharmonicity of G^t_x and the definition of the réduite and because $G^t_x(z) \leq c$ on $\partial B(x,1)$. This proves (8.1).

Lemma 8.2 *Assume that A is sufficiently large and let $K_x = G_x / G(O,x)$. Then*
(a) *for x and y in Γ, $R^F_{G_x}(y) \leq \varepsilon(x,y)\, G(x,y)$ where $\lim_{x\to\zeta, y\to\zeta} \varepsilon(x,y) = 0$,*
(b) *we have $\lim_{x\to\zeta, x\in\Gamma} R^F_{K_x}(0) = R^F_{K_\zeta}(0)$ and $R^F_{K_\zeta}(0) < 1$.*

Recall $F = M \setminus U$. The second part of (b) means that F is minimally thin at ζ and is observed in [A4] under a symmetry assumption which is removed here using Lemma 8.1.

Proof It suffices to prove (a) for $x = z_k$ and $y = z_\ell$ where $z_j \in \Gamma$, $d(0,z_j) = j$, $j \geq 2$.

Assume that $k < \ell$ and let $F_j = \{m \in F\,;\, d(m,\Gamma) = d(m,[z_{j-1},z_{j+1}])\}$, $B_k = \{m \in F\,;\, d(m,[z_k,\zeta]) = d(m,z_k)\}$, $C_\ell = \{m \in F\,;\, d(m,[0,z_\ell]) = d(m,z_\ell)\}$. Note that $R_j = d(z_j,F_j)$ verifies $R_j \geq (A/2)\log(j+1)$ for sufficiently large j.

Using the Harnack principle at infinity ([A4]) several times, the hyperbolicity of M and Lemma 8.1, we get, for $k < j < \ell$,

$$\begin{aligned} R^{F_j}_{G_x}(y) &\leq c\, G(z_j,x)\, R^{F_j}_{G_{z_j}}(y) \leq c'\, G(z_j,x)\, G(y,z_j)\, R^{F_j}_{G_{z_j}}(z_j) \\ &\leq c''\, G(y,x)\, R^{F_j}_{G_{z_j}}(z_j) \leq c''\, G_x(y)\, e^{-\beta R_j} \leq c''\,(j+1)^{-A'}\, G_x(y) \end{aligned}$$

where $A' = \beta A/2$. It is shown similarly that $R^{B_k}_{G_x}(y) \leq c''\,(1+k)^{-A'}\, G(y,x)$ and $R^{C_\ell}_{G_x}(y) \leq c''\,(1+\ell)^{-A'}\, G(y,x)$.

Summing up, we find that

$$R^F_{G_x}(y) \leq R^{B_k}_{G_x}(y) + R^{C_\ell}_{G_x}(y) + \sum_{k<j<\ell} R^{F_j}_{G_x}(y) \leq c''\,[\sum_{k\leq j\leq \ell} (1+j)^{-A'}]\, G(y,x),$$

which proves (a) when $k < \ell$ if $A > 2\beta^{-1}$. The case $\ell \leq k$ is treated similarly.

To prove (b), we first observe that $K_x \le cK_\zeta$ outside $B(x,1)$, $x \in \Gamma$. In fact, $G_x \le c\,[K_\zeta(x)]^{-1} K_\zeta$ on $M \setminus B(x,1)$ since this holds on $\partial B(x,1)$ and G_x is a $\mathcal{L}$ potential. Thus $K_x \le c\,[G(0,x)\,K_\zeta(x)]^{-1} K_\zeta$ outside $B(x,1)$. But from the Harnack inequality at infinity $K_\zeta(x)\,G_x(0) \ge c$ when $x \in \Gamma$ ([A4], p. 99) and the observation follows.

Since $K_x \to K_\zeta$ when $x \to \zeta$, $x \in \Gamma$ ([A4]), $R^F_{K_x}(y) \to R^F_{K_\zeta}(y)$ for all $y \in U$ by dominated convergence (recall that $R^F_{K_x}(y) = \int K_x(z)\,d\mu(z)$ if μ is the harmonic measure of y in U).

By the proof of (a), for x and y on Γ with $d(x) > d(y)$, we have $R^F_{K_x}(y) \le c''\,(1 + d(y))^{1-A'}\,K_x(y)$. Letting $d(x)$ go to infinity and using the above we get

$$R^F_{K_\zeta}(y) \le c''\,(1 + d(y))^{1-A'}\,K_\zeta(y).$$

Thus, $K_\zeta - R^F_{K_\zeta}$ is positive harmonic in U and $\ge \frac{1}{2}$ at $y \in \Gamma$ if $d(y)$ is sufficiently large. It follows from Harnack inequalities that $R^F_{K_\zeta}(0) \le (1-\delta)$ for some $\delta = \delta_M(\theta, p, \varepsilon) > 0$. The proof is complete.

We may now prove properties (i) and (ii) after Theorem 3. From the formula $g(y,x) = G_x(y) - R^F_{G_x}(y)$ and Lemma 8.2, it follows that $g(y,x) \ge 1/2\,G(y,x)$ for x and y on Γ and sufficiently far from O. Using Harnack inequalities this yields (i). Also $g_x(0)/G_x(0) = 1 - R^F_{K_x}(0)$, whence (ii) by (b) in the lemma.

8.2. *Dirichlet problem and harmonic measures for manifolds.* In this subsection and the next, we consider again a general manifold M (with a given reference point $0 \in M$) that verifies only the assumptions of Section 1. Note that if M is hyperbolic then the $\mathcal{L}$-Martin compactification coincides with the compactification with the sphere at infinity, for all $\mathcal{L} \in \mathcal{D}_M(\theta, p, \varepsilon)$, $\theta \ge 1, p > N, \varepsilon > 0$ (ref. [A3], [A4]).

