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Some Remarks on the de Franchis Theorem.

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1. - Introduction.

Let X be a smooth projective curve on $\mathbf{C}$ of genus $g \geq 2$ and let Hol(X) be the set of all surjective holomorphic maps between X and another curve of genus of $g \geq 2$; the classical de Franchis theorem (see [dF] and [S]) assures that Hol(X) is finite.

In 1983 Howard and Sommese gave an explicit, but not sharp, bound for Hol(X):

$$\left(2\sqrt{6}(g-1)+1\right)^{2+2g^2} g^2(g-1)\left(\sqrt{2}\right)^{g(g-1)} + 84(g-1).$$

Notice that a polynomial bound is not possible: in fact in 1986, (see [K]), Kani proved that the size of Hol(X) can not be limited by a polynomial in g and showed that it is bounded by:

$$(g-1)\,2^{2g^2-2}(2^{2g^2-1}-1).$$

In this paper (Th. (4.1)) we improve these bounds. More precisely: let $[x]$ denote the integer part of a real number x, we show that Hol(X) can be bounded by:

$$\exp\left\{(4/3)\ \log(3)(g^2-1) + [\log_2(g)]\ \log(84g) + \log\left(12\sqrt{2}\right)\right\}.$$

We remark that the leading term in the previous expression is: $\exp[(4/3)\ \log(3)g^2]$; while in Kani's bound the leading term was: $\exp[4\log(2)g^2]$.

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Our technique is based on the following idea: firstly we consider only the *primitive* maps; roughly speaking they are maps, from X to another curve, which are not isomorphisms and do not factorize through other maps, (see Definition (3.1)).

We obtain a bound for the number of these maps: $\exp[g^2 \log(3)]$, (see Th. (3.13)), in the following way: we send the primitive maps, injectively, in $H_{1,1}(X^{(2)}, \boldsymbol{R})$, where $X^{(2)}$ is the symmetric product of X; so we get a finite number of homology classes with some numerical properties. We give a metric to $H_{1,1}(X^{(2)}, \boldsymbol{R})$ so that these homology classes become points in a ball of an euclidean space, (the idea is due to Howard and Sommese, but they use the Cartesian product instead of $X^{(2)}$ so they work in a higher dimensional space). So we have to consider a packing problem which we solve by following the method of Kani, (see (3.12)).

From the number of primitive maps we get a bound for the size of $\mathrm{Hol}(X)$ by counting all possible factorizations and isomorphisms in a final calculation, (see §4).

We do not think that our result is the better one: for instance see (4.3) and also remark that the endomorphisms of $J(X)$ induced by maps are very special, by admitting a big kernel; therefore our homology classes are very special too, but we have never considered this fact. Anyway, to obtain a truly better bound, we think that the strategy would be to characterize the endomorphisms of $J(X)$ becoming from maps and not to characterize these homology classes.

2. - Notation and conventions.

curve:	by this term we mean a projective smooth curve on $\boldsymbol{C}$
$X/\sim$:	quotient curve of X by the equivalence relation «$\sim$»
$X \times X$:	Cartesian product of the curve X with itself
F_i:	fibre over a generic point of the i-th factor of $X \times X$
Δ:	diagonal of $X \times X$
T_f:	$\{(P, Q) \in X \times X \mid f(P) = f(Q)\}$ where f is a holomorphic map between two curves X and Y
S_f:	$T_f - \Delta$
$X^{(2)}$:	symmetric product of the curve X with itself
π:	double covering map between $X \times X$ and $X^{(2)}$
R_f:	$\pi_*(S_f)$
b_f:	total branching index of f
d_f:	$\deg(f)$.

In this paper all maps are supposed to be non constant and all curves are supposed to have genus bigger than or equal to 2.

If D and E are divisors on a surface, DE will be their intersection.

3. – Primitive maps.

DEFINITION (3.1). Let $\mathscr{P}(X)$ be the set of all holomorphic maps f between a fixed curve X and another curve Y, $Y \neq X$, such that there is not a third curve Z, $Z \neq X$ and $Z \neq Y$, and two holomorphic maps $h_1: X \to Z$ and $h_2: Z \to Y$ such that $f = h_2 \circ h_1$.

