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Signorini Problem in Hencky Plasticity.

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1. – Introduction.

The goal of this note is to prove the existence of a displacement field for an elasto-plastic body subject to Hencky's law and the Von Mises yield function, when a rigid obstacle is present. In a previous note by B. D. Reddy and F. Tomarelli (see [8]) this problem was solved by taking into account the obstacle through a constraint acting on the whole body. Here we study the rate problem in the strain when the constraint acts only on a part of the boundary.

This last formulation seems interesting both for mechanical reasons (the obstacle plays a role only on the potential contact region) and for mathematical ones: it is a starting point to study the regularity of the solutions (see for instance the paper by G. Fichera [4], about the case of linear elasticity).

The problem is faced as a minimization of the energy functional which is not coercive. For this reason the theory developed by C. Baiocchi, G. Buttazzo, F. Gastaldi and F. Tomarelli in [3] and specialized in [8] is exploited to find sufficient conditions in term of applied loads. The two main steps are:

1) proving the closedness of the boundary constraint in a suitable weak topology,

2) defining a «projection» of functions with bounded deformation into the space of rigid body motions, such that boundary inequalities (and/or inclusions) are preserved.

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2. – Notation and statement of the result.

Consider a body, in its undeformed state, which occupies a bounded domain $\overline{\Omega}$, where Ω is a bounded C^1 open subset of $\{x \in \boldsymbol{R}^3 : x_3 \geqslant 0\}$; here $x = \{x_1, x_2, x_3\}$ denotes the cartesian cohordinates and

$$v = \{v_1(x),\ v_2(x),\ v_3(x)\} \qquad \text{for } x \in \overline{\Omega}$$

denotes any displacement vector field; $\varepsilon(v)$ is the linear strain tensor

$$\varepsilon_{i,j}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad i,j = 1,2,3;$$

$B_\rho(x)$ is the 3-dimensional open disk of radius ρ and center x. $\boldsymbol{M}^3$ is the set of real symmetric 3×3 matrices; the deviator α^D of any $\alpha \in \boldsymbol{M}^3$ is given by

$$\alpha^D_{i,j} = \alpha_{i,j} - \frac{1}{3}\,\alpha_{k,k}\,\delta_{i,j}\,,$$

henceforth the summation convention is understood unless differently specified; $a \wedge b$ denotes the cross product of $a, b \in \boldsymbol{R}^3$.

A vector field of body forces in Ω and a surface traction vector field in $\partial\Omega$ are given by mean of the potential energy associated to the generic displacement v (see [8]):

$$Lv = \int_\Omega (l \cdot v + G : \varepsilon(v)) \tag{2.1}$$

where $l \cdot v = l_i v_i$, $G : \varepsilon = G_{i,j}\varepsilon_{i,j}$ and

$$l \in L^3(\Omega, \boldsymbol{R}^3), \qquad G^D \in C^0_0(\Omega, \mathbf{M}^3), \qquad G_{i,i} \in L^2(\Omega). \tag{2.2}$$

In the Hencky model for perfect plasticity the deformation energy is given, for regular v, by

$$E(v) = \int_\Omega \left(\varphi(\varepsilon^D(v)) + \frac{\chi}{2}(\operatorname{div} v)^2\right) \tag{2.3}$$

where $\varphi : \boldsymbol{M}^3 \to \boldsymbol{R}$ is defined by

$$\varphi(s) = \begin{cases} \mu|s|^2 & \text{if } |s| < k/2\mu; \\ k|s| - k^2/4\mu & \text{if } |s| \geqslant k/2\mu \end{cases} \tag{2.4}$$

and χ, $\mu \in \boldsymbol{R}$, $\chi > 0$ are given constants.

