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On the zeros of polynomials

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Abstract. In this paper we extend a classical result due to Cauchy and its improvement due to Datt and Govil to a class of lacunary type polynomials.

Keywords. Moduli of zeros; ring shaped region; lacunary type polynomials.

1. Introduction and statement of results

A classical result due to Cauchy [1] concerning the bounds for the moduli of the zeros of a polynomial $P(z)$ can be stated as

Theorem A. *If*

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n and

$$M = \max|a_j|, \quad j = 0, 1, 2, \ldots, n-1,$$

then all the zeros of $P(z)$ *lie in a circle*

$$|z| \leqslant 1 + M. \tag{1}$$

In the literature [3–6], there exists some improvements and generalizations of Cauchy's theorem. Recently Datt and Govil [2] have obtained the following improvement to Theorem A.

Theorem B. *If*

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n and

$$A = \max|a_j|, \quad j = 0, 1, 2, \ldots, n-1,$$

then all the zeros of $P(z)$ *lie in a ring shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}(1+An)} \leqslant |z| \leqslant 1 + \lambda_0 A \tag{2}$$

where λ_0 *is the unique root of the equation*

$$x = 1 - \frac{1}{(1+Ax)^n}$$

in the interval $(0, 1)$. *The upper bound in* (2) *is best possible and is attained for the* *polynomial*

$$P(z) = z^n - A(z^{n-1} + z^{n-2} + \cdots + z + 1).$$

The purpose of this paper is to extend the above results for a class of lacunary type polynomials. We start by proving the generalization of Theorem A.

Theorem 1. *If*

$$P(z) = a_n z^n + a_p z^p + \cdots + a_1 z + a_0, \quad 0 \leqslant p \leqslant n-1$$

is a polynomial of degree n and

$$M = \max \left|\frac{a_j}{a_n}\right|, \quad j = 0, 1, \ldots, p,$$

then all the zeros of $P(z)$ lie in $|z| < K$, where K is a unique positive root of the trinomial equation

$$x^{n-p} - x^{n-p-1} - M = 0. \tag{3}$$

For $p = n-1$, this reduces to Theorem A.

The following corollary is obtained by taking $p = n-2$ in Theorem 1.

COROLLARY 1

If

$$P(z) = a_n z^n + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n and

$$M = \max \left|\frac{a_j}{a_n}\right|, \quad j = 0, 1, \ldots, n-1,$$

then all the zeros of $P(z)$ lie in the circle

$$|z| < \frac{1 + \sqrt{(1 + 4M)}}{2}.$$

From Corollary 1, we can easily deduce Corollary 2.

COROLLARY 2

If

$$P(z) = a_n z^n + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n, such that

$$|a_j| \leqslant |a_n|, \quad j = 0, 1, 2, \ldots, n-2,$$

then all the zeros of $P(z)$ lie in

$$|z| \leqslant \frac{1 + \sqrt{5}}{2}.$$

Next we present the following generalization of Theorem B.

Theorem 2. *If*

$$P(z) = z^n + a_p z^p + \cdots + a_1 z + z_0, \quad 0 \leqslant p \leqslant n-1$$

is a polynomial of degree n and

$$A = \max|a_j|, \quad j = 0, 1, \ldots, p$$

then $P(z)$ has all its zeros in the ring shaped region

$$\frac{|a_0|}{2(1+A)^{n-1}\{1+(p+1)A\}} \leqslant |z| \leqslant 1 + \alpha_0 A \tag{4}$$

where α_0 is the unique root of the equation

$$x = 1 - \frac{1}{(1+Ax)^{p+1}}$$

in the interval $(0, 1)$.

The upper bound $1 + \alpha_0 A$ in (4) is best possible and is attained for the polynomial

$$P(z) = z^n - A(z^p + z^{p-1} + \cdots + z + 1).$$

2. Lemma

For the proof of Theorem 2, we need the following lemma.

Lemma. *Let*

$$f(x) = x - \left[\frac{1}{(1+Ax)^{n-p-1}} - \frac{1}{(1+Ax)^n}\right]$$

where n is a positive integer and $A > 0$. If $(p+1)A > 1$, then $f(x)$ has a unique root in the interval $(0, 1)$.

