Ann. Univ. Ferrara - Sez. VII - Sc. Mat.
Vol. XLVII, 117-144 (2001)

Diffusive-Dispersive Traveling Waves and Kinetic Relations III. An Hyperbolic Model of Elastodynamics.

NABIL BEDJAOUI (*)(**) - PHILIPPE G. LEFLOCH (*)

SUNTO - Questa è la terza parte di una serie di lavori dedicati all'esistenza, unicità, monotonia e proprietà asintotiche delle soluzioni d'onda di propagazione per leggi di conservazione diffusive-dispersive. In questa parte, l'attenzione è focalizzata su un modello iperbolico non convesso di due leggi di conservazione che sorgono in elastodinamica non lineare, che tengono conto della viscosità non lineare e dei termini di capillarità. Da una parte, utilizzando le tecniche precedentemente sviluppate, studiamo le proprietà delle corrispondenti onde d'urto classiche e non classiche e le loro corrispondenti relazioni cinetiche. Diverse nuove proprietà sono state trovate per questo modello (iperbolico). Innanzitutto, qui distinguiamo tra una funzione cinetica ed una funzione cinetica inversa, quest'ultima essendo sempre definita globalmente ma possibilmente non sempre globalmente iinvertibile. In secondo luogo, mostriamo che onde d'urto con ampiezza sufficientemente piccola sono sempre classiche, per un valore fissato del rapporto tra diffusione e dispersione. In ultimo, determiniamo il comportamento asintotico della funzione cinetica per onde d'urto aventi sia ampiezza grande sia piccola.

ABSTRACT - This is the third part of a series devoted to the existence, uniqueness, monotonicity, and asymptotic properties of the traveling wave solutions of diffusive-dispersive conservation laws. In this part, we focus attention on a nonconvex hyperbolic model of two conservation laws arising in nonlinear elastodynamics and including nonlinear viscosity and capillarity terms. On one hand, using the techniques developed earlier, we study the properties of the corresponding classical and nonclassical shock waves and their corresponding kinetic relation. Several new features are

(*) Indirizzo dell'autore: Centre de Mathématiques Appliquées & Centre National de la Recherche Scientifique, UMR 7641, Ecole Polytechnique, 91128 Palaiseau Cedex, France. E-mail: lefloch@cmap.polytechnique.fr

(**) Indirizzo dell'autore: INSSET, Université de Picardie, 48 rue Raspail, 02109 Saint-Quentin, France. E-mail: bedjaoui@cmap.polytechnique.fr

found for this (hyperbolic) model: First of all, we distinguish here between a kinetic function and an inverse kinetic function; the latter is always globally defined but may fail to be globally invertible. Second, we show that shock waves with sufficiently small amplitude are always classical, for a fixed ratio of diffusion and dispersion. Third, we determine here the asymptotic behavior of the kinetic function for both shocks with large and small amplitudes.

1. – Introduction.

Consider the following system of two conservation laws in one-space dimension arising, for instance, in nonlinear elastodynamics:

$$(1.1)\qquad \begin{aligned} &\partial_t v - \partial_x \sigma(w) = \beta \partial_x (b(w)\, \partial_x v) - \alpha \partial_x (a_1(w)\, \partial_x (a_2(w)\, \partial_x w)), \\ &\partial_t w - \partial_x v = 0 . \end{aligned}$$

Here $v \in \mathbb{R}$ and $w > -1$ represent the velocity and the deformation gradient of some solid material or fluid, respectively. The viscosity function $b(w)$ and the capillarity functions $a_1(w)$ and $a_2(w)$ are assumed to be bounded below by some positive constants b_0 and a_0, respectively:

$$(1.2)\qquad b(w) \geqslant b_0 > 0 , \qquad a_1(w), a_2(w) \geqslant a_0 > 0 \qquad \text{for all } w > -1 .$$

The parameters β, $\alpha > 0$ measure the relative strengths of the viscosity and capillarity terms in the material. Finally, the stress-function σ depends on the material under consideration. A typical set of assumptions is:

$$(1.3)\qquad \begin{aligned} &w\sigma''(w) > 0 , \qquad \sigma'(0) > 0 , \\ &\lim_{w \to -1} \sigma(w) = -\infty , \qquad \lim_{w \to +\infty} \sigma'(w) = +\infty . \end{aligned}$$

The partial differential equations (1.1) are hyperbolic, and admit two real and distinct wave speeds, $-c(w)$ and $c(w)$, where c is the sound speed defined by

$$c(w) := \sqrt{\sigma'(w)} \qquad \text{for all } w > -1 .$$

This paper is the third part of a series (see [3, 4]) devoted to traveling solutions associated with diffusive-dispersive conservation laws. This activity takes its root in pioneering work by Slemrod [13, 14] concerning the effect of capillarity in fluids and solids. Our purpose here is to study the traveling waves of the system (1.1), that is the solutions depending only on $y := x - \lambda t$ for some speed λ and connecting two constant states at infinity. The equations satisfied by a traveling wave solution $y \mapsto (v(y), w(y))$ read

$$\text{(1.4)} \qquad \begin{aligned} &\lambda v_y + \sigma(w)_y = -\beta(b(w)\, v_y)_y + \alpha(a_1(w)(a_2(w)\, w_y)_y)_y, \\ &\lambda w_y + v_y = 0\,, \end{aligned}$$

and

$$\text{(1.5)} \qquad \begin{aligned} &v_y(y),\ w_y(y),\ w_{yy}(y) \to 0 && \text{when } |y| \to \infty\,, \\ &v(y) \to v_-, \quad w(y) \to w_- && \text{when } y \to -\infty\,, \\ &v(y) \to v_+, \quad w(y) \to w_+ && \text{when } y \to +\infty\,, \end{aligned}$$

where v_-, w_-, v_+, w_+ are constants. Set also

$$v_0 := v_-, \qquad w_0 := w_-,$$

and let us search for all of the right-hand states (v_+, w_+) attainable by a traveling wave leaving from (v_0, w_0). By integration of (1.4) over the interval $(-\infty, y]$ and using (1.5), we obtain

$$\text{(1.6)} \qquad \begin{aligned} &\lambda(v - v_0) + \sigma(w) - \sigma(w_0) = -\beta b(w)\, v_y + \alpha a_1(w)(a_2(w)\, w_y)_y, \\ &\lambda(w - w_0) + v - v_0 = 0\,. \end{aligned}$$

The aim is solving the system of second-order, ordinary differential equations (1.5)-(1.6), under the assumptions (1.2)-(1.3) on the coefficients.

In view of (1.5)-(1.6), the shock speed λ is determined by the jump condition

$$\lambda(w - w_0) + v - v_0 = \lambda(v - v_0) + \sigma(w) - \sigma(w_0) = 0\,,$$

so that

$$\text{(1.7)} \qquad \lambda^2 = \begin{cases} \dfrac{\sigma(w) - \sigma(w_0)}{w - w_0} & \text{if } w \neq w_0, \\ \sigma'(w_0) & \text{if } w = w_0. \end{cases}$$

Recall also that, from (1.6), one can deduce a second-order differential equation in w only:

$$\text{(1.8)} \qquad -\lambda^2(w - w_0) + \sigma(w) - \sigma(w_0) = \lambda\beta b(w)\, w_y + \alpha a_1(w)(a_2(w)\, w_y)_y.$$

Finally, setting $z = a_2(w)\, w_y$, we arrive at a first-order system in the plane (w, z):

$$
(1.9) \qquad
\begin{aligned}
& a_2(w)\, w_y = z\,, \\
& \alpha a_1(w)\, z_y = -\lambda\beta \frac{b(w)}{a_2(w)} z + g(w, \lambda) - g(w_0, \lambda)\,,
\end{aligned}
$$

where

$$
g(w, \lambda) := \sigma(w) - \lambda^2 w\,.
$$

The boundary conditions (1.5) read

$$
(1.10) \qquad
\begin{aligned}
& w_y(y), z_y(y) \to 0 && \text{when } |y| \to \infty\,, \\
& w(y) \to w_-\,, \quad z(y) \to 0 && \text{when } y \to -\infty\,, \\
& w(y) \to w_+\,, \quad z(y) \to 0 && \text{when } y \to +\infty\,,
\end{aligned}
$$

Concerning the problem (1.9)-(1.10), recall that the existence of traveling wave solutions was established first by Schulze and Shearer [12] in the case of the cubic stress-function

$$
(1.11) \qquad \sigma(w) = w(w^2 + 1) \qquad \text{for all } w
$$

and for constant viscosity and capillarity $a_1(w) := a_2(w) := b(w) := 1$ for all w. Importantly, it was observed therein that some of the trajectories of the system do not satisfy the standard Lax and Liu entropy conditions, but instead are *nonclassical undercompressive waves.* By definition, the propagating discontinuities associated with such traveling waves have fewer incoming characteristics than observed with classical compressive waves (generated by zero viscosity limits).

Our objective in this paper is to establish general existence and uniqueness results concerning the traveling wave solutions of the problem (1.9)-(1.10) for viscosity and capillarity functions and stress-functions satisfying solely (1.2)-(1.3). We are also interested in investigating their asymptotic behavior in the viscosity dominant and capillarity dominant limits.