Proposition 8.3 *Assume that the hypothesis of Theorem 1 holds and let $\tilde{M} = M \bigcup \partial M$ be a compactification of M such that ∂M contains at least two points and such that the Dirichlet problem $\mathcal{L}_1(u) = 0$ in M and $u = f$ in ∂M is solvable for $f \in \mathcal{C}(\partial M; \mathbb{R})$ with $u \in \mathcal{C}(\tilde{M}; \mathbb{R})$. The similar Dirichlet problem for $\mathcal{L}_2$ is then also solvable and the corresponding harmonic measures μ^j_x, $x \in M, j = 1,2$ verify $c^{-1}\,\mu^1_x \le \mu^2_x \le c\,\mu^1_x$ where $c = c(\mathcal{L}_1, \mathcal{L}_2) > 0$.*

Remarks 1. If h is a $\mathcal{L}_1$-solution with boundary value 1, $\inf_{x\in M} h(x) > 0$ by the available minimum principle. Uniqueness for the Dirichlet problem with respect to $\mathcal{L}_1$ follows.

2. If the existence of a function $u \in \mathcal{C}(\tilde{M}; \mathbb{R})$ harmonic with respect to $\mathcal{L}_2$ and ≥ 1 in M is assumed from the start, Proposition 8.3 follows from Theorem 1 along familiar barrier arguments.

Proof of Proposition 8.3 Observe first that if $C^{-1}\,G^1(x,y) \le G^2(x,y) \le C\,G^1(x,y)$ when $d(x,y) \ge 1$, then for each nonnegative $\mathcal{L}_1$-harmonic function u in

M there is a $\mathcal{L}_2$-harmonic function v with $C^{-1}u \leq v \leq Cu$. To see this, consider the réduite $p_\rho = R_u^{B_\rho}(x)$ (with respect to $\mathcal{L}_1$) where $B_\rho = B(0,\rho)$. This function is the G^1-potential of a positive measure μ_ρ on $\partial B(0,\rho)$ and $v_\rho = G^2\mu_\rho$ verifies $C^{-1}p_\rho \leq v_\rho \leq Cp_\rho$ in $B(0,\rho-1)$. Since $p_\rho = u$ on $B(0,\rho)$, any cluster value v of v_ρ when $\rho \to \infty$ has the desired property.

Let u be continuous > 0 in $\tilde{M}$, $\mathcal{L}_1$-harmonic in M. Let $\mathcal{L}'_\rho \in \mathcal{D}_M(\theta,p)$ be such that $\mathcal{L}'_\rho = \mathcal{L}_1$ in $B(0,\rho)$ and $\mathcal{L}'_\rho = \mathcal{L}_2$ on $M \setminus B(0,\rho)$. If $\delta \in (0,1)$, by Theorem 1 and the above observation, there exists a $\mathcal{L}'_\rho$-harmonic function $w = w_\rho$ in M with $(1+\delta)^{-1}u \leq w \leq (1+\delta)u$ in M if ρ is large. By a standard extension result ([Her], Lemme 13.1) there is a $\mathcal{L}_2$-superharmonic function σ_1 in M and a positive measure μ with compact support in M such that $\sigma_1 - w = G^2(\mu)$ in $M \setminus B(0,\rho+2)$. Since $\mathrm{card}(\partial M) \geq 2$, the assumptions on $\mathcal{L}_1$ imply that there is a barrier with respect to $\mathcal{L}_1$ at each $\zeta \in \partial M$. Thus $G^1(\mu)$ and hence also $G^2(\mu)$ vanishes at infinity in M. In particular $(1+2\delta)^{-1}\sigma_1 \leq u \leq (1+2\delta)\sigma_1$ near infinity and the upper envelope $\bar{v}$ given by the Perron method for $\mathcal{L}_2$ and the boundary value $f = u_{|\partial\tilde{M}}$ verifies $\bar{v} \leq (1+2\delta)^2 u$ near infinity. Hence $\lim_{x\to\zeta} \bar{v}(x) \leq f(\zeta)$ for $\zeta \in \partial M$ and there is a similar lower bound for the Perron lower function. The corollary follows.

8.3. *The Martin boundary.* We denote by $\widehat{M}_\mathcal{L}$ the Martin compactification of M with respect to $\mathcal{L} \in \mathcal{D}_M(\theta,p,\varepsilon_0)$ ($p > N$, $\theta \geq 1$ and $\varepsilon_0 > 0$) and we let $\Delta_\mathcal{L} = \widehat{M}_\mathcal{L} \setminus M$. The minimal part of $\Delta_\mathcal{L}$ is denoted $\Delta^1_\mathcal{L}$ (see [A4] for definitions and references). When $\mathcal{L}$ is submarkovian (i.e. when $\mathcal{L}(1) \leq 0$), the $\mathcal{L}$ harmonic measure $\mu_x^\mathcal{L}$ of $x \in M$ is defined as follows. If u is the largest harmonic minorant of 1 in M and if ν is the unique positive borel measure on $\Delta_1^\mathcal{L}$ such that $u = K_\nu := \int K_\zeta(.)\,d\nu(\zeta)$ where K is the $\mathcal{L}$-Martin kernel with normalization at O, then $d\mu_x^\mathcal{L}(\zeta) = K_\zeta(x)\,d\nu(\zeta)$.