By relaxation of the functional (2.3) one obtains the finite energy space (on this subject we refer to the book [9] by R. Temam and the works [1], [2] by G. Anzellotti and M. Giaquinta):

$$P(\Omega) = \{v \in L^1(\Omega, \boldsymbol{R}^3) \colon \varepsilon(v) \text{ is an } \boldsymbol{M}^3 \text{ valued Radon measure and } \operatorname{div} v \in L^2(\Omega)\} \tag{2.5}$$

endowed with the norm

$$\|v\|_{P(\Omega)} = \|v\|_{L^1(\Omega, \boldsymbol{R}^3)} + \|\operatorname{div} v\|_{L^2(\Omega)} + \int_\Omega |\varepsilon^D(v)| . \tag{2.6}$$

The deformation energy is still denoted with $E(v)$ given by (2.3) for any $v \in P(\Omega)$, provided $\int \varphi(\varepsilon^D(v))$ is suitable interpreted when $\varepsilon^D(v)$ is only a measure (see [1]). We assume that there is a portion Γ of the boundary such that

$$\Gamma \subset \partial\Omega \cap \{x_3 = 0\} \text{ is a smooth nonempty 2 dimensional open set.} \tag{2.7}$$

Then

$$\exists \hat{x} \in \Gamma \text{ and } r > 0 \text{ s.t.} \tag{2.8}$$

$$B_r(\hat{x}) \cap \Gamma = B_r(\hat{x}) \cap \{x_3 = 0\}, \qquad B_r(\hat{x}) \cap \Omega = B_r(\hat{x}) \cap \{x_3 > 0\}.$$

We define the convex cone of admissible displacements:

$$K_\Gamma = \{v \in P(\Omega) \colon v_3(x) \geqslant 0 \quad \text{a.e. in } \Gamma\} \tag{2.9}$$

in (2.9) a.e. means almost everywhere with respect to the 2-dimensional Lebesgue measure. Moreover, abusing notation, in the above definition (and whenever this does not create any ambiguity), we denote v and its interior trace at the boundary by the same symbol. We recall that

$$p(\Omega) \subsetneqq BD(\Omega)$$

where $BD(\Omega)$ is the usual space of functions with bounded deformation (see [7], [9]) endowed with the norm

$$\|v\|_{BD(\Gamma)} = \|v\|_{L^1(\Omega)} + \int_\Omega |\varepsilon(v)|,$$

and that there is a constant C, depending only on Ω, such that

$$\|v\|_{L^1(\partial\Omega, \boldsymbol{R}^3)} \leqslant C \|v\|_{BD(\Omega)}, \qquad \forall v \in BD(\Omega). \tag{2.10}$$

We look for an equilibrium of the body by minimizing the total energy over the admissible displacement fields.

THEOREM 1. Assume (2.2), (2.8) and

$$\int_\Omega l_3\,dx < 0 = \int_\Omega l_1\,dx = \int_\Omega l_2\,dx, \tag{2.11}$$

$$\int_\Omega l \wedge (x - \hat{x})\,dx = 0, \tag{2.12}$$

$$\|G^D\|_{L^\infty} + C(\Omega,\Gamma)\,\|l\|_{L^3(\Omega,\mathbf{R}^3)} < k, \tag{2.13}$$

where the constant $C(\Omega,\Gamma)$ is defined in Lemma 4.2. Then the functional

$$\int_\Omega \left(\varphi(\varepsilon^D(v)) + \frac{\chi}{2}(\operatorname{div} v)^2\right) - Lv$$

achieves a finite minimum over K_Γ. ■

About the meaning of the compatibility (2.11), (2.12) and the safe load condition (2.13) we refer to [8]. We recall the following statement.

THEOREM 2. Let Y be the dual of a separable Banach space; $T \subset Y$ a sequentially w^* *closed* convex cone and $\Lambda: Y \to \mathbf{R} \cup \{+\infty\}$ a convex seq. w^* l.s.c. map(1). If $0 \in T \cap \operatorname{dom}\Lambda$ and

$$\|y_n\|_Y \to 0 \quad \forall \text{ sequence } \{y_n\}_n \text{ s.t.} \quad y_n \xrightarrow{w^*} 0 \text{ and } \Lambda(y_n) \to \Lambda(0), \tag{2.14}$$