We recall that nonclassical undercompressive shock waves (for hyperbolic model such as (1.1)-(1.3)) as well as phase transitions (for hyperbolic-elliptic equations) have drawn a lot of attention. For references from material science, see for instance [13, 14, 15, 1, 2, 16, 9, 8, 6, 7, 10, 12, 4]. In particular, the importance of the so-called *kinetic relation* in characterizing propagating phase transitions was recognized by Abeyaratne and Knowles [1, 2] and Truskinovsky [15, 16]. The first mathematical formulation of the kinetic relation is due to LeFloch [9]. Hayes and LeFloch [6, 7] have extended the concept of kinetic relation to strictly hyperbolic systems of conservation laws such as (1.1)-(1.3).

The goal of the present paper is establishing the existence, and studying the properties, of the kinetic function associated with the model (1.1)-(1.3). The results differ in significant respects from those obtained first for scalar conservation laws [3] and for an hyperbolic-elliptic model of phase transitions [4]. Precisely, our main results are the following ones:

(1) The existence of *nonclassical traveling waves* and of the *kinetic function* are established by relying on the techniques developed in the first part [3].

(2) The existence and properties of the *classical shocks* are also established.

(3) The *asymptotic behaviors* of the kinetic function both in the neighborhood of zero and in the large are determined, as well as its limits when $\beta \to 0$ or $\alpha \to 0$.

(4) These results provide a precise description of the shock curve generated by the hyperbolic model (1.1)-(1.3).

Furthermore, some examples will be studied for which some important parameters or functions can be determined explicitly.

We make here the additional remarks:

(1) It is easier to determine first the «inverse» of the kinetic function. The issue of inverting this function in order to recover the kinetic function is delicate and is discussed below.

(2) Our analysis demonstrates that *there exists a threshold value for the shock strength,* $|w_+ - w_-|$, *below which the traveling wave is always classical.*

Interestingly, the property (2) above was also satisfied in the case of a cubic flux studied by Schulze and Shearer [12]. Note also that the case when only viscosity is taken into account is covered by the analysis in the present paper, simply by setting $\alpha = 0$. The case $\beta = 0$ correspond to a singular limit for which we refer to [3, 4].

An outline of this paper follows. In Section 2, we state some preliminary results. In Section 3, we show that the results in [3] apply and yield an *inverse kinetic function* which is globally defined on $\mathbb{R}$. Section 4 is devoted to defining the kinetic function from its inverse. Finally in Section 5, we treat a few examples for which explicit formulas are available.

2. – Notations.

By definition, an *equilibrium point* for the system (1.9) is a pair (w, z) for which the vector field in the right-hand side of (1.9) vanishes. Clearly $z = 0$ for such a point and we can focus on the component w. In view of (1.2), a left-hand state w_0 and a speed λ being fixed, there is at most three equilibria w (including w_0, itself) satisfying

$$g(w, \lambda) = g(w_0, \lambda). \tag{2.1}$$

Assuming for definiteness that

$$w_0 > 0$$

and that the speed remains in the range where three equilibria exist (see below for the precise conditions), we denote them by w_2, w_1, and w_0 with the convention that

$$-1 < w_2 \leq w_1 \leq w_0. \tag{2.2}$$

We want to study the system (1.9)-(1.10) for a fixed left-hand state w_0, by using the speed λ or the right-hand states $w_+ = w_1$ or $w_+ = w_2$ as parameters. Throughout this paper, for definiteness, we focus attention on waves propagating to the left, that is

$$\lambda < 0.$$

We will need some notation concerning the graph of the function σ. In view of (1.3), for any $w \neq 0$ there exists a unique line passing through the point of the graph with coordinate w and being tangent to the graph at some other point, whose coordinate is denoted by $\varphi^\natural(w) \neq w$. In other words we have

$$\sigma'(\varphi^\natural(w)) = \frac{\sigma(w) - \sigma(\varphi^\natural(w))}{w - \varphi^\natural(w)} \quad \text{for all } w \neq 0,\ w > -1. \tag{2.3}$$

Note that $w\varphi^\natural(w) < 0$ and, by continuity, $\varphi^\natural(0) = 0$. Thanks to (1.3), the map $\varphi^\natural : (-1, \infty) \to (-1, \infty)$ is monotone decreasing and onto, and so is invertible. Its inverse function, denoted by $\varphi^{-\natural}$, satisfies

$$\sigma'(w) = \frac{\sigma(w) - \sigma(\varphi^{-\natural}(w))}{w - \varphi^{-\natural}(w)} \quad \text{for all } w \neq 0,\ w > -1. \tag{2.4}$$

For each $w_0 > -1$ we also set

$$\lambda^{\natural}(w_0) = -\sqrt{\sigma'(\varphi^{\natural}(w_0))}, \qquad \lambda^{-\natural}(w_0) = -\sqrt{\sigma'(w_0)},$$

which, for each fixed w_0, determine an interval comprising all shock speeds λ in (1.7).

Setting $a(w) = \dfrac{a_2(w)}{a_1(w)}$, we can define the function

$$\text{(2.5}a\text{)} \quad H(w_0, w) = \int_{w_0}^{w} (g(s, \lambda) - g(w_0, \lambda))\, a(s)\, ds\,, \qquad w, w_0 \in (-1, \infty)$$

with λ given by

$$\text{(2.5}b\text{)} \qquad \lambda = \lambda(w_0, w) := \begin{cases} -\sqrt{\dfrac{\sigma(w) - \sigma(w_0)}{w - w_0}} & \text{if } w \neq w_0, \\ -\sqrt{\sigma'(w_0)} & \text{if } w = w_0. \end{cases}$$

It is not difficult to check that:

LEMMA 2.1. *If there exists a traveling wave solution of* (1.9)-(1.10) *connecting* $w_- = w_0$ *to some* $w_+ = w$, *then necessarily*

$$H(w_0, w) \leq H(w_0, w_0) = 0\,,$$

where the inequality is strict if $w \neq w_0$ *and* $\beta > 0$.

LEMMA 2.2. *There exists a function* $\varphi^{\flat}_{\infty} : (-1, \infty) \to (-1, \infty)$, *strictly monotone decreasing and onto such that for all* $w_0 \neq 0$

$$\operatorname{sgn}(w_0)\varphi^{-\natural}(w_0) \leq \operatorname{sgn}(w_0)\, \varphi^{\flat}_{\infty}(w_0) < \operatorname{sgn}(w_0)\, \varphi^{\natural}(w_0)$$

and

$$H(w_0, w) = 0 \;\; \text{and} \;\; w \neq w_0 \qquad \text{iff} \qquad w = \varphi^{\flat}_{\infty}(w_0)\,.$$

Moreover, for all $w_0 \neq 0$ *and all* w, *we have*

$$H(w_0, w) < 0 \quad \text{iff} \quad \operatorname{sgn}(w_0)\, w < \operatorname{sgn}(w_0)\, \varphi^{\flat}_{\infty}(w_0) \;\; \text{or} \;\; \operatorname{sgn}(w_0)\, w \geq \operatorname{sgn}(w_0)\, w_0\,.$$

Geometrically, the function $\varphi^{\flat}_{\infty}$ corresponds to the «equal-area» condition (the line connecting w_0 to w cuts the graph of σ in two equal areas). It is also associated to the *maximal negative* entropy dissipation. Combining the above two lemmas, we deduce for instance that, if there exists a traveling wave connecting $w_0 > 0$ (for definiteness) to w, then

$$w \leqslant \varphi^{\flat}_{\infty}(w_0) \text{ or } w \geqslant w_0.$$

Among these traveling waves, some correspond to *classical shock waves* which satisfy the standard Liu entropy condition, that is for which the line connecting w_0 to w does not cut the graph of σ (except at the end points):

$$w \leqslant \varphi^{-\natural}(w_0) \text{ or } w \geqslant w_0.$$

Then, by definition, *nonclassical trajectories* satisfy

$$w \in [\varphi^{-\natural}(w_0), \varphi^{\flat}_{\infty}(w_0)].$$

Based on the function $\varphi^{\flat}_{\infty}$ we also define a unique function $\varphi^{\sharp}_{\infty}$ by the two conditions (for $w_0 \neq 0$) $\varphi^{\sharp}_{\infty}(w_0) \neq \varphi^{\flat}_{\infty}(w_0)$ and

$$\frac{\sigma(w_0) - \sigma(\varphi^{\sharp}_{\infty}(w_0))}{w_0 - \varphi^{\sharp}_{\infty}(w_0)} = \frac{\sigma(w_0) - \sigma(\varphi^{\flat}_{\infty}(w_0))}{w_0 - \varphi^{\flat}_{\infty}(w_0)}.$$

Let us also set $\lambda_{\infty}(0) = 0$ and, for $w_0 \neq 0$,

$$\lambda_{\infty}(w_0) = -\sqrt{\frac{\sigma(w_0) - \sigma(\varphi^{\flat}_{\infty}(w_0))}{w_0 - \varphi^{\flat}_{\infty}(w_0)}},$$

which is the *maximal admissible speed* for the range of right-hand states w comprised between $\varphi^{\sharp}_{\infty}(w_0)$ and $\varphi^{\natural}(w_0)$, at least. Recall that $\lambda^{\natural}(w_0)$ is a lower bound for the speeds.