Proposition 8.4 *Under the assumptions of Theorem 1, the Martin compactifications of M with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$ coincide, i.e. there is a homeomorphism $\Phi : \widehat{M}_{\mathcal{L}_1} \to \widehat{M}_{\mathcal{L}_2}$ inducing the identity on M. Also, for x, y in M, the ratio $G_1(x,.)/G_2(y,.)$ of the adjoint Green's functions admits a continuous extension to $\widehat{M}_{\mathcal{L}_1} \setminus \{x,y\}$.*

Remark 8.5 If both operators are submarkovian, the corresponding harmonic measures verify $c^{-1}\mu_x^1 \leq \mu_x^2 \leq c\,\mu_x^1$ for $x \in M$ and some constant $c = c(\mathcal{L}_1,\mathcal{L}_2) > 0$. Moreover, there is a continuous density $f(x,\xi)$ on $M \times \Delta_{\mathcal{L}_1}$ such that $d\mu_x^2(\xi) = f(x,\xi)\,d\mu_x^1(\xi)$.

We need the following simple complement to Lemma 5.2 (see [T], [Pi2] for the first claim).

Lemma 8.5 *Under the assumptions of Lemma 5.2, the identity map in M extends to a homeomorphism $\widehat{M}_{\mathcal{L}_1} \to \widehat{M}_{\mathcal{L}_2}$. If $\zeta \in \Delta_M^{\mathcal{L}_1}$, and if $x \in M$ tends to ζ in $\widehat{M}_{\mathcal{L}_1}$, the ratios $G^2(a,x)/G^1(b,x)$, $(a,b) \in M \times M$, converge to a finite positive and* *continuous function $U_\zeta(a,b)$ on $M \times M$.*

Proof Let V be a component of $U = M \setminus \overline{B}(0,R)$ with a fixed reference point $Q \in V$ and let g denote the $\mathcal{L}_1$-Green's function in V. For $a \in V$,

$$\frac{g_x(a)}{g_x(Q)} = \frac{G_x^1(a) - R_{G_x^1}^1(a)}{G_x^1(Q)} \times \frac{G_x^1(Q)}{G_x^1(Q) - R_{G_x^1}^1(Q)} = \frac{K_x^1(a) - R_{K_x^1}^1(a)}{1 - R_{K_x^1}^1(Q)}$$

where R_u^1 is the réduite with respect to $\mathcal{L}_1$ of the function u over $\overline{B}(0,R)$ and K_x^1 is the $\mathcal{L}$-Martin kernel in M with Q as reference point. When $x \in V$ converges in $\widehat{M}_{\mathcal{L}_1}$ to $\zeta \in \Delta_{\mathcal{L}_1}(M)$,

$$\frac{g_x}{g_x(Q)} \to k_\zeta = \frac{K_\zeta^1(.) - R_{K_\zeta^1}^1(.)}{1 - R_{K_\zeta^1}^1(Q)}.$$

If $k_\zeta = k_{\zeta'}$ for some $\zeta' \in \Delta_{\mathcal{L}_1}(M) \cap \overline{V}$, the uniqueness property of the Riesz decomposition shows that $\zeta = \zeta'$. It follows that a sequence $\{x_j\}$ in V with $d(x_j, O) \to +\infty$ converges in $\widehat{M}_{\mathcal{L}_1}$ if and only if $g_{x_j}/g(Q, x_j)$ converges in V. Interchanging $\mathcal{L}_1$ and $\mathcal{L}_2$, it is seen that $\{x_j\}$ converges in $\widehat{M}_{\mathcal{L}_1}$ iff it converges in $\widehat{M}_{\mathcal{L}_2}$, which proves the first claim of the lemma. For $b \in M$, $x \in V$ and using the same notations as before

$$\frac{g(Q,x)}{G^1(b,x)} = \frac{G_x^1(Q) - R_{G_x^1}^1(Q)}{G_x^1(b)}$$
$$= \frac{G_x^1(Q) - R_{G_x^1}^1(Q)}{G_x^1(Q)} \times \frac{G_x^1(Q)}{G_x^1(b)} = [K_x^1(b)]^{-1}\,[1 - R_{K_x^1}^1(Q)].$$

Hence for each compact $K \subset M$,

$$\frac{g(Q,x)}{G^1(b,x)} \to \frac{1 - R_{K_\zeta^1}^1(Q)}{K_\zeta^1(b)} \quad \text{when } x \to \zeta,$$

uniformly with respect to $b \in K$. Using the similar properties for G^2 and g, it follows that uniformly with respect to $(b, b') \in K \times K$

$$\lim_{x \to \zeta} \frac{G^1(b,x)}{G^2(b',x)} = \frac{1 - R_{K_\zeta^2}^2(Q)}{1 - R_{K_\zeta^1}^1(Q)} \times \frac{K_\zeta^1(b)}{K_\zeta^2(b')}.$$

Proof of Proposition 8.4 Let R be a (large) positive real and let $\mathcal{L}$ denote the operator in $\mathcal{D}_M(\theta,p)$ which coincide with $\mathcal{L}_1$ on $B(0,R)$ and with $\mathcal{L}_2$ on $U = M \setminus B(0,R)$. For each given $\delta > 0$, Theorem 1 and Theorem 2 imply that if R is large the Green's function G for $\mathcal{L}$ exists and $(1+\delta)^{-1} \leq G(x,a)/G^1(x,a) \leq (1+\delta)$

if $d(x,a) \geq 1$. Using Lemma 8.5, it follows that if K is compact in M and if x, y belong to a small neighborhood (in $\widehat{M}_{\mathcal{L}_1}$) of $\zeta \in \Delta_{\mathcal{L}_1}$, then $(1+2\delta)^{-1} \leq [G^2(a,x)/G^1(a,x)] : [G^2(a,y)/G^1(a,y)] \leq (1+2\delta)$ for $a \in K$. This means that $G^2(.,x)/G^1(.,x)$ converges uniformly in K when $x \to \zeta$.