$$\Lambda^\infty(y) \geq 0 \quad \forall y \in T, \tag{2.15}$$

$$T \cap \ker \Lambda^\infty \quad \text{is a subspace;} \tag{2.16}$$

where Λ^∞ is the recession functional of Λ (see [3]) and

$$\operatorname{dom}\Lambda = \{v \in P(\Omega) : \Lambda(v) < +\infty\},$$
$$\ker\Lambda^\infty = \{v \in P(\Omega) \text{ s.t. } \Lambda^\infty(v) = 0\},$$

then Λ achieves a finite minimim over T. ■

PROOF. See [3], [8]. ■

OUTLINE OF THE PROOF OF THEOREM 1. We check the assumptions of Theorem 2, with the choices $Y = P(\Omega)$ (which is the dual of a separable Banach space (see the paper [10] by R. Teman and G. Strang), $T = K_\Gamma$ that is a

(1) From now, on seq. and l.s.c. stand respectively for sequentially and lower semicontinuous.

convex cone, and

$$\begin{cases} \Lambda(v) = E(v) - Lv & \forall v \in K_\Gamma\,, \\ \Lambda(v) = +\infty & \forall v \in P(\Omega)\setminus K_\Gamma\,. \end{cases} \tag{2.17}$$

The compactness (2.14) is fulfilled. Assume to know that

$$\exists \delta > 0 : \Lambda^\infty(v) \geq \delta \int_\Omega |\varepsilon^D(v)| \qquad \forall v \in K_\Gamma\,, \tag{2.18}$$

then checking the necessary condition (2.15) is trivial and the compatibility (2.16) can be deduced by (2.11), (2.12) with the same argument used in Theorem 4 of [8].

So the only nontrivial assumptions to prove are: the inequality (2.18) and the fact that K_Γ is seq. w^* closed in $P(\Omega)$. This two points are shown in the following sections (see Lemma 3.1 and Lemma 4.3). ∎

3. – Closedness of the set of admissible deformations.

In $P(\Omega)$, which is a nonreflexive space the strongly closed convex sets are weakly closed but not necessarily closed. Hence a Signorini (or Dirichlet) boundary condition cannot be preserved in general by w^* convergence of sequences. An idea to overcome this difficulty could be to relax the constraint, say, trying to minimize

$$E(v) - Lv + \int_\Gamma (v^3)^- \, dx_1 \, dx_2$$

where $(v_3)^- = \max(0, -v_3)$. In fact this procedure has been widely used when minimizing functionals with linear growth at infinity: for instance, G. Anzellotti and M. Giaquinta (when dealing with a Dirichlet condition in a similar context (see Th. 2.3 of [1])) introduced a relaxation of the tangential component of the datum but proved that the normal component can be prescribed without relaxing it. Since in our case v_3 on Γ is exactly the normal component (up to the sign) of v, we can hope to impose the obstacle without any relaxation: and actually this works, due to the fact that the divergence lies in $L^2(\Omega)$.

LEMMA 3.1. *K_Γ is sequentially $w^* P$ closed.*

PROOF. Consider the Banach space (endowed with the natural norm)

$$L^{3/2}_{\mathrm{div}}(\Omega = \{v \in L^{3/2}(\Omega, \boldsymbol{R}^3) : \operatorname{div} v \in L^{3/2}(\Omega)\},$$
$$\|v\|_{L^{3/2}_{\mathrm{div}}(\Omega)} = \|v\|_{L^{3/2}(\Omega,\boldsymbol{R}^3)} + \|\operatorname{div} v\|_{L^{3/2}(\Omega)}$$

then

$$P \subset \{v \in L^{3/2}(\Omega, \boldsymbol{R}^3) : \operatorname{div} v \in L^2(\Omega)\} \subset L^{3/2}_{\mathrm{div}}(\Omega) \tag{3.1}$$