Modulo some trivial rescaling, the traveling trajectories depend only upon the ratio

$$\delta := \sqrt{\alpha}/\beta \in [0, \infty].$$

To state our result, for each left-hand state w_0 we define the *1-shock set* generated by the equation (1.9)-(1.10) by

$$S^1_{\delta}(w_0) :=$$

$$:= \{w/\text{there is atraveling wave satisfying (1.9)-(1.10) with } w_- = w_0 \text{ and } w_+ = w\}.$$

When searching for traveling wave solutions, one first identifies all of the (nonclassical) states w_2 (see (2.1)-(2.2)).

3. - Inverse kinetic function.

In this section, we strongly rely on the following observation. The system (1.9) reduces to the (scalar) model studied in [3], provided the following transformation is applied in which the index s refer to the «scalar case«:

$$\begin{aligned} &\lambda = -\sqrt{\lambda_s}, \quad \lambda\beta = -\beta_s, \\ &w_\pm = -w_{s\mp}, \quad \sigma(w) = -\sigma_s(-w_s), \\ &a_i(w) = a_{i,s}(-w_s),\ i = 1, 2, \quad b(w) = b_s(-w_s). \end{aligned} \tag{3.1}$$

Indeed, using (3.1), the equation (1.8) transforms into

$$\sigma_s(w_s) - \sigma_s(w_{s+}) - \lambda_s(w_s - w_{s+}) = \beta_s b_s(w_s)(w_s)_y + \alpha a_{1,s}(w_s)(a_{2,s}(w_s)(w_s)_y)_y. \tag{3.2}$$

The existence and properties of traveling wave solutions for the equation were investigated in [3]. One important difference however is that the description in [3] yields the kinetic function «after applying (3.1)» as a function of the right-hand state, rather than of the left-hand state. We will call this function the *inverse kinetic function*. The later is more natural when one is concerned with solving the Riemann problem for the hyperbolic system associated with (1.1). Another source of difficulty in applying [3] is the fact that the diffusion parameter β_s in (3.2) is actually proportional to the shock speed ($\beta_s = -\lambda\beta$). Furthermore, without loss of generality and by a straightforward rescaling of the traveling wave, we can assume that

$$\alpha = 1\,. \tag{3.3}$$

Define first the function

$$G(w; w_0, \lambda) := \int_{w_0}^{w} (g(s, \lambda) - g(w_0, \lambda))\, a(s)\, ds\,. \tag{3.4}$$

Observe that $\partial_w G(w; w_0, \lambda) = 0$ iff (2.1) holds, i.e. w is an equilibrium point. Recall that to each w_0 and speed λ (in some interval) we associate (see (2.1)-(2.2)) two other equilibria w_1 and w_2.

LEMMA 3.1. *Fixing $w_0 > 0$ and $\lambda < 0$ in the interval*

$$\lambda \in (\lambda^{-\natural}(w_0), \lambda^{\natural}(w_0)),$$

the function $\tilde{G}(w) := G(w, w_0, \lambda)$ satisfies

$$\begin{aligned} &\tilde{G}'(w) < 0 \quad \textit{for all } w < w_2 \textit{ or } w \in (w_1, w_0), \\ &\tilde{G}'(w) > 0 \quad \textit{for all } w \in (w_2, w_1) \textit{ or } w > w_0. \end{aligned} \tag{3.5}$$

Moreover, if $\lambda \in (\lambda_\infty(w_0), \lambda^\natural(w_0))$, *we have*

$$\tilde{G}(w_0) = 0 < \tilde{G}(w_2) < \tilde{G}(w_1).$$

If $\lambda = \lambda_\infty(w_0)$ *then*

$$\tilde{G}(w_0) = \tilde{G}(w_2) = 0 < \tilde{G}(w_1).$$

If $\lambda \in (\lambda^{-\natural}(w_0), \lambda_\infty(w_0))$, *then*

$$\tilde{G}(w_2) < 0 = \tilde{G}(w_0) < \tilde{G}(w_1).$$

Observe that the functions G and H are related in the following way:

$$G(w; w_0, \lambda) = H(w_0, w) \quad \text{iff} \quad \lambda = \lambda(w_0, w). \tag{3.6}$$

In view of Lemma 2.1, one must have $H(w_0, w) \leq 0$ for the existence of a traveling wave connecting w_0 to w. Thus, from Lemma 3.1 we conclude that

(3.7) if there exists a trajectory connecting w_0 to w_2 then $\lambda \in [\lambda^{-\natural}(w_0), \lambda_\infty(w_0)]$.

Fix a propagation speed $\lambda < 0$ and a left-hand state $w_0 > 0$, and search for trajectories connecting w_0 to the associated equilibrium w_2 introduced in Section 2. According to our earlier discussion, we necessarily have

$$w_2 \in [\varphi^{-\natural}(w_0), \varphi^\flat_\infty(w_0)], \qquad \lambda \in [\lambda^{-\natural}(w_0), \lambda_\infty(w_0)], \tag{3.8}$$

conditions are assumed throughout this section.

The eigenvalues of the system (1.9) are found to be

$$\mu = \frac{1}{2a_1(w)a_2(w)}\left(-\beta\lambda b(w) \pm \sqrt{\beta^2\lambda^2 b(w)^2 + 4a_1(w)a_2(w)(\sigma'(w) - \lambda^2)}\right).$$

Specifically (for $\beta \neq 0$) we set

$$\begin{aligned} \underline{\mu}(w, \lambda, \beta) &= \frac{\beta\lambda b(w)}{2a_1(w)a_2(w)}\left(-1 + \sqrt{1 + \frac{4a_1(w)\,a_2(w)(\sigma'(w) - \lambda^2)}{\beta^2\lambda^2 b(w)^2}}\right), \\ \overline{\mu}(w, \lambda, \beta) &= \frac{\beta\lambda b(w)}{2a_1(w)\,a_2(w)}\left(-1 - \sqrt{1 + \frac{4a_1(w)\,a_2(w)(\sigma'(w) - \lambda^2)}{\beta^2\lambda^2 b(w)^2}}\right). \end{aligned} \tag{3.9}$$

LEMMA 3.2 (Equilibrium points). *Fix some* w_- *and* λ *and let* w *be an equilibrium point of* (1.9).

If $\sigma'(w) - \lambda^2 > 0$ *then* w *is a saddle point having two real eigenvalues:* $\underline{\mu} < 0 < \overline{\mu}$.

If $\sigma'(w) - \lambda^2 < 0$, then $Re(\underline{\mu})$ and $Re(\overline{\mu})$ are both negative and w is referred to as a stable point. If furthermore $\beta^2\lambda^2 b(w)^2 + 4a_1(w)\, a_2(w)(\sigma'(w) - \lambda^2) > 0$ then w corresponds to a stable node with two real negative eigenvalues $\underline{\mu} < \overline{\mu} < 0$. Otherwise, if $\beta^2\lambda^2 b(w)^2 + 4a_1(w)\, a_2(w)(\sigma'(w) - \lambda^2) > 0$, w is a stable spiral with two complex conjugate eigenvalues with negative real parts.

We first have:

THEOREM 3.3 (Existence of nonclassical trajectories). *Given two states $w_0 > 0$ and $w_2 < 0$ corresponding to a propagation speed λ satisfying*

$$\lambda = -\sqrt{\frac{\sigma(w_2) - \sigma(w_0)}{w_2 - w_0}} \in (\lambda^{-\natural}(w_0),\, \lambda_\infty(w_0)]$$

or equivalently $\lambda \in (\lambda^{\natural}(w_2), \lambda_\infty(w_2)]$, there is a unique diffusion $\beta = \beta(w_0, w_2) \geqslant 0$ such that w_0 can be connected to w_2 by a traveling wave solution of (3.2).

The proof given in the scalar case [3] extends immediately by using the transformation (3.1). This transformation exchanges the role of w_0 and w_2 in [3].

LEMMA 3.4. *Define*

$$\Delta = \{(w_0, w_2) \in \mathbb{R}_+ \times \mathbb{R}_- / w_2 \in (\varphi^{-\natural}(w_0),\, \varphi^{\flat}_{\infty}(w_0)]\}$$

$$= \{(w_0, w_2) \in \mathbb{R}_+ \times \mathbb{R}_- / w_0 \in (\varphi^{\natural}(w_2),\, \varphi^{\flat}_{\infty}(w_2)]\}$$

and consider the function

$$\Delta \ni (w_0, w_2) \mapsto \beta(w_0, w_2)$$

which associates the (unique) value β such that there is a (nonclassical) traveling wave connecting w_0 to w_2 (Theorem 3.3).

Then, for each fixed w_2, $\beta(w_0, w_2)$ is a strictly monotone decreasing function of w_0, mapping the interval $(\varphi^{\natural}(w_2), \varphi^{\flat}_{\infty}(w_2)]$ onto some interval $[0, \beta^{\natural}(w_2))$ where the upper bound $\beta^{\natural}(w_2)$ is finite.

Following the terminology in [3], the value $\beta^{\natural}(w_2)$ is called the *critical diffusion* at w_2: Nonclassical trajectories arriving at w_2 exist only when $\beta \leqslant \beta^{\natural}(w_2)$.