In particular, K_x^2 converges when $x \to \zeta$ in $\widehat{M}_{\mathcal{L}_1}$ and the identity extends continuously to $\tilde{\Phi} : \widehat{M}_{\mathcal{L}_1} \to \widehat{M}_{\mathcal{L}_2}$. Interchanging $\mathcal{L}_1$ and $\mathcal{L}_2$, Proposition 8.4 follows.

9. Applications to elliptic operators in euclidean domains

To simplify the exposition, we have discussed above (§6) the case of $\mathbb{R}^N$ itself and we shall restrict here to bounded domains. Note, however, that the results below in 9.4 for divergence type operators are valid without the boundedness assumption on Ω.

9.1. Let Ω be a bounded domain in $\mathbb{R}^N$ and for $\theta \geq 1$, $0 < \alpha \leq 1$, let $\Lambda_\Omega(\theta,\alpha)$ denote the class of elliptic operators L in Ω of the form

$$L(u)(x) = \sum a_{ij}(x)\, u_{ij}(x) + \sum b_j(x)\, u_j(x) + \gamma(x)\, u(x) \tag{9.1}$$

where the coefficients satisfy the following conditions. For $x \in \Omega$ and $\xi \in \mathbb{R}^N$,

$$\theta^{-1}\,|\xi|^2 \leq \sum a_{ij}(x)\,\xi_i\,\xi_j, \quad \sum |a_{ij}(x)| \leq \theta, \tag{9.2}$$

$$\sum |a_{ij}(x) - a_{ij}(y)| \leq \theta^{-1}\left(\frac{|x-y|}{\delta(x)}\right)^{\alpha} \quad \text{if } y \in \Omega \text{ and } d(x,y) \leq \tfrac{1}{2}\,\delta(x), \tag{9.3}$$

$$\sum |b_j(x)| \leq \theta\,\delta(x)^{-1}, \quad |\gamma(x)| \leq \theta\,\delta(x)^{-2}, \tag{9.4}$$

where $\delta(x) = d(x, \partial\Omega^c)$. If $\tilde{\delta}$ is a standard regularization of δ, if M is the Riemannian manifold (Ω, g) where $g(x,dx) = \tilde{\delta}^{-2}\,|dx|^2$ (Example 1.2.2), equipped with the frame $X_j = \tilde{\delta}(x)\,e_j$, $1 \leq j \leq N$, where $(e_1,\ldots,e_N)$ is the standard basis of $\mathbb{R}^N$, the operator

$$\mathcal{L} = \tilde{\delta}^2 L = \sum a_{ij}(x) X_i X_j + \sum_i [\tilde{\delta}(x)\, b_i - \sum_j a_{ij}(x)\,\tilde{\delta}_j'(x)\,]X_i + \tilde{\delta}(x)^2\gamma(x)$$

is in $\Lambda_M(\theta',\alpha)$ for some $\theta' \geq 1$ (see definitions in Section 7). Moreover, $\mathcal{L} \in \Lambda_M(\theta',\alpha,\varepsilon)$ iff $L + \varepsilon\,\tilde{\delta}(x)^{-2}$ admits a positive supersolution. We let $\Lambda_\Omega(\theta,\alpha,\varepsilon) = \{L \in \Lambda_\Omega(\theta,\alpha);\ \mathcal{L} \in \Lambda_M(\theta',\alpha,\varepsilon)\,\}$.

Recall from [A3] §8 that $L \in \Lambda_\Omega(\theta,\alpha)$ is in $\Lambda_\Omega(\theta,\alpha,\varepsilon)$ for some $\varepsilon > 0$ if there is a Green's function for L in Ω, if $\delta(x)\big(\sum_i |b_i(x)|\big) + \delta(x)^2\gamma^+(x) = o(1)$ when $\delta(x) \to 0$ and if one of the following conditions is also satisfied

(i) the region Ω is uniformly regular (in the sense of [A2]) and the coefficients a_{ij} are (globally) Hölder continuous in Ω,

(ii) there is a constant $c > 1$ such that for $x \in \partial\Omega$ and $r > 0$ there exists $y \in \Omega^c$ with $|y - x| \le c\,r$ and $B(y, c^{-1} r) \subset \Omega^c$.

If $L(1) \le 0$ the Green's function existence condition is implied by the others ([A3]). Also, in this case Ω is Dirichlet regular with respect to L. (See [A2] Theorem 4 and its proof.)

9.2. *John domains of Hölder type.* Let $0 < \beta \le 1$. We say that Ω is a John domain of Hölder type β, if there is a point $O \in \Omega$ and a constant $c_0 = c_0(\Omega) > 0$ such that each point $a \in \Omega$ can be joined to O by a rectifiable path $\Gamma(t)$, $0 \le t \le 1$, with $\Gamma(0) = a$, $\Gamma(1) = O$, $\Gamma \subset \Omega$ and

$$\delta(\Gamma(t))^\beta \ge c_0\,\ell(t) \qquad (9.5)$$

where $\ell(t)$ is the length of $\Gamma([0,t])$. For $\beta = 1$ we recover the John domains (ref. [NV]). For general β, the simplest examples are provided by the Hölder domains of exponent β.

For such domains we have the following (compare [HS], [A1]).