It is well known (about the notation and results for Sobolev functions we refer to [6]) that there is a (unique) trace operator γ defining the normal component along the boundary, such that, denoting by ν the ouward normal to $\partial\Omega$,

$$\begin{cases} \gamma : L^{3/2}_{\mathrm{div}}(\Omega) \to W^{-(2/3),(3/2)}(\partial\Omega), \\ \gamma v = v \cdot \nu, & \forall v \in C^{\infty}(\bar{\Omega}, \boldsymbol{R}^3) \\ \exists C : \|\gamma v\|_{W^{-(2/3),(3/2)}(\partial\Omega)} \leq C\|v\|_{L^{3/2}_{\mathrm{div}}(\Omega)}, & \forall v \in L^{3/2}_{\mathrm{div}}(\Omega). \end{cases} \tag{3.2}$$

Since the restriction maps $W^{-(2/3),(3/2)}(\partial\Omega)$ into $W^{-(2/3),(3/2)}(\Gamma)$, we get that

$$(\gamma v)|_{\Gamma} \in W^{-(2/3),(3/2)}(\Gamma,), \qquad \forall v \in L^{3/2}_{\mathrm{div}}(\Omega).$$

Coming back to our problem, abusing notation, we write simply v_3 instead of $-(\gamma v)|_{\Gamma}$. So

$$\|v_3\|_{W^{-(2/3),(3/2)}(\Gamma)} \leq C'\|v\|_{L^{3/2}_{\mathrm{div}}(\Omega)} \leq C''\|v\|_{P(\Omega)}$$

where C', C'' depend only on Ω, Γ.

Consider now a sequence v^n in K_{Γ}, $w^*\, P(\Omega)$ converging to some v. The Banach-Steinhaus Theorem together with (3.1) give that both $\|v^n\|_{P(\Omega)}$ and $\|v^n\|_{L^{3/2}_{\mathrm{div}}(\Omega)}$ are bounded uniformly in n. Since $L^{3/2}_{\mathrm{div}}(\Omega)$ is reflexive, we get, up to subsequences,

$$v^n \to v \quad \text{weakly in } L^{3/2}_{\mathrm{div}}(\Omega). \tag{3.3}$$

Set now

$$\mathcal{K}_{\Gamma} = \{v \in L^{3/2}_{\mathrm{div}}(\Omega) : v_3 \geq 0 \text{ in } \mathcal{D}'(\Gamma)\}.$$

$\mathcal{K}_{\Gamma}$ is obviously convex, and (due to (3.2)) strongly closed in $L^{3/2}_{\mathrm{div}}(\Omega)$. Hence $\mathcal{K}_{\Gamma}$ is also weakly closed in $L^{3/2}_{\mathrm{div}}(\Omega)$. From (3.1) we get

$$v^n \in \mathcal{K}_{\Gamma} \quad \forall n \tag{3.4}$$

(3.3) and (3.4) give

$$v \in \mathcal{K}_\Gamma . \tag{3.5}$$

But we already know that $v \in P(\Omega)$. Then $v \in K_\Gamma$. ∎

REMARK 3.2. Since $P(\Omega)$ too is endowed with a linear and strongly continuous trace operator $\tau: P(\Omega) \to L^1(\partial\Omega, \boldsymbol{R}^3)$ (see Theorem 1.5i of [5]), which is uniquely determined by the following

$$\begin{cases} \tau v = v|_\Omega & \forall v \in C^\infty(\bar{\Omega}, \boldsymbol{R}^3), \\ \int_\Omega (\varepsilon(v): \varphi + v \cdot \operatorname{div} \varphi) = \int_{\partial\Omega} (\tau v \otimes \nu): \varphi \\ \qquad\qquad \forall v \in BD(\Omega, \boldsymbol{M}^3), \forall \varphi \in C^1(\Omega, \boldsymbol{M}^3) \cap C^0(\bar{\Omega}, \boldsymbol{M}^3), \end{cases} \tag{3.6}$$

in the argument of the proof of Lemma 3.1 we used implicitely (referring to (3.2), (3.6)), the following identity

$$\gamma v = (\tau v) \cdot \nu \qquad \forall v \in P(\Omega) \tag{3.7}$$

and denoted by $-v_3$ their restriction to Γ.

