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Orthogonal and spin bundles over hyperelliptic curves

By

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1. Introduction

Let $\Sigma x_i^2 = 0$, $\Sigma \lambda_i x_i^2 = 0$ be two quadrics in the projective space of dimension $(2g+1)$. Assume that λ_i's are distinct so that these two quadrics determine a *generic* pencil of quadrics. One can associate to this pencil, in a natural way, a hyperelliptic curve X of genus g, namely, the double of the projective line parametrising the pencil with ramification at the points corresponding to the singular quadrics. One can easily check that the variety M_d of d dimensional vector subspaces which are isotropic for both the quadrics is nonsingular. M_d is nonempty if $d \leq g$ and in [1] we gave an interpretation of M_g and M_{g-1} as the Jacobian and the moduli space of vector bundles of rank 2 and fixed determinant of odd degree over X respectively. One might naturally ask if one could obtain similar interpretation for M_d for other d as well. In particular, does $M_1 =$ the intersection of two quadrics have an interpretation as a moduli-space of some sort of bundles over X? This paper answers, among other things, this question. In fact, we give a solution of a more general question, namely, the question of interpreting M_d modulo $\Pi SO(n_i)$, when the λ_i's are allowed to have multiplicities n_i.

The idea of associating a hyperelliptic curve to a pencil of quadrics is nothing new and goes back, at least to Weil [9], who considered the relationship between the intersection of two quadrics (M_1 in our notation) and the curve X from the point of view of diophantine equations over finite fields. He computed the zeta function of M_1 in terms of that of X and verified the Weil conjectures for this variety. Incidentally he also raised the specific question of the geometric relationship between M_1 and X, which we have investigated here. It must also be mentioned that Gauthier [2] has already noticed the identity of the space M_g and the Jacobian of X, although as far as the present author is aware, no proof has been published till recently [6, 1]. The author is thankful to A. Weil for bringing to his attention these references.

The method of proof is quite simple and is not basically different from that in [1]. However, the greater generality leads to conceptual clarity and we have also made a few technical simplifications. If E is a bundle to which the action of i lifts, the bundle does not nevertheless go down to a bundle on $X/i = \mathbf{P}^1$. The obstruction to this is the lack of descent data at the fixed points of i, namely, the

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Weierstrass points of X. In other words, i may not act as identity on the fibres of E at the Weierstrass points. To rectify this, one makes a modification of E at these points to obtain a new bundle E' which does go down. If one keeps track of the additional structures needed to recover E from this bundle on $\mathbf{P}^1$ obtained by descent from E', one can hope for a classification of the bundles started with.

Another description of the method is as follows. Suppose we wish to classify vector bundles of fixed large degree. Consider pairs (E, η) where E is such a vector bundle and η is a trivialisation of E on a sufficiently large divisor D. Then the restriction $H^0(X, E) \to H^0(D, E)$ is injective and using the trivialisation of E on D, we associate to E, a point of a suitable Grassmannian. Thus one could even *construct* the moduli space of vector bundles by passing to the quotient of the Grassmannian in k^{dn} by $\Pi GL(d)$. The special situation in our case then allows us to describe the image precisely.

2. Clifford bundles

If V is a vector space of finite dimension and Q is a non-degenerate quadratic form on V, then the *Clifford group* $\Gamma(Q)$ is the subgroup of the group $C(Q)^{\times}$ of units in the Clifford algebra $C(Q)$, defined by

$$\Gamma(Q) = \{x \in C(Q)^{\times} : xVx^{-1} \subset V\}.$$

Let $\Gamma^+(Q)$ be the group of even elements in $\Gamma(Q)$. The multiplicative group $k^{\times}$ is clearly contained in $\Gamma^+(Q)$. It is actually the centre of $\Gamma(Q)$ if V is of even dimension. Then the spinor norm gives a surjective homomorphism $Nm : \Gamma(Q) \to k^{\times}$ whose kernel is called the reduced Clifford group and denoted $\Gamma_0(Q)$. Let $\Gamma_0^+(Q) = \Gamma_0(Q) \cap \Gamma^+(Q)$. By definition, we have a natural (vector) representation of all these groups in the orthogonal group of Q. The kernel of this representation of Γ is $k^{\times}$. We first notice

Lemma 2.1. If E is an $O(2n)$-bundle on a scheme X, then the obstruction to its liftability to a $\Gamma_0(2n)$-bundle (resp. $\Gamma(2n)$-bundle) is an element of $H^2(X, \mathbf{Z}/2)$ (resp. $H^2(X, \mathcal{O}_X^{\times})$. In particular, any $O(2n)$ bundle on a projective nonsingular curve can be lifted to a $\Gamma(2n)$-bundle. Similar statements are valid for the odd dimensional case when O is replaced by SO and Γ is replaced by Γ^+.

Proof. This is a trivial consequence of the exact sequences

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/2 & \to & \Gamma_0(2n) & \to & O(2n) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & k^{\times} & \to & \Gamma(2n) & \to & O(2n) & \to & 1, \end{array} \tag{2.1}$$

and the induced exact sequence of sheaves

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{Z}/2 & \to & \underline{\Gamma_0(2n)} & \to & \underline{O(2n)} & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \mathcal{O}_X^{\times} & \to & \underline{\Gamma(2n)} & \to & \underline{O(2n)} & \to & 1, \end{array} \tag{2.2}$$

in view of [7]. In fact, the obstruction to the $\Gamma(2n)$-lifting is actually in the image of $H^2(X, \mathbf{Z}/2)$ in $H^2(X, \mathcal{O}_X^\times)$. Hence our assertion.

If P is a $\Gamma(n)$-bundle and ξ is a line bundle, then using the action of $k^\times$ on $\Gamma(n)$, we get a new $\Gamma(n)$-bundle which we will denote $\xi \circ P$. From the fact that $Nm(\lambda a) = \lambda^2 Nm(A)$ for $\lambda \in k^\times$, $A \in \Gamma(n)$ we obtain

(2.3) $\quad Nm(\xi \circ P) = \xi^2 \otimes NmP.$

Lemma 2.4. If P, P' are $\Gamma(2n)$-bundles on X, giving rise to isomorphic $O(2n)$-bundles, then there exists a line bundle ξ such that $P' \cong \xi \circ P$. If E is an $O(2n)$-bundle and P any lift of E to a $\Gamma(2n)$-bundle, then NmP is of even or odd degree according as E can or cannot be lifted to a $\Gamma_0(2n)$-bundle. Similar results hold for the odd dimensional case with $O(n)$ replaced by $SO(n)$.

Proof. The first assertion follows from the sequence 2.2 and [7]. From this it is clear that (a) the parity of deg (NmP) is independent of the lift P, and (b) if deg NmP is even we might, in view of 2.3, as well assume that NmP is trivial by replacing P by a suitable $\xi \circ P$. If NmP is trivial, then P is a $\Gamma_0(2n)$-bundle as is seen from the sequence

$$1 \to \Gamma_0(2n) \to \Gamma(2n) \xrightarrow{Nm} \mathcal{O}^\times \to 1.$$

Definition 2.5. An $O(2n)$-bundle on X is said to be of *even* or *odd type* according as it is or is not liftable to a $\Gamma_0(2n)$-bundle.