PROOF. The proof is a direct application of the results in [3] and is based on the transformation (3.1). We have

$$\beta_s(w_2, w_0) = -\lambda(w_0, w_2)\,\beta(w_0, w_2). \tag{3.10}$$

Fixing $w_2 < 0$, let w_0 and w_0^* two reals in $(\varphi^{\natural}(w_2), \varphi^{\flat}_{\infty}(w_2)]$ with $w_0 < w_0^*$. Then, the obvious extension of Theorem 3.7 in [3] for values $w_2 < 0$ and $w_0 > 0$ gives us

$$\beta_s(w_2, w_0) > \beta_s(w_2, w_0^*).$$

On the other hand,

$$0 > \lambda > \lambda^*$$

and then, using (3.10), we get

$$\beta(w_0, w_2) > \beta(w_0^*, w_2). \qquad \blacksquare$$

Observe that we do not recover exactly the same properties as the ones in the scalar case to the presence of the shock speed in the transformation

$$\beta(w_0, w_2) = \frac{\beta_s(w_0, w_2)}{|\lambda(w_0, w_2)|}.$$

Observe that both $\beta_s(w_0, w_2)$ and $|\lambda(w_0, w_2)|$ are decreasing functions with respect to w_2, so that no conclusion can be deduced about the monotonicity of $w_2 \to \beta(w_0, w_2)$. So at this stage, we can only define the *inverse kinetic function* $\psi^{\flat} = \psi^{\flat}(w, \beta)$ by inverting the relation

$$w_0 \in (\varphi^{\natural}(w_2), \varphi^{\flat}_{\infty}(w_2)] \to \beta = \beta(w_0, w_2) \in [0, \beta^{\natural}(w_2)).$$

We obtain

$$\beta \in [0, \beta^{\natural}(w_2)) \to w_0 = \psi^{\flat}(w_2, \beta) \in (\varphi^{\natural}(w_2), \varphi^{\flat}_{\infty}(w_2)], \tag{3.11a}$$

extended by continuity, according to the scalar case applied on (3.2), by

$$\beta \in [\beta^{\natural}(w_2), \infty) \to w_0 = \varphi^{\natural}(w_2). \tag{3.11b}$$

Observe that $\psi^{\flat}$ need not be a monotone function of the variable w_2. To obtain the actual kinetic function $\varphi^{\flat}$, we will need to inverse it; see Section 4.

Based on [3], it is immediate to state the following results:

THEOREM 3.5. *(The inverse kinetic function.)*
There exists a Lipschitz continuous inverse kinetic function

$$\psi_\delta^\flat : (-1, \infty) \to (-1, \infty),$$

which satisfies

(3.12) $\operatorname{sgn}(w)\,\varphi_\infty^\flat(w) < \operatorname{sgn}(w)\,\psi_\delta^\flat(w) \leq \operatorname{sgn}(w)\,\varphi^\natural(w)$ *for all* $w \neq 0, w \in (-1, \infty)$.

For each right-hand side $w \in (-1, \infty)$, *there exists a (unique) nonclassical traveling wave connecting the left-hand value* $\psi_\delta^\flat(w)$ *to* w.

The function $\psi_\delta^\sharp$ is defined from $\psi_\delta^\flat$ by

$$\frac{\sigma(w) - \sigma(\psi_\delta^\sharp(w))}{w - \psi_\delta^\sharp(w)} = \frac{\sigma(w) - \sigma(\psi_\delta^\flat(w))}{w - \psi_\delta^\flat(w)} \quad \text{for all } w \neq 0,$$

with the constrain

$$\operatorname{sgn}(w)\,\psi_\delta^\flat(w) \leq \operatorname{sgn}(w)\,\varphi^\natural(w) \leq \operatorname{sgn}(w)\,\psi_\delta^\sharp(w).$$

THEOREM 3.6. *(The backward 1-shock set.)*
Setting

$S_\delta^{1'}(w_2) := \{w/$*There is a traveling wave satisfying* (1.9)-(1.10) *with* $w_- = w$ *and* $w_+ = w_2\}$,

we have

$$(3.13) \qquad S_\delta^{1'}(w_2) = \begin{cases} \{\psi_\delta^\flat(w_2)\} \cup (\psi_\delta^\sharp(w_2), w_2] & \textit{for } w_2 \geq 0, \\ [w_2, \psi_\delta^\sharp(w_2)) \cup \{\psi_\delta^\flat(w_2)\} & \textit{for } w_2 \leq 0. \end{cases}$$

We now discuss the proof of Theorem 3.6 decomposed in two lemmas below. Given $w_2 > 0$ and $\beta > 0$, we study the existence of the (classical and nonclassical) traveling waves of (1.9)-(1.10) connecting $w_- = w_1$ to $w_+ = w_2$. The shock speed λ here lies in the interval $\lambda \in (\lambda^{-\natural}(w_2), \lambda^\natural(w_2))$.

Recall that, by Lemma 3.4 and for $\beta < \beta^\natural(w_2)$, there exists a unique nonclassical traveling wave of (1.9)-(1.10) connecting some point $w_0 = \psi(w_2, \beta)$ to w_2 and associated with a speed $\lambda = \lambda(\beta, w_2)$ or equivalently, with the notation (3.1), $\lambda_s = \lambda_s(\beta_s, w_2)$.

LEMMA 3.7. *For each $\beta < \beta^\natural(w_2)$ and each speed λ satisfying $\lambda^{-\natural}(w_2) < \lambda < \lambda(\beta, w_2)$, there exists a traveling wave connecting $w_- = w_1$ to $w_+ = w_2$.*

For each $\beta \geq \beta^\natural(w_2)$ and each speed $\lambda^{-\natural}(w_2) < \lambda < \lambda^\natural(w_2)$, there exists a traveling wave connecting $w_- = w_1$ to $w_+ = w_2$.

PROOF OF LEMMA 3.7. We first treat the case $\beta < \beta^\natural(w_2)$. The region $\lambda^{-\natural}(w_2) < \lambda < \lambda^\natural(w_2)$ corresponds to the region $\lambda_s(\beta_s, w_2) < \lambda_s < \sigma'_s(w_2)$. On the other hand, the parameter $\beta_s = -\beta\lambda = \beta\sqrt{\lambda_s}$ satisfies $\beta_s > \beta\sqrt{\lambda_s(\beta_s, w_2)} = \beta_s(w_2, \psi(w_2, \beta))$. Thus, in view of the monotonicity of the function $w_0 \mapsto \beta_s(w_2, w_0)$ and Theorem 5.1 in [3] applied to (3.2), there exists a connection from w_1 to w_2.

Now, if $\beta \geq \beta^\natural(w_2)$ then, for all $\lambda_s^\natural(w_2) \leq \lambda_s < \sigma'_s(w_2)$ we have

$$\beta_s = \beta\sqrt{\lambda_s} \geq \beta^\natural(w_2)\sqrt{\lambda_s} \geq \beta^\natural(w_2)\sqrt{\lambda_s^\natural(w_2)} = \beta_s^\natural(w_2).$$

Again, in view of the same theorem in [3], there exists a traveling wave of (1.9)-(1.10) connecting w_1 to w_2. ■

LEMMA 3.8. *If $\lambda(\beta, w_2) < \lambda < \lambda^\natural(w_2)$, then there is no traveling wave connecting $w_- = w_1$ to $w_+ = w_2$.*

PROOF OF LEMMA 3.8. The proof is similar to the one of the previous lemma. Suppose that $\beta < \beta^\natural(w_2)$. Then $\lambda(\beta, w_2) < \lambda < \lambda^\natural(w_2)$ corresponds to the interval $\lambda_s^\natural(w_2) < \lambda_s < \lambda_s(\beta_s, w_2)$, and the parameter $\beta_s = \beta\sqrt{\lambda_s}$ satisfies $\beta_s < \beta\sqrt{\lambda_s(\beta_s, w_2)} = \beta_s(w_2, \psi(w_2, \beta)) < \beta_s(w_2, w_0)$. The proof in the case $w_2 > 0$ is completed by relying on the monotonicity of the function $w_0 \mapsto \beta_s(w_2, w_0)$ and on Theorem 5.2 in [3] applied on (3.2). The case $w_2 < 0$ is handled similarly. ■

4. - Kinetic function and asymptotic properties.

This section is devoted to defining the kinetic function from its inverse and to deriving several important properties of it. The main result is:

THEOREM 4.1 (The kinetic function). *Consider the traveling wave solutions of* (1.9)-(1.10) *under the assumptions* (1.2)-(1.3) *and that the ratio of the dispersion to the diffusion $\delta = \sqrt{a}/\beta$ belongs to the interval $[0, \infty)$. Then there exist some interval $(w_{\min}(\delta), w_{\max}(\delta))$ containing* 0 *and a Lipschitz con-*

tinuous kinetic function

$$\varphi^\flat_\delta : (w_{\min}(\delta),\ w_{\max}(\delta)) \subset (-1,\ \infty) \to (-1,\ \infty),$$

which is strictly monotone decreasing and satisfies

$$\operatorname{sgn}(w_0)\, \varphi^{-\natural}(w_0) \leq \operatorname{sgn}(w_0)\, \varphi^\flat_\delta(w_0) < \operatorname{sgn}(w_0)\, \varphi^\flat_\infty(w_0) \tag{4.1}$$

$$\textit{for all } w_0 \neq 0,\ w_0 \in (w_{\min}(\delta),\ w_{\max}(\delta)).$$

For each $w_0 \in (w_{\min}(\delta), w_{\max}(\delta))$, *there exists a (unique) nonclassical traveling wave connecting* w_0 *on the left-hand side to the right-hand value* $\varphi^\flat_\delta(w_0)$.