Theorem 9.1 *Assume that Ω is a John domain Ω of Hölder type $\beta > 0$. Let L_1, L_2 belong to a class $\Lambda_\Omega(\theta,\alpha,\varepsilon)$, $\varepsilon > 0$, and let G_j denote the Green's function of L_j in Ω. Suppose that for some bounded nondecreasing function $\Phi : (0,+\infty) \to \mathbb{R}_+$ and all $x \in \Omega$,*

(i) $\sum_{ij} |a^1_{ij}(x) - a^2_{ij}(x)| + \delta(x)\left(\sum_j |b^1_j(x) - b^2_j(x)|\right) + \delta(x)^2\,|\gamma_1(x) - \gamma_2(x)| \le \Phi(\delta(x))$,

(ii) *Φ satisfies the Dini condition*

$$\int_0^1 \frac{\Phi(t)}{t^{2-\beta}}\,dt < +\infty$$

where we have used obvious notations for the coefficients of L_j. Then, for x and y in Ω,

$$c^{-1}\,G_1(x,y) \le G_2(x,y) \le c\,G_1(x,y), \qquad (9.6)$$

where $c = c(\Omega, L_1, L_2) > 0$. If Ω is Dirichlet regular with respect to L_1, it is also L_2-Dirichlet regular and μ^j_x, $x \in \Omega$, the corresponding harmonic measures in Ω verify

$$c^{-1}\,\mu^1_x \le \mu^2_x \le c\,\mu^1_x. \qquad (9.7)$$

Remarks 9.2 1. The theorem still holds if in condition (i), $|b_j^1(x) - b_j^2(x)|$ is replaced by the mean $\delta(x)^{-N} \int_{B(x,\delta(x)/2)} |b_j^1(y) - b_j^2(y)|\,dy$ and similarly for $|\gamma_1(x) - \gamma_2(x)|$.

2. Let $\delta > 0$. Using Theorem 1′, we see that if $\int_0^1 \frac{\Phi(t)}{t^{2-\beta}}\,dt$ is sufficiently small depending on Ω, β, θ, α, then we may take $c \leq 1 + \delta$.

Proof Let $M = (\Omega, \tilde{\delta}^{-2}\,|dx|^2)$ be the Riemannian manifold attached to Ω as above, and let $d(x) = d_M(O, x)$. If Γ is a path $\Gamma : [0,1] \to \Omega$ connecting $a \in \Omega$ to O with (9.5), we obviously have $\delta(\Gamma(t))^\beta \geq c_1\,(\delta(a)^\beta + \ell(t)\,)$ with $c_1 = c_1(c_0, \beta)$. Therefore,

$$d(a) \leq c \int_0^1 \frac{d\ell(t)}{\delta(\Gamma(t))} \leq c' \int_0^1 \frac{d\ell(t)}{(\delta(a)^\beta + \ell(t))^{1/\beta}},$$

and $d(a) \leq \dfrac{c'}{1-\beta}\,\dfrac{\beta}{\delta(a)^{1-\beta}}$ if $\beta < 1$, and $d(a) \leq c' \log\left(\dfrac{1}{\delta(a)}\right)$ if $\beta = 1$.

On the other hand, it is easily checked that (i) implies that (in M) the operators $\mathcal{L}_j = \tilde{\delta}^2 L_j$ are such that $\mathrm{dist}'_\infty(\mathcal{L}_1, \mathcal{L}_2)(a) \leq \Phi(c\,\delta(a))$ (away from O) and hence, since Φ is nondecreasing,

$$\mathrm{dist}'_\infty(\mathcal{L}_1, \mathcal{L}_2)(a) \leq \Phi\Big(\frac{c}{d(a)^{1/(1-\beta)}}\Big) = \Psi(d(a))$$

if $\beta \neq 1$. Thus, with the notation of Section 6 and for $\beta < 1$, $\mathrm{dist}'_\infty(\mathcal{L}_1, \mathcal{L}_2) \prec \Psi$ in $\Lambda_M(\theta', \alpha)$ (away from O, for some large θ'). But (ii) means that Ψ is integrable over $(0, +\infty)$, so that we may apply Theorem 1′ (Section 7), and since the Green's function $\mathcal{G}_j$ in M of $\mathcal{L}_j$ is related to G_j by the formula $\mathcal{G}_j(x,y) = \tilde{\delta}(y)^{N-2}\,G_j(x,y)$, (9.6) follows. The case $\beta = 1$ is handled similarly. The claim on the harmonic measures follows from the "nondivergence" version of Proposition 8.3.

9.3. *Localization.* Assume that Ω is such that $0 \in \partial\Omega$ and $\Omega \bigcap B(0,\rho) = \{x \in B(0,\rho)\,;\ x_n > f(x_1, \ldots, x_{N-1})\,\}$ for some Lipschitz function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ with $f(0) = 0$ and $\rho > 0$. Let

$$U = \{x \in B(O,\rho) \cap \Omega\,;\ \|(x_1, \ldots, x_{N-1})\| < \delta(x)\,g(\delta(x))\}$$

where g is decreasing on $(0,\rho)$ and such that $\log(1/s) = o(g(s)^\varepsilon)$ when $s \to 0$ for each $\varepsilon > 0$. If the assumptions of Theorem 9.1 hold with $\beta = 1$ and (i) restricted to $x \in U$ then (9.6) holds for x and y in $S = \{(0, \ldots, O, t),\ 0 < t < \rho/2\,\}$. This may be deduced from the nondivergence variant of Theorem 3 (Section 8).