Remark 2.6. An $SO(n)$-structure on an $O(n)$-bundle E with trivial determinant is given by a trivialisation of $\overset{n}{\Lambda}E$, such that its square is the trivialisation of $(\overset{n}{\Lambda}E)^2$ given by the 'discriminant', namely, the lift $\overset{n}{\Lambda}E \to \overset{n}{\Lambda}E^*$ of the given quadratic form $E \to E^*$. Accordingly, even when $\overset{n}{\Lambda}E = \det E$ is only an element of order 2, we could still talk of a *special orthogonal structure*, when we mean an isomorphism $\det E \to \alpha$ such that its square $(\det E)^2 \to \alpha^2 \simeq 1$, is the trivialisation given by the discriminant. Also, if P is a $\Gamma(n)$-bundle, we talk of a *special $\Gamma(n)$-structure* when the associated $O(n)$-bundle E comes provided with a special orthogonal structure. If $\det E$ is trivial, this simply means that P is a $\Gamma(n)^+$-bundle.

3. Orthogonal bundles over hyperelliptic curves

We will make the convention that when we deal with orthogonal bundles of odd rank, we assume that it has trivial determinant. Generally, we will deal with the case of even rank, the odd case being similar.

Let E be an orthogonal bundle over a hyperelliptic curve with a lift $E \approx i^*E$ of the involution i. If P is a lift of E to a Γ-bundle, then we must have $P \approx \alpha \circ i^*P$ for some line bundle α by lemma 2.4. Taking norms on both sides and using 2.3, we get $\alpha^2 \otimes NmP \cong i^*NmP$ and hence $(\alpha \otimes NmP)^2 \approx h^{\deg NmP}$. In particular, α is of degree 0. Let us then assume given an isomorphism

$$P \approx \beta \circ (NmP)^{-1} \circ i^*P, \text{ with } \beta^2 = h^{\deg NmP}.$$

It is clear that β is essentially unique, at least if NmP is fixed. We now recall

Lemma 3.1. There is a one-one correspondence between line-bundles η of fixed even (respectively odd) degree such that $\eta^2 \approx h^{\deg \eta}$, and the set of partitions $W = S \cup T$ of the set of Weierstrass points into subsets of even (resp. odd) cardinality.

Proof. See ([1] lemma 2.1).

If an isomorphism $i^*E \to E$ is given, then at every fixed point of i, namely, a Weierstrass point, we obtain an involution of E_w preserving the quadratic form q_w on E_w. Let E_w^+, E_w^- be the eigenspaces corresponding to the eigenvalues ± 1. Then E_w^+, E_w^- are clearly orthogonal for q_w. In particular, q_w is nondegenerate on both these spaces.

We now seek to prove that for such an $O(2n)$-bundle E, the line bundle β with $i^*P \approx \beta^{-1} \otimes NmP \circ P$ is the unique line bundle with $i^*\beta \approx \beta$ whose associated partition of W (lemma 3.1) is determined by the integers $r_w = \dim E_w^+$. Before we do this, let us normalise the isomorphism $\varphi : P \to \alpha \circ i^*P$ given above. We will of course require that this should induce the given isomorphism $E \to i^*E$. This requirement fixes φ up to an automorphism of the $\Gamma(2n)$-bundle P inducing the identity map on E. This means that φ is determined upto a nonzero scalar factor. We now wish to further specify φ by requiring that it be an involution in the following sense. Firstly, we have an isomorphism $Nm\varphi : NmP \to \alpha^2 \circ i^* NmP$. Writing $\alpha = \beta \otimes NmP$, this gives an isomorphism $\beta^{-2} \to NmP \otimes i^*NmP$. On the other hand $i^*\varphi$ gives an isomorphism $i^*P \to i^*\beta \otimes i^*NmP \circ P$ and hence $\beta \otimes NmP\, i^*P \approx \beta \otimes i^*\beta \otimes NmP \otimes i^*NmP \circ P$ and using $Nm\varphi$ also $\beta \otimes NmP \circ i^*P \approx \beta^{-1} \otimes i^*\beta \circ P$. Composing this with φ we get an isomorphism $P \to \beta^{-1} \otimes i^*\beta \circ P$ which induces identity on E. We require that the involutive lift $\beta \approx i^*\beta$ is so chosen that the above composite is the Identity. We shall call this the *normalised isomorphism*.

Let then $\varphi : P \to \beta \circ NmP \circ i^*P$ be a normalised lift of the given involutive isomorphism $E \to i^*E$.

If we trivialise β_w at $w \in W$, we get a bijection $\varphi_w : P_w \to P_w$ which satisfies $\varphi_w(\xi s) = \varphi_w(s)(Nms)\cdot s$. Since P_w is a set on which $\Gamma(2n)$ acts simply transitively we see that φ_w gives rise to an element g_w of $\Gamma(2n)$ which is determined upto the equivalence relation

$$g \sim g' \text{ if there exists } s \in \Gamma(2n) \text{ with } sgs^{-1}\cdot Nms = g'.$$

Moreover g_w gives rise to a reflection in an r_w-dimensional subspace in the vector representation, and satisfies in view of our normalisation $g_w^2 = \epsilon_w Nmg_w$, where $\epsilon_w = \pm 1$ according as the lift of i to β_w is ± 1 at w. Thus we have the following situation. E_w^+ is an r_w-dimensional subspace of E_w and $g_w \in \Gamma(2n, q_w)$ which represents the reflection in E_w^+ satisfies $g_w^2 = \epsilon_w Nmg_w$.

Lemma 3.2. If $g \in \Gamma(2n)$ represents the reflection in a nondegenerate subspace of dimension r, then $g^2 = Nmg$ if $r \equiv 1$, $2n$ and $g^2 = -Nmg$ if $r \equiv 3$, $2n - 2 \bmod 4$.

Proof. Let $e_1, \ldots, e_{2n}$ be an orthonormal basis for q_w such that $e_1, \ldots, e_r$ is a basis for E_w^+. All the elements e_i belong to $\Gamma(2n)$ and in particular the element $e_1, \ldots e_r$ as well. The induced transformation on E_w is $x \mapsto (e_1, \ldots e_r)\, x\, (e_1 \ldots e_r)^{-1} =$

$e_1 \dots e_r x e_r \dots e_1$ since $e_r^2 = q_w(e_r) = 1$ by choice. If x is e_i, $i > r$, then $x \mapsto (-1)^r x$ and if x is e_i, $i \leq r$, then $x \to (-1)^{r-1} x$. Thus $g = e_1 \dots e_r$ or $e_{r+1} \dots e_{2n}$ $\in \Gamma(2n)$ represents reflection in E_w^+ according as r is odd or even. Now $(e_1 \dots e_r)^2 = (-1)^{r(r-1)/2}$, while $Nm(e_1 \dots e_r) = 1$. If r is odd, $g^2 = (-1)^{r(r-1)/2} Nmg$. This proves the assertion.