In view of Theorem 4.1, a function

$$\varphi^\sharp_\delta : (w_{\min}(\delta),\ w_{\max}(\delta)) \subset (-1,\ \infty) \to (-1,\ \infty),$$

can be uniquely characterized by the conditions

$$\frac{\sigma(w_0) - \sigma(\varphi^\sharp_\delta(w_0))}{w_0 - \varphi^\sharp_\delta(w_0)} = \frac{\sigma(w_0) - \sigma(\varphi^\flat_\delta(w_0))}{w_0 - \varphi^\flat_\delta(w_0)} \tag{4.2a}$$

and

$$\operatorname{sgn}(w_0)\, \varphi^\natural(w_0) \leq \operatorname{sgn}(w_0)\, \varphi^\sharp_\delta(w_0) \leq \operatorname{sgn}(w_0)\, w_0, \tag{4.2b}$$

for all $w_0 \neq 0,\ w_0 \in (w_{\min}(\delta), w_{\max}(\delta))$. Observe that (4.1) and (4.2) imply also

$$\operatorname{sgn}(w_0)\, \varphi^\sharp_\infty(w_0) < \operatorname{sgn}(w_0)\, \varphi^\sharp_\delta(w_0) \leq \operatorname{sgn}(w_0)\, w_0 \qquad \text{if } w_0 \neq 0\,. \tag{4.3}$$

THEOREM 4.2 (The 1-shock set). *Under the same assumptions as in Theorem* 4.1, *we have*

$$S^1_\delta(w_0) = \tag{4.4}$$

$$= \{\varphi^\flat_\delta(w_0)\} \cup \{w \in (w_{\min}(\delta), w_{\max}(\delta)) / \operatorname{sgn}(w)\, w \geq \operatorname{sgn}(w)\, w_0 > \operatorname{sgn}(w)\, \varphi_\delta^{-\sharp}(w)\}\,.$$

Here $\varphi_\delta^{-\flat}$ *is the inverse function of* $\varphi^\flat_\delta$, *but*

$$\varphi_\delta^{-\sharp} = \varphi^\sharp_\delta \circ \varphi_\delta^{-\flat}.$$

THEOREM 4.3 (Asymptotic properties). *Under the same assumptions as in Theorem* 4.1, *the shock speed*

$$\lambda_\delta^\flat(w_0) := \lambda(w_0, \varphi_\delta^\flat(w_0))$$

is strictly monotone decreasing for $w_0 \in (0, w_{\max}(\delta))$ *and strictly monotone increasing for* $w_0 \in (w_{\min}(\delta), 0)$.

There exists a Lipschitz continuous function $\kappa^\natural : (-1, \infty) \to [0, \infty)$, $w_0 \mapsto \kappa^\natural(w_0)$ *such that* $\kappa^\natural(0) = 0$ *and*

$$\text{(4.5)} \qquad \begin{aligned} &\varphi_\delta^\flat(w_0) \to \varphi_\infty^\flat(w_0) \quad \textit{as } \delta \to \infty, \\ &\varphi_\delta^\flat(w_0) = \varphi^{-\natural}(w_0) \quad \textit{provided } \delta\kappa^\natural(w_0) \le 1. \end{aligned}$$

Furthermore, the bounds of the interval $(0, w_{\max}(\delta))$ *satisfy*

$$\text{(4.6)} \qquad w_{\min}(\delta) \to -1, \qquad w_{\max}(\delta) \to +\infty \qquad \textit{when } \delta \to 0.$$

Other asymptotic properties will be stated later in this Section (Theorem 4.4).

The function $\varphi_\delta^\flat$, called the *kinetic function* associated with the model (1.1), completely characterizes the dynamics of the nonclassical shock waves of the underlying hyperbolic conservation law. Observe that the monotonicity conditions derived in Theorem 2.3 ensure the existence and uniqueness of a solution of the corresponding Riemann problem. As will become clear below, this monotonicity property could be violated in the large, and this is the main reason why the kinetic function is only defined on some possibly finite interval $(w_{\min}(\delta), w_{\max}(\delta))$. More precisely, as we have shown in Section 3, the Lipschitz continuous *inverse kinetic function* ψ is well-defined globally:

$$w_2 \in (-1, +\infty) \mapsto \psi(w_2, \beta) = w_0,$$

where w_0 denotes the right-hand state of the nonclassical trajectory if $\beta < \beta^\natural(w_2)$ and, otherwise, ψ is defined by

$$\psi(w_2, \beta) = \varphi^\natural(w_2) \qquad \text{for all } \beta \ge \beta^\natural(w_2).$$

On the other hand, we have set

$$\text{(4.7)} \qquad \beta^\natural(w_2) = \frac{\beta_s^\natural(w_2)}{\sqrt{\sigma'(\varphi^\natural(w_2))}}.$$

In [3] we already derived the following local behavior of the kinetic function:

$$\begin{aligned} &\beta_s^{\natural}(w_2) \sim c\,|w_2| \quad \text{if } \sigma'''(0) > 0\,, \\ &\beta_s^{\natural}(w_2) = o(w_2) \quad \text{if } \sigma'''(0) = 0\,, \end{aligned} \tag{4.8}$$

where c is a constant defined by

$$c = \frac{a_1(0)\, a_2(0)\, \sigma'''(0)}{4\sqrt{2}b(0)}\,.$$

This is equivalent to saying

$$\begin{aligned} &\beta^{\natural}(w_2) \sim c'\,|w_2| \quad \text{if } \sigma'''(0) > 0\,, \\ &\beta^{\natural}(w_2) = o(w_2) \quad \text{if } \sigma'''(0) = 0\,, \end{aligned} \tag{4.9}$$

with the constant $c' = c/\sqrt{\sigma'(0)}$.

We deduce from (4.3) that, for each fixed $\beta > 0$, we have $\beta > \beta^{\natural}(w_2)$ in a neighborhood of $w_2 = 0$ at least. So $\psi(w_2, \beta) = \varphi^{\natural}(w_2)$ for all sufficiently small $|w_2|$. It follows that the inverse kinetic function ψ is strictly monotone decreasing in some interval of the form $(w_{\min}(\delta), w_{\max}(\delta))$ where $w_{\min}(\delta) < 0 <$ $< w_{\max}(\delta)$.

More precisely, since $\psi < \varphi^{\natural}$ for $w_2 > 0$, and $\psi > \varphi^{\natural}$ for $w_2 < 0$, and thanks to the monotonicity property of $\varphi^{\natural}$, we can choose $w_{\min}(\delta)$ and $w_{\max}(\delta)$ such that for all $w_0 \in \psi((w_{\min}(\delta), w_{\max}(\delta)))$, the set $\psi^{-1}(\{w_0\})$ contains exactly one element. The kinetic function $\varphi_\delta^{\flat}$ then is well-defined in the interval $(w_{\min}(\delta), w_{\max}(\delta)) = (\psi(w_{\max}(\delta)), \psi(w_{\min}(\delta))$. The proof of Theorem 4.1 is completed.

Note that, if $\beta < \sup\{\beta^{\natural}(w_2), w_2 \in (-1, +\infty)\}$, we can always choose $w_{\min}(\delta)$ and $w_{\max}(\delta)$ such that the interval $(w_{\min}(\delta), w_{\max}(\delta))$ contains a «truly nonclassical» region in both intervals $(w_{\min}(\delta), 0)$ and $(0, w_{\max}(\delta))$. This means that for some w_2 we indeed have $\psi(w_2) \neq \varphi^{\natural}(w_2)$. Further properties on the asymptotic behavior of $\beta^{\natural}(w_2)$ when $w_2 \to -1$ and $w_2 \to +\infty$ are derived in Theorem 4.4 below.

Finally observe that Theorem 4.2 is just a restatement of Theorem 3.6, obtained by exchanging the roles played by the states w_- and w_+.

This completes the discussion of the proofs of Theorems 4.1 to 4.3.

To conclude this section we establish a *lower bound* for the critical threshold diffusion $\beta^\natural$. This result provides some further information on the kinetic function in the large, not stated in Seetion 3.

Now, setting

$$d(w) = \frac{b(w)}{a_1(w)},$$

we have

THEOREM 4.4. (1) *For all* $w_2 < 0$ *the following lower bound on the critical diffusion holds*

$$\beta^\natural(w_2) \geqslant T(w_2) := \frac{1}{\sqrt{2}} \frac{\sqrt{G(\varphi^\natural(w_2)) - G(w_2))}}{|\lambda^\natural(w_2)|(D(\varphi^\natural(w_2)) - D(w_2))}, \tag{4.10}$$

where $D'(w) = d(w)$.