The validity of (3.7) follows from the following argument. γ is unique due to (3.2) and the density of $C^\infty(\bar{\Omega}, \boldsymbol{R}^3)$. On the other hand $C^\infty(\bar{\Omega}, \boldsymbol{R}^3)$ is not dense in $P(\Omega)$ with respect to the strong topology. Nevertheless the space $C^\infty(\Omega, \boldsymbol{R}^3) \cap P(\Omega) \cap LD(\Omega)$ is dense with respect to an intermediate topology σ (here $LD(\Omega) = \{v \in L^1(\Omega, \boldsymbol{R}^3): \varepsilon(v) \in L^1(\Omega, \boldsymbol{M}^3)\}$, see Theorem II.3.4 of [9]):

$$v^n \xrightarrow{\sigma} v \quad \text{iff} \quad \left\{ \begin{array}{c} v^n \to v \quad \text{in } L^1(\Omega, \boldsymbol{R}^3) \\ \operatorname{div} v^n \to \operatorname{div} v \quad w L^2(\Omega) \\ \int_\Omega \varepsilon(v^n): \varphi \to \int_\Omega \varepsilon(v): \varphi \quad \forall \varphi \in \mathcal{D}(\Omega, \boldsymbol{M}^3) \\ \int_\Omega |\varepsilon(v^n)| \to \int_\Omega |\varepsilon(v)| \end{array} \right\}.$$

Moreover (see [9], Theorem II.3.1) the operator r is continuous (with values in $L^1(\partial\Omega, \boldsymbol{R}^3)$) with respect to the topology σ (while it is not weakly continuous). Then, for any $v \in P(\Omega)$ there is $\{v^n\}_n$ s.t. $v^n \in C^\infty(\Omega, \boldsymbol{R}^3) \cap P \cap LD$ and v^n σ converges to v; plugging $\varphi_{i,j} = \psi \delta_{i,j}$ in (3.6) and recalling that γ extends to $LD(\Omega)$ since $C^\infty(\bar{\Omega}, \boldsymbol{R}^3)$ is dense in $LD(\Omega)$ (see [9]; th. I.1.3)

$$\int_{\partial\Omega}(\tau v^n)\cdot\nu\psi = \int_{\Omega}(v^n\nabla\psi + \psi\operatorname{div} v^n) = \int_{\partial\Omega}(\gamma v^n)\psi, \qquad \forall n, \ \forall\psi\in C^\infty(\bar\Omega)$$

and passing to the limit we get (3.7). ■

For the sake of completeness we prove (3.2): more in general we give the following statement.

PROPOSITION 3.3. If $1\le p<+\infty$ there is a linear continuous map γ s.t.

$$\begin{cases}\gamma: L^p_{\mathrm{div}}(\Omega)\to W^{-(1/p),p}(\partial\Omega,\boldsymbol{R}^3), \\ \gamma v = v\cdot\nu, & \forall v\in C^\infty(\bar\Omega,\boldsymbol{R}^3) \\ \exists C \text{ s.t. } \|\gamma v\|_{W^{-(1/p),p}(\partial\Omega,\boldsymbol{R}^3)}\le C\|v\|_{L^p_{\mathrm{div}}(\Omega)}, & \forall v\in L^p_{\mathrm{div}}(\Omega).\end{cases} \tag{3.8}$$

PROOF. For any v in $C^\infty(\bar\Omega,\boldsymbol{R}^3)$ and w in $W^{(1/p),p'}(\partial\Omega)$ we have

$$\left|\int_{\partial\Omega}(v\cdot\nu)w\right| = \left|\int_{\Omega}(v\cdot\nabla w + w\operatorname{div} v)\right| \le \|v\|_{L^p}\|\nabla w\|_{L^{p'}} + \tag{3.9}$$
$$+\|\operatorname{div} v\|_{L^p}\|w\|_{L^{p'}} \le \|v\|_{L^p_{\mathrm{div}}}\|w\|_{W^{1,p'}}(\Omega) \le C\|v\|_{L^p_{\mathrm{div}}}\|w\|_{W^{(1/p),p'}(\partial\Omega)}$$

since there is a linear continuous extension map from $W^{(1/p),p'}(\partial\Omega)$ into $W^{1,p'}(\Omega)$. From (3.9), the density of $C^\infty(\bar\Omega,\boldsymbol{R}^3)$ in L^p_{div} and the Hahn-Banach Theorem, there is a linear map γ satisfying (3.8). ■