We have now proved

Proposition 3.3. Let E be an orthogonal vector bundle of dimension $2n$ to which the action of i lifts. If P is a Clifford bundle lifting E, then $P \approx \beta \circ (NmP) \circ i^*P$, where β is the unique line bundle of degree $= -$ degree NmP with $i^*\beta \approx \beta$ associated to the partition of W given by the subset

$$S = \{w \in W : r_w \equiv 1,\ 2n \bmod 4\},$$

$$T = \{w \in W : r_w \equiv 3,\ 2n - 2 \bmod 4\},$$

where r_w is the dimension of the fixed subspace of E_w under the action of i. In particular, if E is of even (resp. odd) type, then $\# S$ is even (resp. odd).

Remark 3.4. We call attention to a subtlety regarding special orthogonal structures. Suppose α is a line bundle with an action of i on it. Let E be an orthogonal bundle of odd type with $i^*E \approx E$. Then i^*E also comes with a special orthogonal structure whenever E is provided with one. However, we claim that the isomorphism $\varphi : i^*E \to E$ cannot preserve this special orthogonal structure. For, if it does, we would have a commutative diagram

$$\begin{array}{ccc} i^*\det E & \xrightarrow{\det \varphi} & \det E \\ {\scriptstyle i^*\eta}\downarrow & & \downarrow{\scriptstyle \eta} \\ i^*\alpha & \xrightarrow{\ i\ } & \alpha \end{array}$$

Restricting to a fibre at $w \in W$, we get a commutative diagram

$$\begin{array}{ccc} \det E_w & \xrightarrow{\det \varphi_w} & \det E_w \\ {\scriptstyle \eta_w}\downarrow & & \downarrow{\scriptstyle \eta_w} \\ \alpha_w & \xrightarrow{\ \epsilon_w\ } & \alpha_w \end{array}$$

Now, $\det \varphi_w = \pm 1$ according as $\dim E_w^-$ is even or odd and since E is of odd type, $\{w : \det \varphi_w = 1\}$ is of odd cardinality, while $\{w : \epsilon_w = 1\}$ is of even cardinality (lemma 3.1).

4. Stability of orthogonal bundles

Definition 4.1. Let E be an orthogonal bundle. We say that E is stable (resp. semistable), if every proper isotropic sub-bundle F of E has degree < 0 (resp. ≤ 0).

Proposition 4.2. E is semistable as an orthogonal bundle if and only if it is semistable as a vector bundle.

Proof. It is obvious that semistability (resp. stability) as a vector bundle implies semistability (resp. stability) as an orthogonal bundle. As for the converse, let F be a sub-bundle of E. Consider the sheaf $N = F \cap F^{\perp}$ where $F^{\perp}$ is obtained from F by orthocomplementation. Now N may not be a vector sub-bundle of F, but let N' be the sub-bundle generated by it. Then we have the exact sequence

$$0 \to N' \to F \oplus F^{\perp} \to M' \to 0,$$

where M' is the sub-bundle of E generated by $F + F^{\perp}$. Obviously, $M' = (N')^{\perp}$. Thus $\deg (F \oplus F^{\perp}) = \deg N' + \deg (N')^{\perp} = 2 \deg N'$, since we have the exact sequence $0 \to (N')^{\perp} \to E \to (N')^* \to 0$. Hence $\deg F = \deg N' \leq 0$, since N' is obviously isotropic.

Remarks 4.3.

(i) It is easy to see that our definition of stability and semi-stability coincides with the general notion introduced by Ramanathan [5], for principal bundles with reductive structure group.

(ii) E could be stable as an orthogonal bundle and yet be nonstable as a vector bundle. The reason is, with the notation of proposition 4.2, that the bundle $F + F^{\perp}$ could be the whole of E, and $N' = 0$. When this happens, the bundle E is a direct sum of two orthogonal bundles. Actually, this phenomenon does happen as the following example shows.

Example 4.4. Let E be a vector bundle over X of rank 2. Then ad E, the bundle of endomorphisms of trace 0 is a special orthogonal bundle, the Killing form giving the orthogonal structure. It is easy to see, using the one-one correspondence between isotropic sub-bundles of ad E and sub-bundles of E, that E is stable if and only if ad E is stable as an orthogonal bundle. On the other hand, if $E \approx \alpha \otimes E$ for some line bundle with $\alpha^2 = 1$, then ad E contains α as a proper nondegenerate sub-bundle. Indeed, one can even construct bundles (see [4] with $E \approx \alpha \otimes E$ and $E \approx \beta \otimes E$ for two distinct elements α, β with $\alpha^2 = \beta^2 = 1$, so that ad E becomes a direct sum $\alpha \oplus \beta \oplus \alpha\beta$.

Regarding this, we have

Proposition 4.5. (i) E is stable as an orthogonal bundle if and only if E is an orthogonal direct sum of sub-bundles E_i, which are mutually nonisomorphic with each E_i stable as a vector bundle.

(ii) If Aut E is the bundle of orthogonal automorphisms of a stable bundle E, then E is stable as a vector bundle if and only if $\Gamma(\text{Aut}\, E) = \pm 1$.

Proof. (1) We have proved above that if E is not stable as a vector bundle, then E is the orthogonal direct sum of sub-bundles E_1, E_2 of E. Now clearly E_1, E_2 are stable orthogonal so that by induction on rank E, we see that $E \approx \Sigma E_i$, where E_i are stable as vector bundles. If such a direct sum has repeated summands, say $E_1 \simeq E_2$, then the imbedding $X \mapsto (X, iX)$ of E_1 in $E_1 \oplus E_1$ gives an isotropic sub-bundle of degree zero, contradicting the stability of E. Thus all the E_i's are mutually nonisomorphic. Conversely, if $E = \Sigma E_i$, E_i stable as vector bundles and $E_i \not\sim E_j$, then any sub-bundle $\neq 0$ of degree 0 must be a sum of some of the E_i and hence cannot be isotropic.

(2) If E is stable as a vector bundle, then $\Gamma(\text{Aut } E) \subset k^*$ and hence must be ± 1. The converse is a consequence of (1).

Proposition 4.6. If E is an orthogonal bundle over a curve with an involution i which lifts to E, then E is semistable as an orthogonal bundle if and only if for every i-invariant, isotropic sub-bundle F of E, we have $\deg F \leq 0$.

Proof. If F is any sub-bundle of E, then as in proposition 4.2, we have the sequence

$$0 \to N' \to F \oplus i^*F \to M' \to 0,$$

where N', M' are sub-bundles generated by $F \cap i^*F$ and $F + i^*F$ respectively. Now $\deg N' \leq 0$ by assumption, while the fact that $\deg M' \leq 0$, follows from proposition 4.2. Since $\deg F = \deg i^*F$, this proves our claim.

Let X be a hyperelliptic curve with rational involution i. Let E be a vector bundle on X with a lift of i. For every $w \in W$, the set of Weierstrass points of X, denote by r_w the dimension of the space of fixed points of i on the fibre E_w. Consider the exact sequence

$$(4.7) \qquad 0 \to E' \to E \to \sum_{w \in W} E_w^- \otimes \mathcal{O}_w \to 0,$$

where E_w^- is the eigenspace for i_w corresponding to the eigenvalue -1. The maps $E \to E_w^- \otimes \mathcal{O}_w$ are given by the natural projections $E_w \to E_w^-$. The kernel E' is clearly locally free. Moreover, the involution i acts on E'.