Proof of the Lemma. Consider

$$\begin{aligned}(1+Ax)^n f(x) &= (1+Ax)^n x - (1+Ax)^{p+1} + 1, \quad \text{where } p \leqslant n-1 \\ &= \binom{n}{0} x + \binom{n}{1} Ax^2 + \binom{n}{2} A^2x^3 + \cdots + \binom{n}{n} A^n x^{n+1} \\ &\quad - \left\{\binom{p+1}{1} Ax + \binom{p+1}{2} A^2x^2 + \cdots \right. \\ &\quad \left. + \binom{p+1}{p} A^p x^p + (Ax)^{p+1}\right\} \\ &= (1-(p+1)A)x + \sum_{k=2}^{p+1} \frac{(p+1)!\,A^{k-1}x^k}{k!(p-k+2)!} \\ &\quad \times \left\{k\left(\frac{n(n-1)\cdots(n-k+2)}{(p+1)p\cdots(p-k+3)} + A\right) - (p+2)A\right\} \\ &\quad + \sum_{k=p+2}^{n+1} \binom{n}{k-1} A^{k-1} x^k\end{aligned}$$

$$=(1-(p+1)A)x+\sum_{k=2}^{p+1}\frac{(p+1)!A^{k-1}x^k}{k!(p-k+2)!}\{k(A+l_k)-(p+2)A\}$$

$$+\sum_{k=p+2}^{n+1}\binom{n}{k-1}A^{k-1}x^k, \tag{5}$$

where $l_k=\dfrac{n(n-1)\cdots(n-k+2)}{(p+1)p\cdots(p-k+3)}\geqslant 1$ for all $k=2,3,\ldots,p+1$, as $p\leqslant n-1$. Since $1-(p+1)A<0$, the coefficients of $x^{p+2},\ldots,x^{n+1}$ are positive and $k(A+l_k)-(p+2)A$ are monotonically increasing for $k=2,3,\ldots,p+1$, it follows from Descartes rule of signs that $(1+Ax)^n f(x)=0$ has exactly one positive root. Since

$$f(x)=x-\left[\frac{1}{(1+Ax)^{n-p-1}}-\frac{1}{(1+Ax)^n}\right]$$

then

$$f'(x)=1-\left[\frac{(1+p-n)A}{(1+Ax)^{n-p}}+\frac{nA}{(1+Ax)^{n+1}}\right]. \tag{6}$$

If $(p+1)A>1$, then it is clear from (6) that $f'(0)<0$. Thus there exists a $\delta>0$ such that $f'(x)<0$ in $(0,\delta)$. Also $f(1)>0$, hence $f(x)=0$ has one and only one positive root in $(0,1)$ and the lemma follows.

3. Proof of the theorems

Proof of Theorem 1. We have

$$P(z)=a_nz^n+a_pz^p+\cdots+a_1z+a_0,\quad 0\leqslant p\leqslant n-1,$$

so that for $|z|>1$,

$$|P(z)|\geqslant|a_n||z|^n\left\{1-\left(\frac{|a_p|}{|a_n|}\frac{1}{|z|^{n-p}}+\cdots+\frac{|a_1|}{|a_n|}\frac{1}{|z|^{n-1}}+\frac{|a_j|}{|a_n|}\frac{1}{|z|^n}\right)\right\}.$$

Since $|(a_j/a_n)|\leqslant M\ \forall j=0,1,2,..,n-1$, it follows that

$$|P(z)|\geqslant|a_n||z|^n\left\{1-\frac{M}{|z|^{n-p}}\left(1+\frac{1}{|z|}+\frac{1}{|z|^2}+\cdots+\frac{1}{|z|^p}\right)\right\}$$

$$>|a_n||z|^n\left\{1-\frac{M}{|z|^{n-p}}\left(1+\frac{1}{|z|}+\cdots\right)\right\}$$

$$=|a_n||z|^n\left\{1-\frac{M}{|z|^{n-p-1}}\cdot\frac{1}{(|z|-1)}\right\}$$

$$\geqslant 0,$$

if

$$|z|^{n-p}-|z|^{n-p-1}-M\geqslant 0.$$

This implies

$$|P(z)|>0\quad\text{if } |z|\geqslant K,$$

where K is the (unique) positive root of the trinomial equation defined by (3) in $(1,\infty)$.

Hence all the zeros of $P(z)$ whose modulus is greater than 1 lie in $|z| < K$. Since all those zeros whose modulus is less or equal to 1 already lie in $|z| < K$, the desired result follows.