(2) *In particular, suppose that* $\sigma'(w) \sim Aw^\alpha$ *as* $w \to +\infty$ *and* $d(w) \leqslant d_M$, *and* $a(w) \geqslant a_M$, *where* A, α, a_M, *and* d_M *are positive constants. Then we have*

$$\liminf_{w_2 \to -1} \beta^\natural(w_2) \geqslant \frac{\alpha\sqrt{a_M}}{2d_M(\alpha+1)}.$$

PROOF. We will rely on the equation (1.9) written in the phase plane (w, z):

$$z(w)\frac{dz}{dw}(w) = \tag{4.11}$$

$$= |\lambda^\natural(w_2)|\beta^\natural(w_2)\, d(w)\, z(w) + a(w)(g(w, \lambda^\natural(w_2)) - g(w_2, \lambda^\natural(w_2))).$$

Fix some values $w_2 < 0$ and $\beta = \beta^\natural(w_2)$ and consider the trajectory $z = z(w)$ connecting $w_0 = \varphi^\natural(w_2)$ to w_2. The maximal negative value of the function $w \to$ $\to z(w)$ is achieved at some point $w^* \in (w_2, w_0)$. So we have $z^* := z(w^*) = -$ $- \max_w |z(w)|$. Integrating the equation (4.5) over the interval $[w^*, w_0]$, we get

$$-\frac{z_*^2}{2} + |\lambda^\natural(w_2)|\beta^\natural(w_2)\int_{w_*}^{w_0} |z(w)|\, d(w)\, dw = G(w_0) - G(w_*). \tag{4.12}$$

Since $G(w_0) - G(w_*) \geqslant 0$ we deduce that

$$
(4.13)\qquad
\begin{aligned}
\frac{{z_*}^2}{2} &\leqslant |\lambda^\natural(w_2)|\,|\beta^\natural(w_2)|\,|z_*|\,(D(w_0) - D(w_*)) \\
&\leqslant |\lambda^\natural(w_2)|\,|\beta^\natural(w_2)|\,|z_*|\,(D(w_0) - D(w_2)).
\end{aligned}
$$

In other words, we have the following upper bound for the maximal value z_*:

$$
(4.14)\qquad |z_*| \leqslant 2\,|\lambda^\natural(w_2)|\,|\beta^\natural(w_2)(D(w_0) - D(w_2)).
$$

Next, we integrate (4.11) again, but now on the interval $[w_2, w_0]$:

$$
(4.15)\qquad
\begin{aligned}
0 \leqslant G(w_0) - G(w_2) &= |\lambda^\natural(w_2)|\,|\beta^\natural(w_2)| \int_{w_2}^{w_0} |z(w)|\,d(w)\,dw \\
&\leqslant |\lambda^\natural(w_2)|\,|\beta^\natural(w_2)|\,|z^*|\,(D(w_0) - D(w_2)).
\end{aligned}
$$

Combining (4.14) and (4.15) we conclude that

$$
\beta^\natural(w_2) \geqslant T(w_2),
$$

which establishes the first item of the theorem.

To prove the second claim, we observe that

$$
(4.16)\qquad 2T(w_2)^2 \geqslant \frac{a_M}{\sigma'(w_0)\, d_M^2 (w_0 - w_2)^2} \int_{w_2}^{w_0} (\sigma(w) - \sigma(w_0) - \sigma'(w_0)(w - w_0))\,dw\,.
$$

On the other hand, w_2 and $w_0 = \varphi^\natural(w_0)$ by definition are related by

$$
(4.17)\qquad \sigma(w_2) - \sigma(w_0) - \sigma'(w_0)(w_2 - w_0) = 0\,.
$$

Note that when $w_2 \to -1$, then we also have $w_0 = \varphi^\natural(w_0) \to +\infty$. Clearly, if w_0 were bounded, then (4.17) would imply that $\sigma(w_2)$ remains bounded which is false. Therefore, for sufficiently large w, since $\sigma'(w) \sim Aw^\alpha$ then $\sigma(x) \sim Ax^{\alpha+1}/(\alpha+1)$ and for all w_0 large enough, we deduce from (4.17)

$$
(4.18)\qquad |\sigma(w_2)| \sim \frac{A\alpha}{(\alpha+1)} w_0^{\alpha+1}.
$$

We now estimate the right-hand side of (4.16):

$$\begin{aligned}\int_{w_2}^{w_0}(\sigma(w)-\sigma(w_2)-\sigma'(w_0)(w-w_0))\,dw &\geq \int_0^{w_0}(\sigma(w)-\sigma(w_2)-\sigma'(w_0)(w-w_0))\,dw\\ &\geq \int_0^{w_3}(\sigma(0)-\sigma(w_2)-\sigma'(w_0)(w-w_0))\,dw\\ &\geq \frac{1}{2}\,w_3^2\sigma'(w_0)\end{aligned} \tag{4.19}$$

where w_3 is characterized by

$$\sigma(w_0)-\sigma(0)-\sigma'(w_0)(w_0-w_3)=0\,. \tag{4.20}$$

The latter is equivalent to

$$w_3=w_0-\frac{\sigma(w_0)-\sigma(0)}{\sigma'(w_0)}\,.$$

For w_0 large enough, we have

$$w_3\sim\frac{a}{a+1}\,w_0\,. \tag{4.21}$$

The second claim of the theorem is now a consequence of (4.10), (4.16), (4.19) and (4.21). ■

5. – Examples of kinetic functions.

This section focuses on some polynomial stress-functions. On one hand, in Theorem 5.1, to illustrate our results and based on the work by Schulze and Shearer [12], we describe the properties of the classical and nonclassical trajectories of (1.1) in the case of the cubic flux-function (1.11) (when also $a_1(w) = a_2(w) = b(w) := 1$). Schulze and Shearer's results are reformulated in a convenient form by following the proposed framework in Section 2. For this model example, the kinetic function and the 1-shock set are known explicitly.

On the other hand, in Theorem 5.2, the critical diffusion introduced in our analysis in Section 3 is determined explicitly for a class of stress-functions.

The condition

$$\lim_{w\to -1}\sigma(w) = -\infty$$

no longer holds and, in the rest of this section it is more convenient to consider the problem (1.9)-(1.10) on the whole real line, that is $w \in \mathbb{R}$. The equations (1.9) can be written in the form

$$\alpha w'_{yy} + \beta \lambda w'_y = \sigma(w) - \sigma(w_0) - \lambda^2 (w - w_0) . \tag{5.1}$$

From the definitions in Section 2, it is easily checked that

$$\varphi^{\natural}(w) = -\frac{w}{2}, \quad \varphi^{-\natural}(w) = -2w, \quad \lambda^{\natural}(w) = \frac{3}{4} w^2 + 1, \quad \lambda^{-\natural}(w) = 3w^2 + 1,$$

and

$$\varphi^{\flat}_{\infty}(w) = -w, \qquad \varphi^{\sharp}_{\infty}(w) = 0, \qquad \lambda_{\infty}(w) = w^2 + 1 .$$

For $\lambda < 0$ such that $1 + w_0^2 \leqslant \lambda^2 < 3w_0^2 + 1$ and $w_0 > 0$, there exist exactly two solutions $w_1 < w_2$ (plus w_0) of the cubic equation

$$\lambda^2 = \frac{\sigma(w) - \sigma(w_0)}{w - w_0} = w^2 + w_0 w + w_0^2 + 1 \tag{5.2}$$

given explicitly by

$$w_1 = \frac{1}{2} \left(-w_0 + \sqrt{4\lambda^2 - 3w_0^2 - 4} \right) \tag{5.3a}$$

and

$$w_2 = -\frac{1}{2} \left(w_0 + \sqrt{4\lambda^2 - 3w_0^2 - 4} \right) . \tag{5.3b}$$

Setting $z = u_y$ and using the fact that the right-hand side of (5.1) vanishes exactly at the points w_0, w_1 and w_2, the equation (5.1) is found to be equivalent to

$$w_y = z , \tag{5.4a}$$

$$\alpha z_y = -\lambda \beta z + (w - w_0)(w - w_1)(w - w_2) , \tag{5.4b}$$

with of course

$$\lim_{y \to \pm\infty} w(y) = w_{\pm} , \qquad \lim_{y \to \pm\infty} z(y) = 0 .$$

We are going to derive explicit formulas for the nonclassical shocks and the kinetic function in this case. From the analysis in Section 2, we know that the solutions $w(y)$ of (5.4) connecting w_0 to w_2 are strictly monotone functions of the variable y. Combining the equations (5.4a) and (5.4b), in the phase plan

(w, z) we can write

$$\alpha z(w) \frac{dz}{dw}(w) + \beta\lambda z(w) = (w - w_0)(w - w_1)(w - w_2). \tag{5.5}$$

For simplicity in the notation, we take $\alpha = 1$. Following [12], we search a traveling wave solution of (5.5) in the special (parabolic) form

$$z(w) = k(w - w_0)(w - w_2). \tag{5.6}$$

Plugging (5.6) in (5.5), after some simplification we see that necessarily

$$k = \frac{1}{\sqrt{2}} \tag{5.7a}$$

and

$$\begin{aligned} \beta\lambda &= k(w_0 + w_2) - \frac{w_1}{k} \\ &= -\left(k + \frac{1}{k}\right) w_1 = -\frac{3}{\sqrt{2}} w_1. \end{aligned} \tag{5.7b}$$

Then, from (5.3) and (5.7*b*) we get a relation between the diffusion, the left-hand state, and the shock speed, indeed:

$$\beta = \beta(w_0, \lambda) := \frac{3}{2\sqrt{2}\lambda}\left(w_0 - \sqrt{4\lambda^2 - 3w_0^2 - 4}\right) \tag{5.8}$$

which is defined as long as $\lambda < 0$ and $1 + w_0^2 \leq \lambda^2 < 3w_0^2 + 1$. One can also express λ in term of the left- and right-hand states, that is using (5.2) in (5.8) we have equivalently

$$\beta = \beta(w_0, w_2) := \frac{3}{2\sqrt{2}} \frac{\sqrt{4w_2^2 + 4w_0 w_2 + w_0^2 - 4} - w_0}{\sqrt{w_0^2 + w_0 w_2 + w_2^2 - 1}}. \tag{5.9}$$

The formula (5.9) provides the diffusion for which two given states can be connected by a nonclassical traveling waves.