Lemma 4.8. For every $w \in W$, the involution i acts as Identity on E'_w.

Proof. We have only to show that i acts as Identity on the kernel $E'_w \to E_w$ since it does so on the image which is E_w^+. Now this kernel is $\text{Tor}_1(E_w^- \otimes \mathcal{O}_w, \mathcal{O}_w)$ $= E_w^- \otimes \text{Tor}_1^{\mathcal{O}_x}(\mathcal{O}_w, \mathcal{O}_w)$. From the resolution

$$0 \to L_w^{-1} \to \mathcal{O}_x \to \mathcal{O}_w \to 0,$$

we conclude that $\text{Tor}_1^{\mathcal{O}_x}(\mathcal{O}_w, \mathcal{O}_w)$ is canonically isomorphic to the fibre $(L^{1}{}_w^{-})_w$ $= K_w$, the space of differentials at w. Since i acts as $-$ Identity on K_w, it follows that i acts as Identity on $E'_w = E_w^- \otimes K_w$.

From lemma 4.8, the bundle E' descends to a vector bundle $\bar{E}$ on $X/_i \approx \mathbf{P}^1$. Let us now make the assumption on E :

(A) E satisfies $H^0(X, E \otimes h^{-g}) = 0$ and $H^1(X, E) = 0$. Then we have $H^0(X, E' \otimes h^{-g}) \subset H^0(X, E \otimes h^{-g}) = 0$ by (A) and hence in particular $H^0(\mathbf{P}^1, \bar{E} \otimes h^{-g}) = 0$. Consequently, $\bar{E} \otimes h^{-g}$ is a direct sum of line bundles, each of which is of degree < 0. This implies that $\bar{E} \otimes h^{-(g-1)}$ is a direct sum of line bundles of degree ≤ 0 and hence contained in a trivial bundle. Cononically speaking, we have $\bar{E} \otimes h^{-(g-1)} \subset H^1(\bar{E} \otimes h^{-(g+1)})_{\mathbf{P}^1}$. On the other hand, we have the isomorphism $H^1(\mathbf{P}^1, \bar{E} \otimes h^{-(g+1)}) \simeq H^1(X, E')^{\ddagger}$ in view of

Lemma 4.9. For any vector bundle V on $\mathbf{P}^1$, we have the isomorphism $H^1(\mathbf{P}^1, V)$ $\approx H^1(X, \pi^* V \otimes h^{-(g+1)})^{\ddagger}$.

Proof. The left side is dual to $H^0(\mathbf{P}^1, V^* \otimes h^{-2}) \approx H^0(X, \pi^* V^* \otimes h^{-2})^{\#}$ while the right side is dual to $H^0(X, \pi^* V^* \otimes h^{-(g+1)} \otimes K_X)^{\ddagger}$. But the isomorphism $h^{-(g+1)} \otimes K_X \approx h^{-2}$ commutes with i only upto sign. This proves the assertion.

Thus $\tilde{E} \otimes h^{-(g-1)} \subset H^1(X, E')_{\mathbf{P}^1}$ canonically. From the exact sequence 4.7, we obtain the exact sequence

$$0 \to H^0(X, E)^{\ddagger} \to \Sigma E_w^- \to H^1(X, E')^{\ddagger} \to 0.$$

To check the injectivity $H^0(X, E)^{\ddagger} \to \Sigma E_w^-$, we note that its kernel is contained in the kernel $H^0(X, E) \to \Sigma E_w$ which is $H^0(X, E \otimes L_w^{-1}) = H^0(X, E \otimes h^{-(g+1)}) \subset H^0(E \otimes h^{-g}) = 0$ by assumption (A). As for the surjectivity $\Sigma E_w^- \to H^1(X, E')^{\ddagger}$, this follows from the fact that $H^1(X, E)^{\ddagger} \subset H^1(X, E) = 0$ by (A).

Let us return to orthogonal bundles. If F is a semistable orthogonal bundle, then $E = F \otimes \alpha$ for some i-invariant line bundle α of degree $j = 2g - 1$ satisfies (A). Thus we can associate to such a bundle the subspace $H^0(X, E)^{\ddagger}$ of ΣE_w^-. The orthogonal structure on F gives rise to nondegenerate quadratic forms on F_w^+ and F_w^- (see § 3), and hence an α_w^2-values quadratic form on E_w^-. This fits in the following diagram

$$(4.10)\qquad \begin{array}{ccc} H^0(X, E)^{\ddagger} & \longrightarrow & \Sigma E_w^- \\ \downarrow q & & \downarrow q_w \\ H^0(X, \alpha^2)^{\#} \approx H^0(\mathbf{P}^1, h^j) & \longrightarrow & \Sigma \alpha_w^2 = h_w^j \end{array}$$

Now consider the exact sequence (on $\mathbf{P}^1$)

$$0 \to h^{j-(2g+2)} \to h^j \to h^j \mid \mathcal{O}_w \to 0.$$

Then we get a linear map $\Sigma h_w^j \to H^1(\mathbf{P}^1, h^{j-(2g+2)})$. Clearly, $\dim H^1(\mathbf{P}^1, h^{j-(2g+2)}) = 1$ or 2 according as $j = 2g$ or $2g - 1$.
In either case, we have

Proposition 4.11. The quadratic form on E_w^- with values in h_w^j gives rise to a form on ΣE_w^- or a quadratic map with values in a 2-dimensional space according as $j = 2g$ or $2g - 1$. In any case, the subspace $H^0(X, E) \subset \Sigma E_w^-$ is totally isotropic for the quadratic maps involved.

Proof. This is a simple consequence of the commutativity of 4.10, and the exactness of

$$H^0(\mathbf{P}^1, h^j) \to \Sigma h_w^j \to H^1(\mathbf{P}^1, h^{j-(2g+2)}).$$

Let us further analyse the situation. If $j = 2g$, a simple computation using ([1], proposition 2.2) shows that $H^0(X, E)^{\ddagger}$ has dimension $\frac{1}{2}(\Sigma \dim E_w^-)$. In other words, it is a maximal isotropic space for the quadratic form. If $j = 2g - 1$, $H^0(X, E)^{\ddagger}$ has dimension $= \frac{1}{2}(\sum_w \dim E_w^- - rkE)$. For any $t \in \mathbf{P}^1 - W$, we get a linear map $H^1(\mathbf{P}^1, h^{-3}) \to H^1(\mathbf{P}^1, h^{-2})$ and hence a nondegenerate quadratic form on ΣE_w^-. It is easy to determine the orthocomplement of $H^0(X, E)^{\ddagger}$ for this form. Indeed, it is $H^0(X, E \otimes h)^{\ddagger}$ where the latter space is imbedded by evaluation in $\Sigma E_w^- \otimes h_w$, and h_w is trivialised by means of the section of h vanishing at the point of X lying over t. To see this, we have only to show that the bilinear product of an element in $H^0(X, E)$ and an element in $H^0(X, E\otimes h)^{\ddagger}$ goes to zero in $H^1(\mathbf{P}^1, h^{-2})$. From the diagram

$$\begin{array}{ccccc} H^0(X, h^{2g-1}) & \to & \Sigma h_w^{2g-1} & \to & H^1(X, h^{-3}) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(X, h^{2g}) & \to & \Sigma h_w^{2g} & \to & H^1(X, h^{-2}) \end{array}$$

this is equivalent to showing that q_w maps it in the image of $H^0(X, h^{2g})$ in $\Sigma\, h_w^{2g}$. But this is obvious from the diagram

$$\begin{array}{ccc} H^0(X, E)^{\ddagger}\otimes H^0(X, E\otimes h) & \longrightarrow & \Sigma E_w^- \\ \downarrow & & \downarrow \\ H^0(X, \alpha^2\otimes h)^{\#} = H^0(\mathbf{P}^1, h^{2g}) & & \Sigma h_w^{2g-1} \\ & \searrow \quad \swarrow & \\ & \Sigma h_w^{2g} & \end{array}$$