9.4. *Operators in divergence form.* Similar results hold for operators in divergence form. We briefly describe what is obtained in this case. Set $D_x =$

$\{y \in \Omega\,;\,|x-y| \le \frac{1}{2}\delta(x)\}$ for $x \in \Omega$. For $p > N$ and $\theta \ge 1$, denote $\mathcal{D}_\Omega(\theta,p)$ the class of operators L in the form

$$L(u) = \sum_{1\le i,j\le N} \partial_i(a_{ij}\,\partial_j(u)) + \sum_{1\le j\le N} b_j\,\partial_j u + \sum_{1\le j\le N} \partial_j(b'_j\, u) + \gamma u, \tag{9.8}$$

the coefficients being measurable functions on Ω, with (9.2) and for $x \in \Omega$,

$$\sum_j \delta(x)^{1-N/p}\,(\|b_j\|_{L^p(D_x)} + \|b'_j\|_{L^p(D_x)}) + \delta(x)^{2-2N/p}\,\|\gamma\|_{L^{p/2}(D_x)} \le \theta. \tag{9.9}$$

Set $\mathcal{D}_\Omega(\theta,p,\varepsilon) = \{L \in \mathcal{D}_\Omega(\theta,p)\,;\, L + \varepsilon\delta^{-2}.$ admits a > 0 supersolution in $\Omega\}$.

As before, if $L \in \mathcal{D}_\Omega(\theta,p)$, then $\mathcal{L} = \delta^2 L$ has a natural representation in some class $\mathcal{D}_M(\theta',p)$ where $M = (\Omega, g)$, $g = \tilde\delta(x)^{-2}\,|dx|^2$, and $L \in \mathcal{D}_\Omega(\theta,p,\varepsilon)$ iff $\mathcal{L} \in \mathcal{D}_M(\theta',p,\varepsilon)$. Straightforward calculations show that for $L^{(j)} \in \mathcal{D}_\Omega(\theta,p)$, $j = 1, 2$, the function $\mathrm{dist}_q(\mathcal{L}^{(1)},\mathcal{L}^{(2)})(x)$ related to M and the corresponding operators $\mathcal{L}^{(j)}$ (and a small radius r_0) is estimated by a constant times the expression

$$\sum \|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{\tilde L^q(D_x)} + \delta^{1-N/p} \sum_j (\|b_j^{(1)} - b_j^{(2)}\|_{L^p(D_x)} + \|b'^{(1)}_j - b'^{(2)}_j\|_{L^p(D_x)}) + \delta^{2(1-N/p)}\|\gamma_1 - \gamma_2\|_{L^{p/2}(D_x)}, \tag{9.10}$$

where $\|.\|_{\tilde L^q(D_x)}$ is the L^q norm with respect to the normalized measure $\mu_x = \delta(x)^{-N}\,dx$.

If $L \in \mathcal{D}_\Omega(\theta,p)$ has $b_j = b'_j = 0$ for $1 \le j \le N$ and $\gamma \le 0$ and if Ω is uniformly regular, then $L \in \mathcal{D}_\Omega(\theta,p,\varepsilon)$. This follows immediately from the validity of a version of Hardy's inequality for Ω. (See [A2].) Thus by Remark 1.3 we have the following statement.

Proposition 9.2 *Assume that Ω is uniformly regular and that $L \in \mathcal{D}_\Omega(\theta,p)$ is in the form (9.8) with $L(1) \le 0$ and $\sum_j \delta(x)^{1-N/p}\,(\|b_j\|_{L^p(D_x)} + \|b'_j\|_{L^p(D_x)}) + \delta(x)^{2-2N/p}\,\|\gamma\|_{L^{p/2}(D_x)} \le f(\delta(x))$ for $x \in \Omega$ and a function f in $(0,\infty)$ such that* $\lim_{t\to 0} f(t) = 0$.

Then, $L \in \mathcal{D}_\Omega(\theta,p,\varepsilon)$ for some $\varepsilon > 0$ depending only on Ω, θ, p and f.

The next statement is the variant of Theorem 9.1 for divergence-type operators.

Theorem 9.1′ *Suppose that Ω is a uniformly regular John domain of Hölder type β, $0 < \beta \le 1$, and let $L^{(1)}$, $L^{(2)}$ be members of a class $\mathcal{D}_\Omega(\theta,p,\varepsilon)$, ($p > N$, $\theta \ge 1$, $\varepsilon > 0$). Assume further that when $x \in \Omega$,*

$$\sum_{i,j} |a_{ij}^{(1)}(x) - a_{ij}^{(2)}(x)| + \sum_j \delta^{1-N/p}\,(\|b_j^{(1)} - b_j^{(2)}\|_{L^p(D_x)} + \|b'^{(1)}_j - b'^{(2)}_j\|_{L^p(D_x)}) + \delta^{2(1-N/p)}\,\|\gamma^{(1)} - \gamma^{(2)}\|_{L^{p/2}(D_x)} \le \varphi(\delta(x))$$

where $\delta = \delta(x)$, φ is nondecreasing with $\int_0^1 s^{\beta-2}\varphi(s)\,ds < +\infty$ and where we have used obvious notations for the coefficients of $L^{(j)}$. Then, the Green's functions $G^{(j)}$ of these operators —with respect to Ω— satisfy

$$C^{-1}\,G^{(1)}(x,y) \le G^{(2)}(x,y) \le C\,G^{(1)}(x,y)$$

with $C = C_\Omega(\theta,p,\varepsilon,\varphi) > 0$.

Similar inequalities hold for the L_j-harmonic measures in Ω. Also, in the condition above one may replace the terms involving the $a_{ij}^{(k)}$ by the sum $\sum \|a_{ij}^{(1)} - a_{ij}^{(2)}\|_{\tilde{L}^q(D_x)}$ with q sufficiently large depending on p and θ. Observe that by Proposition 9.2, Corollary 1.2 follows from the particular case where $\beta = 1$ and the L^p norms in the condition above are bounded.