To describe the kinetic function, we now fix some value of diffusion (recall that the capillarity has been normalized to be one) and we distinguish between two regimes.

First of all, considering the equation (5.8), the range of β, for which a nonclassical traveling wave exists, is determined by letting the speed λ varies in the relevant interval $1 + w_0^2 \leq \lambda^2 < 3w_0^2 + 1$. Precisely we find that

$$0 \leqslant \beta < \frac{3w_0}{\sqrt{2(3w_0^2+1)}} =: \beta^{-\natural}(w_0). \tag{5.10}$$

On the other hand, for β fixed in the above interval we can inverse the relation (2.19) and obtain the quadratic equation

$$(9-2\beta^2)\,\lambda^2 + 3\sqrt{2}\beta w_0 \lambda - 9(w_0^2+1) = 0$$

and so, since $\lambda < 0$,

$$\lambda = \lambda(\beta, w_0) = -\frac{3}{\sqrt{2}}\,\frac{\beta w_0 + \sqrt{(18-3\beta^2)\,w_0^2 + 18 - 4\beta^2}}{9-2\beta^2} \tag{5.11}$$

Using $w_0 + w_1 + w_2 = 0$ and (5.7b) we obtain also

$$w_2 = -w_0 - w_1 = -w_0 + \frac{\sqrt{2}}{3}\beta\lambda\,.$$

Finally the kinetic relation is found to be

$$\varphi^{\flat}(w_0, \beta) = \tag{5.12}$$

$$= \frac{w_0(9-\beta^2) + \beta\sqrt{(18-3\beta^2)\,w_0^2 + 18 - 4\beta^2}}{2\beta^2 - 9} \quad \text{when } \beta \leqslant \frac{3w_0}{\sqrt{2(3w_0^2+1)}}\,.$$

Second, for «large» value of the diffusion parameter, that is if (5.10) does not hold, there is actually a connection between w_0 and $w_2 = w_1 = -w_0/2$ (see [12] and the discussion on classical trajectories in Section 4). In this case, the kinetic function is trivial:

$$\varphi^{\flat}(w_0, \beta) = -2w_0 \quad \text{when } \beta \geqslant \frac{3w_0}{\sqrt{2(3w_0^2+1)}}\,. \tag{5.13}$$

Observe that the kinetic function (5.12)-(5.13) is defined on the whole real line and is globally monotone decreasing in w_0, for each β. Furthermore, since $\alpha = 1$ we have $\beta = 1/\delta$ and we can check directly that

$$\varphi^{\flat}_{\delta}(w_0) = \varphi^{\flat}(w_0, \beta) \rightarrow \begin{cases} \varphi^{-\natural}(w_0) = -2w_0 & \text{when } \beta = \dfrac{1}{\delta} \rightarrow \infty\,, \\[2ex] \varphi^{\flat}_{\infty}(w_0) = -w_0 & \text{when } \beta = \dfrac{1}{\delta} \rightarrow 0\,. \end{cases} \tag{5.14}$$

As in Section 2, we denote by $\varphi_\delta^{-\flat}$ the inverse function of $\varphi_\delta^\flat$, and

$$\varphi_\delta^{-\natural} = \varphi_\delta^\natural \circ \varphi_\delta^{-\flat}.$$

Define also

$$\chi = (\varphi_\delta^{-\natural})^{-1} = \varphi_\delta^\flat \circ (\varphi^\natural{}_\delta)^{-1}.$$

On the other hand, since $\varphi_\delta^\natural(w_0) = -w_0 - \varphi^\flat(w_0, \beta)$, and using (5.12) and (5.13) we have

$$\varphi_\delta^\natural(w_0) = \begin{cases} \dfrac{\beta^2 w_0 + \beta\sqrt{(18-3\beta^2)\, w_0^2 + 18 - 4\beta^2}}{9-2\beta^2} & \text{when } \beta \leqslant \dfrac{3w_0}{\sqrt{2(3w_0^2+1)}}, \\[2ex] w_0 & \text{when } \beta \geqslant \dfrac{3w_0}{\sqrt{2(3w_0^2+1)}}. \end{cases}$$

Now, inverting the previous relation we obtain

$$(\varphi_\delta^\natural)^{-1}(w_0) = \begin{cases} -\dfrac{w_0}{2} + \dfrac{1}{2\beta}\sqrt{(18-3\beta^2)\, w_0^2 - 4\beta^2} & \text{when } \beta \leqslant \dfrac{3w_0}{\sqrt{2(3w_0^2+1)}}, \\[2ex] w_0 & \text{when } \beta \geqslant \dfrac{3w_0}{\sqrt{2(3w_0^2+1)}}. \end{cases}$$

Finally, using that $\chi(w_0) + w_0 + (\varphi_\delta^\natural)^{-1}(w_0) = 0$ we obtain

$$\chi(w_0) = \begin{cases} -\dfrac{w_0}{2} - \dfrac{1}{2\beta}\sqrt{(18-3\beta^2)\, w_0^2 - 4\beta^2} & \text{when } \beta \leqslant \dfrac{3w_0}{\sqrt{2(3w_0^2+1)}}, \\[2ex] -2w_0 & \text{when } \beta \geqslant \dfrac{3w_0}{\sqrt{2(3w_0^2+1)}}. \end{cases}$$

THEOREM 5.1. *The* 1-*shock curve issuing from the point* w_0 *is given by*

$$S_\delta^1(w_0) = \begin{cases} (-\infty, \chi(w_0)) \cup \{\varphi_\delta^\flat(w_0)\} \cup [w_0, +\infty) & \text{if } w_0 > 0, \\ (-\infty, w_0] \cup \{\varphi_\delta^\flat(w_0)\} \cup (\chi(w_0), +\infty) & \text{if } w_0 < 0, \end{cases} \tag{5.15}$$

PROOF. We restrict attention again to the case $w_0 > 0$, since the case $w_0 < 0$ is entirely similar. Any right-hand state $w > 0$ is associated with the nonclassical left-hand state $\widehat{w} := \varphi_\delta^{-\flat}(w) < 0$. But (5.7*b*) reads in this case

$$\beta\lambda = \frac{3}{\sqrt{2}}\varphi_\delta^{\natural}(\widehat{w}) = \frac{3}{\sqrt{2}}\varphi_\delta^{-\natural}(w),$$

which implies $\varphi_\delta^{-\natural}(w) < 0$ and then $\varphi_\delta^{-\natural}(w) < w_0$.

Now, if $w \leqslant 0$ the proof is not so direct, since $\widehat{w} > 0$. First, we note that, for all $\beta > 0$ fixed, there exists a unique $\lambda' < 0$ such that

$$\beta\lambda' = -\frac{3}{\sqrt{2}}w_0. \tag{5.16}$$

Consider first the case that $\beta < \frac{3}{\sqrt{2}}\frac{w_0}{\sqrt{3w_0^2+1}}$. Then λ' must satisfy

$$\lambda'^2 > \sigma'(w_0) = 1 + 3w_0^2. \tag{5.17}$$

We deduce that there exists a unique $w_0' > 0$ such that there exists a non-classical connection from w_0' to $w_2' < 0$ with $w_1' = \varphi_\delta^{\natural}(w_0') = w_0$.

Now, if $\beta \geqslant \frac{3}{\sqrt{2}}\frac{w_0}{\sqrt{3w_0^2+1}}$, there is no $\lambda' < 0$ satisfying (5.16) and (5.17). But, setting $w_0' = w_0$ and $w_2' = \varphi_\delta^{\flat}(w_0)$, we can write $w_1' = \varphi_\delta^{\natural}(w_0') = \varphi_\delta^{-\natural}(w_2') = w_0' = w_0$.