In other words, we have shown

Proposition 4.12. The ortho complement of $V = H^0(X, E)^{\ddagger}\subset \Sigma E_w^-$ is $H^0(X, E\otimes h)^{\ddagger}$ for the quadratic form at $t\in\mathbf{P}^1$ and $H^0(X, E\otimes h)^{\ddagger}/H^0(X, E)^{\ddagger}$ is canonically the fibre of $\tilde{\tilde{E}}\otimes h^{-(g-1)}$ at t.

In other words, if the quadratic map $\Sigma E_w^-\to H^1(\mathbf{P}^1, h^{-2})$ is regarded as a bundle with an h-valued quadratic form on the trivial bundle ΣE_w^- over $\mathbf{P}^1$, then $V^{\perp}/V\approx \tilde{\tilde{E}}\otimes h^{-(g-1)}$.

Proof. We have only to trace the isomorphisms

$$\begin{aligned} H^0(X, E\otimes h)^{\ddagger}/H^0(X, E)^{\ddagger} &\approx \ker \Sigma E_w^-/H^0(X, E)^{\ddagger}\to\Sigma E_w^-/H^0(X, E\otimes h) \\ &\simeq \ker H^1(X, E')\to H^1(X, E'\otimes h) \\ &\approx \ker H^1(\mathbf{P}^1, \tilde{\tilde{E}}\otimes h^{-(g+1)})\to H^1(\mathbf{P}^1, \tilde{\tilde{E}}\otimes h^{-g}) \end{aligned}$$

this map being given by the section of h vanishing at t. From the exact sequence (on $\mathbf{P}^1\times\mathbf{P}^1$)

$$\begin{aligned} 0 &\to p_1^*(\tilde{\tilde{E}}\otimes h^{-(g+1)})\to p_1^*(\tilde{\tilde{E}}\otimes h^{-(g+1)})\otimes L_\Delta\to p_1^*(\tilde{\tilde{E}}\otimes h^{-(g+1)}) \\ &\otimes L_\Delta\otimes \mathcal{O}_\Delta\to 0, \end{aligned}$$

where Δ is the diagonal divisor in $\mathbf{P}^1\times\mathbf{P}^1$. On the other hand $L_\Delta|_\Delta\approx K_x^-$ where K_x is the canonical line bundle. Taking direct images on $\mathbf{P}^1$, we get

$$E\otimes h^{-(g+1)}\otimes h^2\approx \text{the kernel required.}$$

5. Classification of i-invariant orthogonal bundles

Theorem 1. Let X be a hyperelliptic curve of genus ≥ 2. Then the space U of semistable orthogonal bundles of rank r with an i-action is isomorphic to the following space. Let $\Sigma\xi_w$ be an orthogonal direct sum of spaces ξ_w with non-degenerate quadratic forms Q_w. Let ΣQ_w and $\Sigma\lambda_w Q_w$ be two quadratic forms, where

λ_w are mutually distinct scalars determined by X. Then the group $\Pi O(Q_w)$ acts on the variety M of subspaces of dimension $\frac{1}{2}(\Sigma \dim \xi_w - r)$ isotropic for both these forms. The quotient (in the sense of geometric invariant theory) is isomorphic to U.

Proof. Let us for convenience reinterpret the pencil of quadrics on $\Sigma\xi_w$. First of all, we will assume the standard quadratic forms Q_w to have values, not in the field, but in the one-dimensional space h_w^{2g-1}. Then the quadratic map $\Sigma\xi_w \to \Sigma h_w^{2g-1}$ gives, on composition with the boundary homomorphism $\Sigma h_w^{2g-1} \to H^{\circ}(\mathbf{P}^1, h^{-3})$ of the sequence

$$0 \to h^{-3} \to h^{2g-1} \to h^{2g-1} \otimes \mathcal{O}_W \to 0,$$

a pencil of quadrics as required.

Fix a line bundle α of degree $2g-1$ with an i-action on it, once for all. Then we have seen in § 4, that we have a natural inclusion $H^0(X, E) \subset \Sigma E_w^-$, where $E = F \otimes \alpha$, F an orthogonal bundle with an i-action satisfying condition (A) in 4.8. Choosing quadratic isomorphisms $E_w^- \to \xi_w$, we associate to E, a subspace of $\Sigma\xi_a$ of dim $= \dim H^0(X, E)^{\ddagger} = \frac{1}{2}(\dim \xi_w - n)$, where $rkE = n$. By proposition 4.11, we see that this subspace is isotropic for the given pencil on $\Sigma\xi_w$. Moreover, the intersection of this space with ξ_w is the kernel of

$$H^0(X, E)^{\ddagger} \to \sum_{w' \neq w} E_{w'},$$

and is hence isomorphic to $H^0(X, E \otimes L_{W-w}^{-1})^{\ddagger} \subset H^0(X, E \otimes L_{W-w}^{-1}) \subset H^0(X, E \otimes h^{-g}) = 0$ by (A). Thus the subspace V associated to F satisfies (A') V is a subspace of $\Sigma\xi_w$ which is isotropic for the pencil of quadrics and $V \cap \xi_w = 0$ for every $w \in W$.

Conversely, if V is a subspace of $\Sigma\xi_w$ satisfying (A'), then the ortho complement of V with respect to any quadric of the pencil is of the same dimension. Indeed this needs to be checked only for quadrics corresponding to points $w \in W \subset \mathbf{P}^1$. The nullity of this quadric is ξ_w and $V \cap \xi_w = 0$ implies that $V^{\perp}$ is of the complementary dimension containing ξ_w. This proves that $V^{\perp}/V$ is a vector bundle on $\mathbf{P}^1$ which comes with a subspace of the fibre $(V^{\perp}/V)_w$, namely, the isomorphic image of ξ_w. Using this, we can construct a bundle E (on X)

$$0 \to V^{\perp}/V \otimes h^{g-1} \to E \to \mathcal{T} \to 0,$$

in which $\mathcal{T}$ is a torsion sheaf concentrated on W with length $(\mathcal{T}) = \dim \xi_w$, and the kernel $(V^{\perp}/V \otimes h^{g-1})_w \to E_w$ is $\xi_w \otimes h_w^{g-1}$. Then $E = F \otimes \alpha$ with F an i-invariant orthogonal bundle. It is then easy to see that $H^0(E \otimes h^{-g}) = 0$, so that we have

Proposition 5.1. Let α be a line bundle on X of degree $2g-1$, invariant under i. There is then a bijection between the set of orthogonal bundles F of rank n together with (a) an i-action and (b) orthogonal isomorphisms $\eta_w (F_w \otimes \alpha_w)^- \to \xi_w$, satisfying $H^1(X, F \otimes \alpha) = 0$, and the space of subspaces V of $\Sigma\,\xi_w$ of dimension $\frac{1}{2}(\Sigma \dim \xi_w - n)$ which are isotropic for the pencil of quadrics $\Sigma Q_w = 0$, $\Sigma\,\lambda_w\,Q_w = 0$, and satisfy $V \cap \xi_w = 0$ for all w.