9.5. *An application to Green's functions and harmonic measures with respect to nondivergence-type elliptic operators.* Suppose that Ω is a uniformly regular John domain of Hölder type β, $0 < \beta \le 1$, and let $L \in \Lambda_\Omega(\theta,\alpha)$ be in the form (9.1) verifying (9.2), $a_{ij} = a_{ji}$, $\gamma \le 0$ and $\delta(x)\sum_i |b_i| + \delta(x)^2\,|\gamma(x)| \le \varphi(\delta)$ where φ is an increasing function on $(0,+\infty)$ such that $\int_0^1 s^{\beta-2}\varphi(s)\,ds < +\infty$. Assume further that for $x \in \Omega$ (recall $\delta(x) = d(x,\Omega^c)$)

$$|a_{ij}(x) - a_{ij}(y)| \le \varphi(\delta(x))\left(\frac{|x-y|}{\delta(x)}\right)^\alpha \quad \text{when } |y-x| \le \tfrac{1}{2}\delta(x). \tag{9.11}$$

Denote G and $\tilde{G}$ the Green's functions of L and $\tilde{L} = \sum_{i,j}\partial_i(a_{ij}\partial_j(.))$, respectively.

Theorem 9.3 *Under the above conditions there is a constant $c \ge 1$ such that*

$$c^{-1}\,\tilde{G}(x,y) \le G(x,y) \le c\,\tilde{G}(x,y)$$

for all x and y in Ω. In particular, (i) *G is quasi-symmetric in the sense that $G(x,y) \le c^2\,G(y,x)$, and* (ii) *$c^{-1}\,\tilde{\mu}_x \le \mu_x \le c\,\tilde{\mu}_x$ if μ_x (resp. $\tilde{\mu}_x$) denotes the harmonic measure of x in Ω with respect to L (resp. $\tilde{L}$).*

Proof We assume as we may that $b_j = 0$, $\gamma = 0$ (Theorem 9.1) and we construct functions a_{ij}^0 by regularising a_{ij} in the usual way, using a fixed Whitney partition of Ω (ref. [Ste]). Standard arguments show that a_{ij}^0 satisfy the uniform ellipticity condition (9.2), and that

$$|\nabla a_{ij}^0(x)| \le c'\,\delta(x)^{-1}\,\varphi(\delta(x)), \qquad |a_{ij}^0(x) - a_{ij}(x)| \le \varphi(\delta(x)). \tag{9.12}$$

In particular, the operator $L^0 = \sum \partial_i(a_{ij}^0\,\partial_j(.)) = \sum a_{ij}^0\,\partial_i\,\partial_j(.) + \sum \partial_i(a_{ij}^0)\,\partial_j(.)$ belongs to (or rather has a representation in) a class $\Lambda_\Omega(\theta',\alpha)$ and by Theorem

9.1 has a Green's function comparable to G. At the same time, it is a formally self-adjoint operator of divergence type with a representation in $\mathcal{D}_\Omega(\theta'',p,\varepsilon)$ for some θ'', any fixed $p > N$, $\varepsilon > 0$ small. By Theorem 9.1′, L^0 and $\tilde{L}$ have Green's functions equivalent in size. The same reasoning applies to harmonic measures, and the theorem follows.

As a consequence we finally show the following.

Theorem 9.4 *Suppose that Ω is a Lipschitz domain. Let L be an elliptic operator in Ω in the form* (9.1) *and such that* (9.2) *and* $\sum_j \delta(x)\,|b_j|+\delta^2(x)|\gamma(x)| \leq \varphi(\delta(x))$ *hold for some nondecreasing function φ verifying the Dini condition $\int_0^1 t^{-1}\,\varphi(t)\,dt < +\infty$. Assume moreover that the a_{ij} are globally Hölder continuous in Ω. Then, the L-harmonic measures μ_x in Ω, $x \in \Omega$, are absolutely continuous with respect to the area measure σ on $\partial\Omega$ and $\mu_x = f_x.\sigma$ with $f_x \in L^2(\sigma)$.*

Proof By Theorem 9.3 we may assume that $b_1 = \cdots = b_N = \gamma = 0$, $a_{ij} = a_{ji}$, and then replace L by $L_1 = \sum_{i,j} \partial_i(a_{ij}\partial_j(.))$. By the main result in [FKJ] (or [D]) we are done.

We may also use an argument based on Theorem 9.1, which we may sketch as follows. If L_1 is in the form $L_1 = \sum_{ij} \partial_i(a_{ij}\partial_j)$ with $a_{ij} = a_{ji}$, and if $P \in \partial\Omega$ is such that the a_{ij} are constant along a direction transverse to $\partial\Omega$ in a neighborhood V of P, it is known that the required property holds in the neighborhood of P. This is observed in [FKJ] and follows easily from the Rellich formula ([N], p. 244).

Pick $P \in \partial\Omega$, a transverse direction ν to $\partial\Omega$ around P and a small ball $B(P,r)$. Let $L_P = \sum_{ij} \partial_i(a^0_{ij}\partial_j(.))$ be the (divergence-type) operator whose coefficients are constant along the parallel to ν in $B(P,r)$ and coincide with those of L_1 on $\partial\Omega \cup B(P,r)^c$. Clearly, $|a_{ij}(x) - a^0_{ij}(x)| \leq \delta(x)^\alpha$ in Ω. Using Theorem 9.1′ again it is seen that the harmonic measures with respect to L and L_P are uniformly comparable on $\partial\Omega$. The result then follows from Theorem 9.1′ and a standard covering argument.

Notes added in proof

1. Analogues of our main results for discrete potential theoretic settings, as well as extensions of Section 7 to more general second-order elliptic operators in nondivergence form will be discussed elsewhere.

2. After the revised version of this paper was sent to the Editors with a new Section 6 inserted, we learned from a letter of Prof. Minoru Murata that he also remarked that (a domain version of) Corollary 6.1 follows from Theorem 1.