We introduce the following equivalence relation in $\mathscr{P}(X)$: two maps f and h of $\mathscr{P}(X)$, between X and another distinct curve X', are equivalent if there exists such a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X' \\ g \updownarrow & & \updownarrow k \\ X & \xrightarrow[\ h\ ]{} & X' \end{array}$$

in which the vertical arrows are isomorphisms, (eventually the identities on X or X').

The set $P(X)$ of the primitive maps will be the set $\mathscr{P}(X)$ modulo the previously defined equivalence relation; so that when we say: «f is a primitive map» we mean that $f \in \mathscr{P}(X)$ is a representative of its equivalence class in $P(X)$.

REMARK (3.2). The map $P(X) \to \mathrm{Div}(X \times X)$ which associates the divisor S_f to every primitive map f, is well defined.

PROPOSITION (3.3). *Let f and h be two maps between X and another curve; if there exists a divisor D of $X \times X$ which is common to S_f and S_h, then f and h are not primitive maps.*

PROOF. We consider the following relation «$\sim$» on X: $P \sim Q$ if and only if $P = Q$ or $(P, Q) \in D$ $\forall P, Q \in X$. As D is a non empty, effective, symmetric divisor of $X \times X$, «$\sim$» is a non trivial equivalence relation on X. If we define $Y = X/\sim$ we get a map $q: X \to Y$; it is easy to see that f and h factorize through q. □

COROLLARY (3.4). *Let f and h be two primitive maps between X and another curve; then $S_f S_h \geq 0$. If we consider the two classes $[S_f]$ and $[S_h]$ in $H_{1,1}(X \times X, \mathbb{Z})$ we also have $[S_f][S_h] = S_f S_h \geq 0$.*

COROLLARY (3.5). *The map* $\{f \in P(X)\} \to \{[S_f] \in H_{1,1}(X \times X, \boldsymbol{Z})\}$ *is injective* (see also [H-S], Lemma 2).

PROOF. If $[S_f] = [S_h]$ for two distinct primitive maps f and h then: $[S_f][S_h] = [S_f][S_f] = (S_f)^2 = (d_f - 1)(2 - 2g) + (d_f - 2)b_f < 2 - 2g < 0$ (see [H-S], p. 431), which is a contradiction to Corollary (3. 4). □

COROLLARY (3.6). *The map* $\{f \in P(X)\} \to \{[R_f] \in H_{1,1}(X^{(2)}, \boldsymbol{Z})\}$ *is injective.* (We recall that $R_f = \pi_*(S_f)$).

PROOF. Let A, B be two symmetric divisors of $X \times X$, we have that $2\{\pi_*(A)\pi_*(B)\} = AB$; therefore $R_f R_h \geqslant 0$ if and only if $S_f S_h \geqslant 0$. Then we can proceed as in the proof of (3.5). □

PROPOSITION (3.7). *Let F be $\pi_*(F_1) = \pi_*(F_2)$, the standard hyperplane of $X^{(2)}$; we have $R_f F = d_f - 1$* (see [H-S], p. 430). *The map*

$$\{f \in P(X)\} \to ([R_f - (d_f - 1)F] \in H_{1,1}(X^{(2)}, \boldsymbol{Z})\}$$

is injective and, obviously, $(R_f - (d_f - 1)F)F = 0$.

PROOF. We have only to show that, if f and h are two distinct primitive maps, then $[R_f - (d_f - 1)F] \neq [R_h - (d_h - 1)F]$. If not, we would have

$$[R_f] = [R_h + (d_f - d_h)F] \quad \text{and} \quad R_f(R_h + (d_f - d_h)F) = (R_f)^2 < 0,$$

hence R_f and $R_h + (d_f - d_h)F$ would have common components, hence S_f and $S_h + (d_f - d_h)(F_1 + F_2)$ would have common components; but it is impossible because S_f and S_h have no common components as f and h are primitive, and S_f, S_h do not contain neither F_1 nor F_2, otherwise f or h would be constant maps. □

REMARK (3.8). (3.5), (3.6), (3.7) are true even if we substitute the usual homology classes with real homology classes.