Clearly, this bijection is compatible with the action of $\Pi O(Q_w)$ on both the sets. The action on (F, η_w) is simply $(F, g_w \circ \eta_w)$ by $g = (g_w)$. On the other hand, clearly, the group $\Pi O(Q_w)$ acts on $\Sigma\xi_w$ and leaves the pencil invariant and hence also acts on the set of isotropic subspaces in question.

To complete the proof of theorem 1, we have to investigate the correspondence between stable (resp. semistable) bundles and stable (resp. semistable) points under the action of $\Pi O(Q_w)$.

We recall

Definition 5.2. Let G be a reductive group acting linearly on a vector space and hence also on the projective space. If M is a projective variety invariant under G, a point $m \in M$ is said to be semistable (resp. properly stable) for the action of G, if it satisfies the following equivalent conditions.

(1) There exists a G-invariant hypersurface not passing through m (resp. and in addition, the isotropy group at m is finite and the orbit of m in the complement of this hypersurface is closed).

(2) For any one parameter group $t \mapsto \lambda(t)$ of G, consider $\lim_{t\to 0} \lambda(t)\, m \in M$. This point is represented by a line in V on which the one parameter group acts as a character $t \mapsto t^{s_m}$, for some $s_m \in \mathbf{Z}$. Then $s_m \geqslant 0$ (resp. $s_m > 0$).

The equivalence of these two conditions is proved in ([3], Chapter 2).

For the action of $\Pi O(Q_w)$ on the space of isotropic subspaces, we have

Proposition 5.3. A subspace V of $\Sigma\xi_w$ is semistable (resp. properly stable) if and only if for every proper family (N_w) of isotropic subspaces of ξ_w, we have

$$\dim\, (\Sigma N_w) \cap V + \dim\, (\Sigma N_w^{\perp}) \cap V \leqslant (\text{resp.} <)\ \dim V.$$

Proof. We will use the equivalent condition (2) in definition 5.2 to prove this. Any one-parameter group $t \mapsto \lambda_t$ of $\Pi O(Q_w)$ can be described as follows. There exist isotropic subspaces N_i of $\Sigma\xi_w$, compatible with this orthogonal decomposition with

$$0 = N_0 \subset N_1 \subset \dots \subset N_r \subset N_r^{\perp} \subset N_{r-1}^{\perp} \subset \dots \subset N_1^{\perp} \subset N_0^{\perp} = \Sigma\xi_w = \bar{W}.$$

λ_t leaves all N_i invariant and induces the character $t \mapsto t^{a_i}$ on N_i/N_{i-1}, $i = 1, \dots, r$ with $a_1 \leqslant a_2 \leqslant \dots \leqslant a_r \leqslant 0$. The action on $N_r^{\perp}/N_r$ is trivial, while on $N_{i-1}^{\perp}/N_i^{\perp}$, it acts as $t \mapsto t^{-a_i}$. Now if V is any isotropic subspace of $\Sigma\xi_w$, then $\lim_{t\to 0} \lambda_t(V)$ is a space on which λ acts as on the associated graded space for the above filtration (N_i). In other words, we have

$$\begin{aligned} s_V &= \sum_{i=1}^{r} a_i \dim \frac{N_i \cap V}{N_{i-1} \cap V} + \sum_{i=1}^{r} (-a_i) \dim \frac{N_{i-1}^{\perp} \cap V}{N_i^{\perp} \cap V} \\ &= \sum_{i=1}^{r} a_i (\dim\, N_i \cap V - \dim\, N_{i-1} \cap V) \\ &\quad + \sum_{i=1}^{r} (-a_i) \left(\dim\, \mathrm{Im}\, V \text{ in } \frac{W}{N_i^{\perp}} - \dim\, \mathrm{Im}\, V \text{ in } \frac{W}{N_{i-1}^{\perp}}\right) \\ &= \sum_{i=1}^{r} (a_i - a_{i+1}) \left(\dim\, N_i \cap V - \dim\, \mathrm{Im}\, V \text{ in } \frac{W}{N_i^{\perp}}\right) \end{aligned}$$

where we make the convention that $a_{r+1} = 0$. Since $a_i - a_{i+1}$ are arbitrary negative integers, we must have

$$\dim \left(N_i \cap V - \dim \operatorname{Im} V \text{in} \frac{W}{N_i^{\perp}}\right)$$

$$= \dim (N_i \cap V) + \dim (N_i^{\perp} \cap V) - \dim V > 0,$$

in order that we may have $s_V < 0$. This proves our assertion.

We will now investigate the equivalence among semistable points which gives the geometric invariant theoretic quotient. According to [3], two semistable points are equivalent if their orbits have a common limit point. We claim

Proposition 5.4. Two semistable subspaces $V_1, V_2 \subset \Sigma\xi_w$ represent the same point in the quotient if and only if there exist isotropic flags $(N_1), (N_2) \subset \Sigma\xi_w$ compatible with the decomposition such that the associated graded spaces V_1', V_2' are in the same G-orbit. Moreover, any semistable subspace V is equivalent to a space $\oplus V_i$ where each V_i is stable in $\Sigma V_i \cap \xi_w$.

Proof. If V is not stable, then there exists an isotropic sub-bundle $N \subset \Sigma\xi_w$, compatible with the decomposition such that

$$\dim \, V \cap N + \dim \, V \cap N^{\perp} = \dim V.$$

Now using induction and replacing V by $V \cap N^{\perp}/V \cap N \subset N^{\perp}/N$, we can prove the last part of the statement. From the analysis of the stability condition, we see that the closure of an orbit of V under a suitable one-parameter group contains the associated graded of V. Finally, it remains to prove that if V_1, V_2 have different stable series, they map on distinct points in the quotient. This is clear, if V_1 is stable since then, the orbit is closed. In general, we use induction on the length of the stable series and semicontinuity of rank in a family of linear transformations.

Lemma 5.5. If a point $V \subset \Sigma\xi_w$ is semistable for the action of $\Pi O(Q_w)$, then $V \cap \xi_w = 0$ for all w.