References

[A1] A. Ancona, *Comparaison des fonctions de Green et des mesures harmoniques pour des opérateurs elliptiques*, C. R. Acad. Sci. Paris **294** (1982), 505–508.

[A2] A. Ancona, *On strong barriers and an inequality of Hardy for domains in* $\mathbb{R}^N$, J. London Math. Soc. **34** (2) (1986), 274–290.

[A3] A. Ancona, *Negatively curved manifolds, elliptic operators and the Martin boundary*, Ann. of Math. **125** (1987), 495–536.

[A4] A. Ancona, *Théorie du potentiel sur les graphes et les variétés*, Ecole d'été de Probabilités de Saint-Flour XVIII - 1988, Lecture Notes in Math. **1427**, Springer-Verlag, Berlin, 1990, pp. 5–112.

[A5] A. Ancona, *Positive harmonic functions and hyperbolicity*, in *Potential Theory, Surveys and Problems*, Lecture Notes in Math. **1344**, Springer-Verlag, Berlin, 1988, pp. 1–23.

[Ar] D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc. **73** (1967), 890–896.

[BNV] H. Berestycki, L. Nirenberg and S. R. S. Varadhan, *The principal eigenvalue and maximum principle for second order elliptic operators in general domains*, Comm. Pure Appl. Math. **47** (1994), 47–92.

[B] M. Brelot, *Axiomatique des fonctions harmoniques*, Les Presses de l'Université de Montréal, 1969.

[C] L. Carleson, *On mappings conformal at the boundary*, J. Analyse Math. **19** (1967), 1–13.

[CZ] M. Cranston and Z. Zhao, *Conditional transformation of drift formula and Potential Theory for* $\frac{1}{2}\Delta + b(.).\nabla$, Comm. Math. Phys. **112** (1987), 613–627.

[D] B. Dahlberg, *On the absolute continuity of elliptic measures*, Amer. J. Math. **108** (1986), 1119–1138.

[FKJ] E. Fabes, C. Kenig and D. Jerison, *Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure*, Ann. of Math. **119** (1984), 121–141.

[FKP] R. A. Fefferman, C. Kenig and J. Pipher, *The theory of weight and the Dirichlet problem for elliptic equations*, Ann. of Math. **134** (1991), 65–124.

[GK] S. J. Gardiner and M. Klimek, *Convexity and subsolutions of partial differential equations*, Bull. London Math. Soc. **18** (1986), 41–43.

[Gia] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Math. Studies **105**, Princeton University Press, Princeton, N.J., 1983.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1977.

[Ham] U. Hamenstädt, *Harmonic measures for compact negatively curved manifolds*, Preprint, Bonn Universität (1995).

[Her] R.-M. Hervé, *Recherche sur la théorie axiomatique des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier **XII** (1962), 415–471.

[HH] M. Hervé and R.-M. Hervé, *Les fonctions surharmoniques associées à un opérateur elliptique du second ordre à coefficients discontinus*, Ann. Inst. Fourier **XIX** (1969), 305–359.

[HS] H. Hueber and M. Sieveking, *On the quotients of Green functions*, Ann. Inst. Fourier **32** (1) (1982), 105–118.

[LU] O. A. Ladyzhenskaia and N. N. Uraltseva, *Linear and Quasi-Linear Equations*, Academic Press, New York and London, 1968.

[Mey] N. G. Meyers, *An* L^p*-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa **17** (1963), 189–206.

[Mok] G. Mokobodzki, *Algèbre des multiplicateurs d'un espace de Dirichlet*, to appear.

[Mu] M. Murata, *Structure of positive solutions to* $(-\Delta + V)u = 0$ *in* R^n, Duke Math. J. **53** (1986), 869–943.

[N] J. Nečas, *Les méthodes directes en théorie des opérateurs elliptiques*, Masson (Paris), Academia (Prague), 1967.

[NV] R. Näkki and J. Väisälä, *John disks*, Exposition. Math. **9** (1991), 3–43.

[Pi1] Y. Pinchover, *Sur les solutions positives d'équations elliptiques et paraboliques*, C. R. Acad. Sci. Paris **302** (1986), 447–450.

[Pi2] Y. Pinchover, *On positive solutions of second order elliptic equations, stability results and classification*, Duke Math. J. **57** (1988), 955–980.

[Pi3] Y. Pinchover, *Criticality and ground states of second order elliptic equations*, J. Differential Equations **80** (1989), 237–250.

[Pi4] Y. Pinchover, *On the equivalence of Green functions of second order elliptic equations in* R^n, Differential Integral Equations **5** (1992), 481–490.

[SC1] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** (1992), 417–450.

[SC2] L. Saloff-Coste, *Parabolic Harnack inequality for divergence form second order differential operators*, Potential Analysis **4** (4) (1995), 429–467.

[Ser] J. Serrin, *On the Harnack inequality for linear elliptic equations*, J. Analyse Math. **4** (1956), 292–308.

[Sta] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier **15** (1) (1965), 189–258.

[Ste] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N.J., 1970.

[T] J. C. Taylor, *The Martin boundaries of equivalent sheaves*, Ann. Inst. Fourier **20** (1) (1970), 433–456.

[W1] K.-O. Widman, *On the boundary behavior of solutions to a class of elliptic partial differential equations*, Ark. Mat. **6** (1967), 485–533.

[W2] K.-O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations*, Math. Scand. **21** (1967), 17–37.

[Z1] Z. Zhao, *Subcriticality, positivity and gaugeability of the Schrödinger operator*, Bull. Amer. Math. Soc. **23** (1990), 513–517.

[Z2] Z. Zhao, *Subcriticality and gaugeability of the Schrödinger operator*, Trans. Amer. Math. Soc. **334** (1992), 75–96.

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