4. – A projection onto the rigid body motion preserving boundary inequalities.

The usual projections onto the space of rigid body motions

$$\boldsymbol{RBM} = \{v\in P(\Omega): v(x) = Ax+b, \ \forall x\in\Omega, \ A \text{ skew-symmetric}\} \tag{4.1}$$

that leave unchanged $\boldsymbol{RBM}$ (see [5], [9]) have to be modified, in order to preserve the boundary constraint. In fact such projections consist in averaging v and its moments over open subsets of Ω, but in our problem there is no nonempty 3 dimensional open ball where the constraint $v_3\ge 0$ holds. Moreover, averaging over Γ only (or any subset of Γ) would destroy and dependence on x_3.

DEFINITION 4.1. $\forall v \in P(\Omega)$, we define $\Pi v \in \mathbf{RBM}$ by

$$(\Pi v)(x) = b(v) + A(v)(x - \hat{x}) = b(v) + a(v) \wedge (x - \hat{x})$$

where $\hat{x}$ is the point satisfying (2.8), while $a(v)$, $b(v)$ in $\boldsymbol{R}^3$, and the skew-symmetric matrix $A(v)$ are given by the following definitions (from now on *no summation convention* is used and r is given by (2.8) too):

$$b_i(v) = \frac{1}{\pi r^2} \int_{\Gamma \cap B_r(\hat{x})} v_i(x)\, dx_1\, dx_2 \qquad i = 1, 2, 3,$$

$$A_{ii}(v) = 0 \qquad i = 1, 2, 3,$$

$$A_{ij}(v) = \frac{3}{r^3} \int_{S_k} [v_i|_{S_k} - b_i(v)]\, dx_i\, dx_j \qquad i < j,$$

$$A_{ij}(v) = -A_{ji}(v) \qquad i > j,$$

where $i \neq k \neq j$, and

$$S_k = B_r(\hat{x}) \cap \{(x - \hat{x})_k = 0\} \cap \{(x - \hat{x})_i > 0\} \cap \{(x - \hat{x})_j > 0\},$$

$v_i|_{S_k}$ is the trace of v_i on S_k from the side $\{(x - \hat{x})_k > 0\}$ ■

LEMMA 4.2. The map $\Pi \colon P(\Omega) \to \boldsymbol{RBM}$ of Definition 4.1 is a linear continuous map with respect to the strong topology, and

$$\Pi v = v \qquad \forall v \in \boldsymbol{RBM}. \tag{4.2}$$

Hence there is a constant $C(\Omega, \Gamma)$ such that

$$\|v - \Pi v\|_{L^{3/2}(\Omega, \boldsymbol{R}^3)} \leq C(\Omega, \Gamma) \int_\Omega |\varepsilon(v)|. \tag{4.3}$$

PROOF. The linearity is trivial. The continuity properties of the trace in $P(\Omega)$ entail

$$\int_\Omega |b(v)| \leq \frac{|\Omega|}{\pi r^2} \|v\|_{L^1(\partial\Omega, \boldsymbol{R}^3)} \leq C\|v\|_{BD(\Omega)},$$

$$\int_\Omega |A(v)(x - \hat{x})| \leq C'\Big(\|v\|_{L_1(\Gamma)} + \sum_{i \neq k} \|v_i|_{S_k}\|_{L^1(S_k)}\Big) \leq C''\|v\|_{BD(\Omega)}.$$

Since $\varepsilon(\Pi v) = 0 \ \forall v$, the continuity of Π follows from the above inequalities. Choose now $v \in \boldsymbol{RBM}$, say

$$v(x) = b + A(x - \hat{x}) \qquad \text{with } A \text{ skew-symmetric;} \tag{4.4}$$

we have to show that $\Pi v = v$.