Proof. Indeed, take $N_{w_0} = V \cap \xi_{w_0}$, and $N_w = 0$ for $w \neq w_0$ in proposition 5.3. Then semstability of V implies that

$$\dim (V \cap \xi_{w_0}) + \dim (V \cap (N_{w_0}^{\perp} + \sum_{w \neq w_0} \xi_w)) \leqslant \dim V.$$

Since V is isotropic, any $v = \Sigma v_w \in V$ should satisfy $v_{w_0} \perp N_{w_0}$. Hence $V \subset N_{w_0}^{\perp} + \sum_{w \neq w_0} \xi_w$. This leads to a contradiction if $N_{w_0} \neq 0$.

In view of lemma 5.5, we claim we have a natural morphism from the open set of semistable points into $\tilde{M}$, where $\tilde{M}$ is the variety over the moduli space M consisting of i-invariant orthogonal bundles E, together with isomorphisms $E_{\tilde{w}} \to \xi_w$ as in proposition 5.1. To see this, we need only check.

Proposition 5.6. A semistable (resp. stable) point V corresponds to a semistable (resp. stable) bundle.

Proof. In fact, if F is such that $H^1(X, F \otimes \alpha) = 0$, and if $H^0\ (F \otimes \alpha)^{\ddagger} \subset \Sigma \xi_w$ is a semistable point, then we have to show that F is semistable. Let $L \subset F$ be any i-invariant isotropic sub-bundle. Then $H^0\ (X,\ L \otimes \alpha)^{\ddagger} \subset H^0(X,\ F \otimes \alpha)^{\ddagger}$ is a subspace contained in $H^0\ (X, F \otimes \alpha)^{\ddagger} \cap (\Sigma (L \otimes \alpha)^{-}_{w})$. Taking $N_w = (L \otimes \alpha)^{-}_{w}$ in the definition of stable points (proposition 5.3), we get

$$\dim\ H^0(X, L \otimes \alpha)^{\ddagger} + \dim\ H^0(X, L^{\perp} \otimes \alpha)^{\ddagger} \leqslant \dim\ H^0(X, F \otimes \alpha)^{\ddagger}.$$

This implies that

$$\dim\ H^0(X, L \otimes \alpha)^{\ddagger} - \dim\ H^1(X, L \otimes \alpha)^{\ddagger} + \dim\ H^0(X, L^{\perp} \otimes \alpha)^{\ddagger}$$
$$- \dim\ H^1(X, L^{\perp} \otimes \alpha)^{\ddagger} \leqslant \dim\ H^0(F \otimes \alpha)^{\ddagger}.$$

Using ([1], proposition 2.2), we get (setting $\dim\ (L \otimes \alpha)^{-}_{w} = a_w,\ rkL = r$)

$$r + \tfrac{1}{2}(\deg L + (2g-1)\, r - \Sigma\, (r - a_w))$$
$$+ n - r + \tfrac{1}{2}(\deg L^{\perp} + (2g-1)(n-r) - \Sigma\, (n - r - r_w + a_w) \leqslant \tfrac{1}{2}\Sigma r_w - n).$$

i.e., $n + \frac{1}{2}\ (\deg L + \deg L^{\perp} + (2g-1)\, n - (2g+2)\, n + \Sigma r_w) \leqslant \frac{1}{2}(\Sigma r_w\, a_w - n).$

This means that $\deg L + \deg L^{\perp} \leqslant 0$, proving our assertion.

We have thus obtained a morphism from the open subset of semistable points into the required moduli space. It is obvious that this morphism is $\Pi O\,(Q_w)$-invariant and hence goes down to a morphism of the quotient variety. Moreover, proposition 5.1 implies that this is an isomorphism of the open set in the quotient corresponding to points representing closed orbits in the set of semistable points (namely stable points), onto an open subset. Hence it follows that there is a surjective, birational morphism of this quotient into the moduli variety.

Now since the moduli variety in question is a component of the action of i on the moduli of orthogonal bundles and since the latter is normal [5], it follows that the former is also normal. Finally to prove the theorem we have only to check that the morphism is injective. Any point in the quotient can be represented by ΣV_i where V_i is stable in $\Sigma V_i \cap \xi_w$. Thus ΣV_i is mapped on the bundle $\Sigma \varphi_i (V_i)$, where φ_i are similar maps defined on $\Sigma V_i \cap \xi_w$. If $\varphi\,(V)$ is S-equivalent to $\varphi\,(V')$ then it is necessary that after a permutation of the factors, we must have $\varphi_i (V_i) \sim \varphi_i (V_i')$. But this implies that V_i and V_i' are in the same orbit, under the group $\Pi O\,(V_i \cap \xi_w)$. This proves that φ is injective and hence that it is an isomorphism by Zariski's main theorem.

6. Spin structures and low dimensional cases

If E is an orthogonal bundle of odd type, then any lift P of E to a Γ-bundle has norm of odd degree. Now the lift of i-action to E gives rise to a line bundle β as in § 3, determined by the behaviour of i at w. For each such lift P of E with fixed norm, we obtain a lift to $\Gamma(q_w)$ of a reflection in the subspace E^{+}_{w} with respect to the quadratic form q_w induced from the quadratic structure on E. As we have seen in proposition 3.3, this induces an orientation on each E^{-}_{w}, and hence we have analogously the

Theorem 2. The moduli of Γ^+-bundles P of fixed norm of odd degree with isomorphisms $i^*P \to \alpha \circ P$ of Γ-bundles is isomorphic to the quotient of the variety of isotropic spaces for the pencil

$$\Sigma Q_w = 0,\ \Sigma \lambda_w Q_w = 0,$$

where Q_w is a nondegenerate form on a space ξ_w, by the group $\Pi SO(Q_w)$.

Proof. What we have to show is only that the space of i-invariant Γ^+-bundles are in (1, 1)-correspondence with i-invariant orthogonal bundles modulo the group $\Pi SO(E_w^-)$. Now the group $\Pi SO(E_w^-)/\Pi O(E_w)$ is isomorphic to $(\mathbf{Z}/2)^W$ and acts on the set of Γ^+-bundles of fixed norm with a fixed associated orthogonal bundle E as follows. An element of $(\mathbf{Z}/2)^W$ can be identified (lemma 3.1) with line bundles whose square is 1 or $h^{\deg N}$. Now if $\alpha^2 = 1$, it acts on a Γ^+-bundle P by sending P to $\alpha \circ P$. If $\alpha^2 \simeq N$, we send P to $\alpha^{-1} \circ N \circ i^*P$. It is easy to see that this action is simply transitive. Indeed, the subgroup $\{\alpha : \alpha^2 = 1\}$ clearly acts on the set of Γ^+-bundles with a fixed associated special orthogonal bundle E. Any element in the other coset twists P into another Γ^+-bundle whose associated special orthogonal bundle is not isomorphic to P (remark 3.4). This proves our assertion. Finally, to check that the morphism from the moduli of Γ^+-bundles to the variety of isotropic spaces is actually an isomorphism (instead of being just bijective), we have to show that the inverse map is an isomorphism. But we know that the composite of the inverse map with the morphism into the moduli of orthogonal bundles is a morphism. Now this follows from the fact that, given any orthogonal bundle in $T \times X$ and $T \in T$, there exists an etale neighbourhood $U \to T$ of t such that the induced bundle on $U \times X$ can be lifted to a Γ^+-bundle.