From now on we confuse the divisors D, S_f, R_f etc. with their homology classes.

We recall that Howard and Sommese consider $H_{1,1}(X \times X, \boldsymbol{R})$ and they give a metric to it (see [H-S], p. 433). We can give a similar metric to $H_{1,1}(X^{(2)}, \boldsymbol{R})$; we remark that $F^2 = 1 > 0$; by the Hodge index theorem (see [G-H], p. 472) we must have $D^2 < 0$ for every real non zero homology class of type (1.1) which satisfies $FD = 0$.

For every $D \in H_{1,1}(X^{(2)}, \boldsymbol{R})$ we set: $D = D_1 + D_2$, where $D_1 = (DF)F$ and $D_2 = D - D_1$, (so that $D_2 F = 0$); the norm $\|D\|$ is defined by:

$$\|D\|^2 = (D_1)^2 - (D_2)^2 .$$

It is a norm since $(D_1)^2 \geqslant 0$ and $-(D_2)^2 \geqslant 0$, with both zero implying $D_1 = 0$ (since $F^2 > 0$) and $D_2 = 0$ by the Hodge theorem. Let m be the dimension of $H_{1,1}(X^{(2)}, \boldsymbol{R})$ as a real, normed, vector space, we call O its origin. For every class R_f, the class $R_f - (d_f - 1)F$ is the orthogonal projection of R_f onto the hyperplane which is orthogonal to the class F and passes through O.

From now on we put: $V = H_{1,1}(X^{(2)}, \boldsymbol{R})$ and we call vectors the classes of V. We have the following:

LEMMA (3.9). *For every vector R_f (f being a primitive map as usual), the angle between R_f and F (in V) is always less than a right angle.*

PROOF. We only need to calculate the cosine of this angle by the previously defined metric. It is: $(d_f - 1)/\|R_f\|\,\|F\| > 0$. □

LEMMA 3.10. *On every line passing through O in V, there exists at most one vector R_f.*

PROOF. Suppose that for two distinct primitive maps f and h we have: $R_h = aR_f$ for some $a \in \boldsymbol{R}$, $a \neq 0$. Then: $0 \leqslant R_f R_h = a(R_f)^2$, but $(R_f)^2 < 0$, (see [H-S] p. 431 and recall the proof of Corollary (3.6)) hence $a < 0$: it is impossible by Lemma (3.9). □

PROPOSITION (3.11). *Let f, h be two distinct primitive maps, then the cosine of the angle between the two vectors $R_f - (d_f - 1)F$ and $R_h - (d_h - 1)F$ in V is less than $1/2$.*

PROOF. We calculate:

$$-[R_f - (d_f - 1)F][R_h - (d_h - 1)F]/\|R_f - (d_f - 1)F\|\,\|R_h - (d_h - 1)F\| \leqslant -R_f R_h +$$

$$+(d_f - 1)(d_h - 1)/\{[(g-1)(d_f - 1) + (d_f - 1)^2][(g-1)(d_h - 1) + (d_h - 1)^2]\}^{1/2} <$$

$$< (d_f - 1)(d_h - 1)/\{[2(d_f - 1)^2][2(d_h - 1)^2]\}^{1/2} = 1/2. \qquad \square$$

Now we need the following:

PACKING LEMMA (3.12) (see [K] p. 194), *Suppose a and b are real numbers with $b < 1$ and $2(1 - a^2) \geqslant 1 - b$, and let $v_0, v_1, \dots, v_N$ be a finite sequence of non zero vectors of a real euclidean vector space V, whose dimension is m.*

Suppose that:

i) $\cos(v_0, v_i) = a$, *for* $1 \leq i \leq N$,

ii) $\cos(v_i, v_j) \leq b$, *for* $1 \leq i < j \leq N$.

Then, putting $c = [2(1-a^2)/(1-b)]^{1/2}$, *we have:*

$$N \leq (c+1)^{m-1} - (c-1)^{m-1}.$$

Now we are ready to prove our:

THEOREM (3.13). *The number of elements of* $P(X)$ *is less than* $\exp[g^2 \log(3)]$.