Finally, in all of the cases, we have

$$\varphi_\delta^{-\natural}(w) = -w - \varphi_\delta^{-\flat}(w) = -w - \widehat{w}. \tag{5.18}$$

Now, by differentiating the above equation, we obtain

$$\varphi_\delta^{-\natural\prime}(w) = -1 - \frac{1}{\varphi_\delta^{\flat\prime}(\widehat{w})} = \frac{-1}{\varphi_\delta^{\flat\prime}(\widehat{w})}(1 + \varphi_\delta^{\flat\prime}(\widehat{w})). \tag{5.19}$$

By a straightforward calculation based on (5.12) and (5.13), we obtain that, for all $\beta > 0$,

$$\varphi_\delta^{\flat\prime}(\widehat{w}) + 1 = \tag{5.20}$$

$$= \begin{cases} -1 & \text{if } \beta \geqslant \dfrac{3}{\sqrt{2}}\dfrac{\widehat{w}}{\sqrt{3\,\widehat{w}^2+1}}, \\[2ex] \dfrac{\beta^2}{2\beta^2-9} + \dfrac{\beta\widehat{w}(18-3\beta^2)}{(2\beta^2-9)\sqrt{(18-3\beta^2)\,\widehat{w}^2+18-4\beta^2}} & \text{if } \beta < \dfrac{3}{\sqrt{2}}\dfrac{\widehat{w}}{\sqrt{3\,\widehat{w}^2+1}}. \end{cases}$$

Then, it is clear that, in all of the cases,

$$\varphi_\delta^{\flat\prime} + 1 < 0, \tag{5.21}$$

which also implies

$$\varphi_\delta^{-\natural\prime}(w) < 0\,. \tag{5.22}$$

Finally, thanks to the last property, we deduce that $\varphi_\delta^{-\natural}(w) > w_0 = \varphi_\delta^{-\natural}(w_2')$ iff $w < w_2' = \chi(w_0)$. ■

Case of the functions in the form: $\sigma_{k,\alpha}(w) = \operatorname{sgn}(w)\,|w|^\alpha + kw$, $\alpha > 1$ and $k \geq 0$. Functions a and b are taken in the simple form $a(w) = b(w) = 1$ This sort of functions defined on the whole real line seems to be very interesting, since

$$\sigma_{0,\alpha}(tw) = t^\alpha \sigma_{0,\alpha}(w)\,, \qquad t > 0 \tag{5.23}$$

This property leads us to the main result

THEOREM 5.2. *The critical diffusion $\beta^\natural_{k,\alpha}(w_2)$ is given by*

$$\beta^\natural_{k,\alpha}(w_2) = C_\alpha \frac{|w_0|^{\frac{(\alpha-1)}{2}}}{\sqrt{\alpha|w_0|^{\alpha-1} + k}}\,, \tag{5.24}$$

where $w_0 = \varphi^\natural(w_2)$ and C_α is a positive constant that only depends on α.

Therefore, for any $\beta > C_\alpha$, the model does not admit nonclassical traveling waves.

PROOF. Given the function $\sigma_{k,\alpha}$ and $w_2 < 0$ (the case $w_2 > 0$ being treated similarly), one can easily see that $w_0 = \varphi^\natural(w_2)$ is independent of $k \geq 0$. So, we can rewrite the equation (1.8) in the form

$$-\sigma'_{k,\alpha}(w_0)(w - w_0) + \sigma_{k,\alpha}(w) - \sigma_{k,\alpha}(w_0) = \lambda^\natural_{k,\alpha}(w_2)\,\beta^\natural_{k,\alpha}(w_2)\,w_y + \alpha w_{yy}. \tag{5.25}$$

Fixing $\alpha > 1$, we set

$$\beta^\natural_{s,k} = -|\lambda^\natural_{k,\alpha}(w_2)|\,\beta^\natural_{k,\alpha}(w_2)\,.$$

We can rewrite (5.25) in the form

$$-\sigma'_{k,\alpha}(w_0)(w - w_0) + \sigma_{k,\alpha}(w) - \sigma_{k,\alpha}(w_0) = -\beta^\natural_{s,k} w_y + \alpha w_{yy}. \tag{5.26}$$

Now, since for $w \in \mathbb{R}$

$$-\sigma'_{k,\alpha}(w_0)(w - w_0) + \sigma_{k,\alpha}(w) - \sigma_{k,\alpha}(w_0) =$$

$$= -\sigma'_{0,\alpha}(w_0)(w - w_0) + \sigma_{0,\alpha}(w) - \sigma_{0,\alpha}(w_0)\,,$$

we obtain that

$$\beta^{\natural}_{s,k} = \beta^{\natural}_{s,0}. \tag{5.27}$$

Consider now the transformation

$$w = \tilde{w} w_0.$$

By setting $\sigma = \sigma_{0,\alpha}$, we have

$$\sigma(w) = w_0^{\alpha} \sigma(\tilde{w}).$$

The equation (5.26) becomes

$$-\sigma'(1)(\tilde{w} - 1) + \sigma(\tilde{w}) - \sigma(1) = w_0^{1-\alpha}(-\beta^{\natural}_{s,k}\tilde{w}_y + \alpha \tilde{w}_{yy}). \tag{5.28}$$

By the transformation $y \to y w_0^{(\alpha-1)/2}$, the last equation becomes

$$-\sigma'(1)(\tilde{w} - 1) + \sigma(\tilde{w}) - \sigma(1) = -\beta^{\natural}_{s,k} w_0^{(1-\alpha)/2}\tilde{w}_y + \alpha \tilde{w}_{yy}. \tag{5.29}$$

On the other hand, the parameter $\tilde{w}_2 = \frac{w_2}{w_0}$ is a negative constant independent of w_2 with $w_0 = \varphi^{\natural}(w_2)$, and is the unique negative solution of

$$\frac{|x|^{\alpha} - 1}{|x| + 1} = \alpha\,. \tag{5.30}$$

We deduce that

$$\beta^{\natural}_{s,k} = C_{\alpha} w_0^{(\alpha-1)/2} \tag{5.31}$$

where, $C_{\alpha} > 0$ is a constant independent of w_0. Finally, using (5.27), (5.31) and the expression of $\sigma'_{k,\alpha}(w_0)$ we obtain (5.24).

REFERENCES

[1] R. ABEYARATNE - J. K. KNOWLES, *Kinetic relations and the propagation of phase boundaries in solids*, Arch. Rational Mech. Anal., **114** (1991), pp. 119-154.

[2] R. ABEYARATNE - J. K. KNOWLES, *Implications of viscosity and strain gradient effects for the kinetics of propagating phase boundaries in solids*, SIAM J. Appl. Math., **51** (1991), pp. 1205-1221.

[3] N. BEDJAOUI - P. G. LEFLOCH, *Diffusive-dispersive traveling waves and kinetic relations I. Nonconvex hyperbolic conservation laws*, J. Differential Equations, to appear.

[4] N. BEDJAOUI - P. G. LEFLOCH, *Diffusive-dispersive traveling waves and kinetic relations II. An hyperbolic-elliptic model of phase transitions*, Proc. Roy. Soc. Edinburgh, to appear.

[5] J. GUCKENHEIMER - P. HOLMES, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Applied Math. Sc. 42, Springer Verlag, New York, 1983.

[6] B. T. HAYES - P. G. LEFLOCH, *Nonclassical shocks and kinetic relations: Scalar conservation laws*, Arch. Rational Mech. Anal., **139** (1997), pp. 1-56.

[7] B. T. HAYES - P. G. LEFLOCH, *Nonclassical shocks and kinetic relations: Strictly hyperbolic systems*, SIAM J. Math. Anal., **31** (2000), pp. 941-991.

[8] D. JACOBS - W. R. MCKINNEY - M. SHEARER, *Traveling wave solutions of the modified Korteweg-deVries Burgers equation*, J. Differential Equations, **116** (1995), pp. 448-467.

[9] P. G. LEFLOCH, *Propagating phase boundaries: Formulation of the problem and existence via the Glimm scheme*, Arch. Rational Mech. Anal., **123** (1993), pp. 153-197.

[10] P. G. LEFLOCH, *An introduction to nonclassical shocks of systems of conservation laws, Proceedings of the «International School on Theory and Numerics for Conservation Laws»*, Freiburg/Littenweiler (Germany), 20-24 October 1997, ed. D. Kröner, M. Ohlberger and C. Rohde, Lecture Notes in Computational Science and Engineering, (1998), pp. 28-72.

[11] P. G. LEFLOCH, *Hyperbolic Systems of Conservation Laws: The theory of classical and nonclassical shock waves*, ETH Lecture Notes series, Birkhäuser, to appear.

[12] S. SCHULZE - M. SHEARER, *Undercompressive shocks for a system of hyperbolic conservation laws with cubic nonlinearity*, J. Math. Anal. Appl., **229** (1999), pp. 344-362.

[13] M. SLEMROD, *Admissibility criteria for propagating phase boundaries in a van der Waals fluid*, Arch. Rational Mech. Anal., 81 (1983), pp. 301-315.

[14] M. SLEMROD, *Dynamic phase transitions in a van der Waals fluid*, J. Differential Equations, 52 (1984), pp. 1-23.

[15] L. TRUSKINOVSKY, *Dynamics of non-equilibrium phase boundaries in a heat conducting nonlinear elastic medium*, J. Appl. Math. and Mech. (PMM), 51 (1987), pp. 777-784.

[16] L. TRUSKINOVSKY, *Kinks versus shocks, in «Shock induced transitions and phase structures in general media»*, R. Fosdick, E. Dunn, and H. Slemrod ed., IMA Vol. Math. Appl. 52, Springer-Verlag (1993).

Pervenuto in Redazione l'8 novembre 2000.
In forma definitiva il 26 gennaio 2001.