By using lemma 2.1 and the Kunneth decomposition, we can pass to such a neighbourhood U that the induced bundle on $U \times X$ has the obstruction to Γ_0-lifting in $H^1(X, \mathbf{Z}/2)$. But its image in $H^2(X, \mathcal{O}^*)$ is zero and hence the obstruction to Γ-liftability of the induced bundle is also zero, since it is in the image of $H^2(X, \mathcal{O}^*) \to H^2(U \times X, \mathcal{O}^*)$. This proves theorem 2.

We will now make a few remarks on the dimensions of ξ_w, although these are implicit.

Remark 6.1. When we deal with orthogonal bundles F of even rank, then the corresponding integers (dim F_w^-) have the property: $\{w : \dim F_w^- \text{ is odd}\}$ is of even cardinality. Thus we see that F has even or odd rank according as $\Sigma \dim \xi_w$ is even or odd. This also follows from ([1], proposition 2.2) since $\frac{1}{2}((2g-1)n - \Sigma \dim E_w^-)$ is an integer, where $E = F \otimes \alpha$, $\deg \alpha = 2g - 1$ By construction, ξ_w is isomorphic to E_w^-.

We shall now consider special cases. The simplest is when all the spaces ξ_w are one-dimensional or 0. In that case, there are no proper isotropic subspaces contained in ξ_w. Moreover, the group $\Pi O(Q_w)$ is a finite group. If we deal with $\Gamma(n)$-structures is such a case, then the moduli space is actually isomorphic to the variety of isotropic spaces. In other words, we have

Theorem 3. The moduli space of i-invariant orthogonal bundles E of rank $2n$ with Γ^+-structures such that $\dim((E \otimes \alpha)_w^-) = 1$ for all w, where α is an i-invariant line bundle of degree $2g - 1$, is isomorphic to the variety of $(g + 1 - n)$-dimension subspaces of k^{2g+2} which are isotropic to the quadrics

$\Sigma X_i^2 = 0,\ \Sigma \lambda_i X_i^2 = 0.$

In particular, the intersection of the above two quadrics is itself a moduli space of $\Gamma^+(2n)$-bundles of the above type.

6.2. *Case of rank* 2. Let us further specialise to the case rank $= 2$. In this case, i-invariance of orthogonal bundles of determinant 1 is automatic. Thus we get: *The Jacobian of X is isomorphic to the variety of g-dimensional subspaces of* k^{2g+2}, *isotropic for* $\Sigma X_i^2 = 0$ *and* $\Sigma \lambda_i X_i^2 = 0$. See [1, 2 and 6].

6.3. *Case of rank* 4. In this case, $\Gamma^+(4)$ is isomorphic to the subgroup of $GL(2) \times GL(2)$ consisting of elements (A, B) with $\det A \cdot \det B = 1$. The homomorphism into $SO(4)$ is given by $(A, B) \mapsto A \otimes B$. On the other hand, $\Gamma(4)$ is the split extension of $\mathbf{Z}/(2)$ by $\Gamma^+(4)$ given by the action $(A, B) \to (B, A)$, by the non-trivial element of $\mathbf{Z}/(2)$. Thus a $\Gamma^+(4)$-bundle is essentially a pair of bundles M, N with $\det M \otimes \det N =$ trivial bundle. But a $\Gamma(4)$-bundle does not distinguish between M and N. Firstly, we have: *The moduli space of vector bundles of rank* 2 *and fixed determinant of odd degree is isomorphic to the variety of* $(g-1)$*-dimensional subspaces of* k^{2g+2}, *isotropic for* $\Sigma X_i^2 = 0$ *and* $\Sigma \lambda_i X_i^2 = 0$. The group of even changes of sign in the coordinates correspond to tensorisation by line bundles of order 2.

6.4. *Case of rank* 6. If R is a four-dimensional vector space, $\overset{2}{\Lambda} R$ is a six-dimensional space equipped with a ΛR-valued quadratic form. Thus we get a homomorphism of the subgroup $G = \{(\lambda, g) : \lambda^2 = \det g\}$ of $k^x \times GL(R)$ into $SO(6)$ by sending (λ, g) to $\lambda^{-1} \circ \overset{2}{\Lambda} g$. It is easy to see that $G = \Gamma^+(6)$. Thus a $\Gamma^+(6)$-bundle is the same as the data consisting of a line bundle η and a vector bundle V of rank 4 with $\eta^2 \approx \det V$. Also $Nm(\eta, V) = \eta$ so that (η, V) is of odd type if and only if V has degree $\equiv 2 \pmod 4$. If α is any line bundle, $\alpha \circ (\eta, V) = (\alpha^2 \otimes \eta, \alpha \otimes V)$. Two $\Gamma^+(6)$-bundles (η, V), (η', V') are isomorphic as $\Gamma(6)$-bundles if and only if $\eta = \eta'$ and $V = V'$ or $\eta' = \eta^{-1}$ and $V' = \eta^{-1} \otimes V^*$. Hence an isomorphism $(\eta, V) \approx i^*(\eta, V) \circ Nm(\eta, V) \circ \beta^{-1}$ for some β with $\beta^2 = h^{-\deg V}$ yields $V \approx i^* V \otimes \eta \otimes \beta^{-1}$, or $V \approx i^* V^* \otimes \beta^{-1}$. Thus we deduce: Consider stable vector bundles of rank 4 and fixed determinant of degree 2. Assume given an isomorphism $i^* V \approx V^* \otimes \beta^{-1}$ with $\beta^2 = h$. The moduli of such objects can thus be described as the variety of subspaces isotropic for a pencil of quadrics.

Remark 6.5. The constructions involved in the proof of the main theorems seem to be of a very general nature. For one thing, one can obtain results similar to [1, Theorem 3] for spin bundles with an i-action. Furthermore, it is also possible to get results for symplectic bundles, etc. However one ought to be able to understand these results in a group scheme theoretic set-up. In other words, given a bundle with an i-action, we modify the bundle at the Weierstrass points so that it goes down to $\mathbf{P}^1$. By keeping track of the additional structures on the quotient bundle, one would expect to classify bundles on X with i-action.

References

[1] Usha V Desale and Ramanan S 1976 Classification of vector bundles of rank 2 on hyper-elliptic curves; *Inventiones Math.* **38** 161–185

[2] Luc Gauthier 1954–55 Footnote to a footnote of André Weil. *Univ. de Politec. Turino Rend. Sem. Math.* **14** 325–328.

[3] Mumford D 1965 *Geometric invariant theory,* (Springer-Verlag)

[4] Narasimhan M S and Ramanan S 1975 Generalised Prym varieties as fixed points; *J. Indian Math. Soc.* p. 1–19

[5] Ramanathan A, Stable principal bundles on a compact Riemann surface, Springer Lecture Notes (to appear)

[6] Reid M *Intersection of two or more quadrics,* Thesis, Cambridge

[7] Serre J P *Cohomologie Galoisienne,* (Paris : Hermann)

[8] Seshadri C S 1967 Space of unitary vector bundles on a compact Riemann surface; *Ann. Math.* 85 303–336.

[9] A Weil 1954 Footnote to a recent paper; *Am. J. Math.* 76 347–350

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