PROOF. By Proposition (3.7) and Lemma (3.10) we can say that the number of primitive maps, from X to another curve, is less than the cardinality N of a certain set of non zero vectors in V, no two of them lying on the same line, each of them belonging to the hyperplane orthogonal to F which passes through O.

By Proposition (3.11) we can apply the packing Lemma (3.12) to this set of vectors of V with the adjunction of $v_0 = F$. We have: $a = 0$ and $b = 1/2$ and $m = g^2 + 1$, because $V = H_{1,1}(X^{(2)}, \mathbf{R})$ is the Poincarè dual of $H^{1,1}(X^{(2)}, \mathbf{R})$, whose dimension is $g^2 + 1$. So our theorem follows from (3.11). □

4. - The main theorem.

THEOREM 4.1. *The number of holomorphic maps from a curve X to another curve, (both of genus bigger than or equal to 2) is less than:*

$$\exp\left\{(4/3)\ \log(3)(g^2-1) + [\log_2(g)]\ \log(84g) + \log\left(12\sqrt{2}\right)\right\}.$$

PROOF. By (3.13) we know that the number of surjective maps between X and another curve Y, $Y \neq X$, eventually composed by an isomorphism of X, but not by an isomorphism of Y, is less than:

$$\exp[\log(3) g^2]\, 84(g-1).$$

The target curve has genus $g/2$ at most, we can repeat the previous calculation for this curve and so on.

We put $s = [\log_2(g) - 1]$ if g is not a power of 2, $s = \log_2(g) - 2$ if g is a power of 2. A map from X can be broken out into $s+1$ different primitive maps at most; at each level we must compute all possible isomorphisms. So that the number we are looking for is less than:

$$\{\exp[\log(3)g^2]\}\,84(g-1)\cdot\{\exp[\log(3)(g/2)^2]\}\,84[(g/2)-1]\cdot\ldots$$
$$\ldots\cdot\{\exp[\log(3)(g/2^s)^2]\}\,84[(g/2^s)-1)]\cdot 48.$$

The last factor 48 is necessary to count the isomorphisms of genus 2 curve, they are 48 at most, (see [ACGH] p. 46).

Now by looking at:

$$g^2+(g/2)^2+(g/4)^2+\ldots+(g/2^s)^2<(4/3)(g^2-1),$$

$$(g-1)(g/2-1)(g/4-1)\ldots(g/2^s-1)=$$
$$=(g-1)(g-2)(g-4)\ldots(g-2^s)\,2^{s(s+1)/2}<g^{s+1}/\sqrt{8}$$

we get our assertion. □

REMARK (4.2). For g small (e.g. for $g=3$) the previous calculation can be sharpened, but we recall that we are looking for a bound true for *all* g.

REMARK (4.3). We believe that our bound for primitive maps is good; in spite of this, our final bound is still not sharp: in fact look at the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X'' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ X' & \xrightarrow[\ h\ ]{} & X''' \end{array}$$

in which X, X', X'' and X''' are four *distinct* curves, and f, g, h, k are four holomorphic maps; in our calculation the map $h\circ g=k\circ f$ counts twice, and we are not able to control this case.

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SUMMARY

Let X be a smooth projective curve defined on $\boldsymbol{C}$. The number of holomorphic maps from a fixed X to another curve, (both of genus bigger than or equal to two), is finite by the classical de Franchis theorem. In this paper we get an explicit bound for this number, depending on the genus of X only. Our bound is better than all the previously given ones (by Howard-Sommese and Kani).

SOMMARIO

Sia X una curva liscia proiettiva definita su C. Il numero delle applicazioni olomorfe esistenti tra una X fissata ed un'altra curva, (entrambe di genere maggiore od uguale a due), è finito in base al classico teorema di de Franchis. In questo lavoro noi otteniamo, per tale numero, un limite superiore esplicito, dipendente solo dal genere di X. La nostra stima è migliore di tutte quelle date precedentemente (da Howard-Sommese e da Kani).

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