$b(v)$ is defined by averaging $(b + A(x - \hat{x}))$ over $\Gamma \cap B_r(\hat{x})$; such set is contained in $\{(x - \hat{x})_3 = 0\}$ and is symmetric with respect to $(x - \hat{x})_h$ for $h = 1, 2$, while $(x - \hat{x}) \to A(x - \hat{x})$ is an odd function. Then $\int_{\Gamma \cap B_r(\hat{x})} A(x - \hat{x}) = 0$ and

$$b(v) = b. \tag{4.5}$$

Now fix $i \neq k \neq j, i < j$. By using (4.4), (4.5) and the fact that

$$\int_{S_k} (x - \hat{x})_j \, dx_i \, dx_j = \frac{r^3}{3} \qquad \text{for } i \neq j \neq k \neq i,$$

in the Definition 4.1, we get

$$A_{ij}(v) = \frac{3}{r^3} \int_{S_k} [v_i|_{S_k} - b_i(v)] \, dx_i \, dx_j = \frac{3}{r^3} \int_{S_k} \sum_{h=1}^{3} A_{ih}(x - \hat{x})_h \, dx_i \, dx_j =$$

$$\left(\int_{S_k} (x - \hat{x})_j \, dx_i \, dx_j \right)^{-1} \int_{S_k} A_{ij}(x - \hat{x})_j \, dx_i \, dx_j = A_{ij}.$$

Hence $A_{ij}(v) = A_{ij}$ for $i < j$, and summarizing

$$A_{ij}(v) = A_{ij} \qquad \forall i, j \tag{4.6}$$

(4.5), (4.6) together prove (4.2). Due to (4.2) and the continuity of Π, (4.3) is a consequence of Theorem 1.5ii of [5]. ■

LEMMA 4.3. The assumptions of Theorem 1 imply (2.18).

PROOF. Referring to (2.17) we have

$$\Lambda^\infty(v) = +\infty \qquad \forall v \text{ s.t. div } v \not\equiv 0.$$

So, it is enough proving (2.18) $\forall v \in K_\Gamma$ s.t. $\operatorname{div} v \equiv 0$. Obviously

$$b_3(v) \geqslant 0 \qquad \forall v \in K_\Gamma. \tag{4.7}$$

Now (2.11), (2.13), (4.3) and (4.7) give

$$(4.8)\qquad \int_\Omega l\cdot v = \int_\Omega l\cdot(v-\Pi v) + \int_\Omega l\cdot[b(v)+a(v)\wedge(x-\hat{x})]$$

$$\leq \|l\|_{L^3(\Omega,\mathbf{R}^3)}\|v-\Pi v\|_{L^{(3/2)}(\Omega,\mathbf{R}^3)} \leq C(\Omega,\Gamma)\|l\|_{L^3(\Omega,\mathbf{R}^3)}\int_\Omega|\varepsilon(v)|$$

$$= C(\Omega,\Gamma)\|l\|_{L^3(\Omega,\mathbf{R}^3)}\int_\Omega|\varepsilon^D(v)| \qquad \forall v\in K_\Gamma \text{ s.t. } \operatorname{div} v\equiv 0.$$

The safe load condition (2.13) and (4.8) together give (2.18). ■

REMARK 4.4. The operator Π introduced by Definition 4.1 preserves not only inequalities on Γ, but also inclusions:

given any closed convex subset $Q\subset\mathbf{R}^3$, one gets

$$v(x)\in Q \text{ a.e. } x\in\Gamma \Rightarrow b(v)\in Q. \qquad ■$$

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ABSTRACT

Sufficient conditions are given, in order to have an equilibrium displacement field for an elasto-plastic body satisfying a constraint at the boundary.

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