Proof of Theorem 2. We shall first prove that $P(z)$ has all its zeros in $|z| \leqslant 1 + \alpha_0 A$, and for this it is sufficient to consider the case when $(p+1)A > 1$ (for if $(p+1)A \leqslant 1$, then on $|z| = R > 1$, $|P(z)| \geqslant R^n - (p+1)A|z|^p \geqslant R^n - R^p > 0$).

If

$$A = \max|a_j|, \quad j = 0, 1, \ldots, p$$

and

$$P(z) = z^n + a_p z^p + a_{p-1} z^{p-1} + \cdots + a_0, \quad 0 \leqslant p \leqslant n-1,$$

then

$$\begin{aligned} |P(z)| &\geqslant |z|^n \left\{1 - \left(\frac{|a_p|}{|z|^{n-p}} + \frac{|a_{p-1}|}{|z|^{n-p+1}} + \cdots + \frac{|a_0|}{|z|^n}\right)\right\} \\ &\geqslant |z|^n \left\{1 - \frac{A}{|z|^{n-p}}\left(\frac{|z|^{p+1} - 1}{(|z|-1)|z|^p}\right)\right\} \\ &= |z|^n - A\left(\frac{|z|^{p+1} - 1}{|z| - 1}\right). \end{aligned}$$

Hence for every $\alpha > 0$, we have on $|z| = 1 + A\alpha$,

$$|P(z)| \geqslant (1 + A\alpha)^n - \frac{(1 + A\alpha)^{p+1} - 1}{\alpha} > 0$$

if

$$\alpha(1 + A\alpha)^n > (1 + A\alpha)^{p+1} - 1$$

which implies

$$\begin{aligned} \alpha &> \frac{(1 + A\alpha)^{p+1}}{(1 + A\alpha)^n} - \frac{1}{(1 + A\alpha)^n} \\ &= \frac{1}{(1 + A\alpha)^{n-p-1}} - \frac{1}{(1 + A\alpha)^n}. \end{aligned} \tag{7}$$

Thus if α_0 is the unique root of the equation

$$x = \frac{1}{(1 + Ax)^{n-p-1}} - \frac{1}{(1 + Ax)^n}, \quad \text{(by above lemma)},$$

in $(0, 1)$, then every $\alpha > \alpha_0$ satisfies (7) and hence $|P(z)| > 0$ on $|z| = 1 + A\alpha$ which implies that $P(z)$ has all its zeros in

$$|z| \leqslant 1 + A\alpha_0. \tag{8}$$

Next we prove that $P(z)$ has no zero in

$$|z| < \frac{|a_0|}{2(1 + A)^{n-1}\{1 + (p+1)A\}}.$$

Let us denote the polynomial $g(z)$ by $(1 - z)P(z)$, then

$$\begin{aligned} g(z) &= a_0 + \sum_{j=1}^{p} (a_j - a_{j-1}) z^j + z^n - a_p z^{p+1} - z^{n+1} \\ &= a_0 + h(z) \quad \text{(say)} \end{aligned}$$

if

$$R = 1 + A$$

then

$$\begin{aligned}\max_{|z|=R} |h(z)| &\leqslant R^{n+1} + R^n + |a_p| R^{p+1} + \sum_{j=1}^{p} |a_j - a_{j-1}| R^j \\ &\leqslant R^n [R + 1 + A + 2Ap] \\ &= 2R^n [R + Ap] \\ &= 2(1+A)^n [1 + (p+1)A]. \end{aligned} \tag{9}$$

Hence on $|z| \leqslant R$,

$$\begin{aligned}|g(z)| = |a_0 + h(z)| &\geqslant |a_0| - |h(z)| \\ &\geqslant |a_0| - \frac{|z|}{(1+A)} \max_{|z|=R>1} |h(z)| \quad \text{(by Schwarz lemma)} \\ &\geqslant |a_0| - \frac{|z|}{(1+A)} 2(1+A)^n \{1 + (p+1)A\} \quad \text{(by (9))} \\ &> 0,\end{aligned}$$

if

$$|z| < \frac{|a_0|}{2(1+A)^{n-1}[1 + (p+1)A]}.$$

Hence all the zeros of $P(z)$ lie in

$$|z| \geqslant \frac{|a_0|}{2(1+A)^{n-1}[1 + (p+1)A]}. \tag{10}$$

Combining (8) and (10), we get all the zeros of $P(z)$ to be in the ring shaped region

$$\frac{|a_0|}{2(1+A)^{n-1}[1 + (p+1)A]} \leqslant |z| \leqslant 1 + A\alpha_0.$$

This completes the proof of Theorem 2.

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