ASYMPTOTIC EXPANSION OF THE MULTIVARIATE BERNSTEIN POLYNOMIALS ON A SIMPLEX

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Abstract

In this note we study the local behaviour of the multi-variate Bernstein polynomials B_n on the d-dimensional simplex $S \subset \mathbb{R}^d$. For function f admitting derivatives of sufficient high order in $x \in S$ we derive the complete asymptotic expansion of $B_n f$ as n tends to infinity. All the coefficients of n^{-k} that only depend on f and x are calculated explicitly. It turns out that combinatorial numbers play an important role. Our results generalize recent formulae due to R. Zhang in a way.

1 Introduction

For each function f defined on $[0, 1]$, the classical Bernstein polynomials $B_n (n=0,1,2,\cdots)$ are given by

$$B_n(f;\ x) = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} f\left(\frac{\nu}{n}\right), \qquad x \in [0,\ 1]. \tag{1}$$

Bernstein[8] already proved that, for $q \in \mathbb{R}$ and $f \in C^{2q}[0,1]$, the univariate Bernstein polynomials satisfy the asymptotic relation

$$B_n(f;\ x) = f(x) + \sum_{s=1}^{2q} \frac{T_{n,s}(x)}{s!\,n^s} f^{(s)}(x) + o(n^{-q}), \qquad n \to \infty, \tag{2}$$

where

$$T_{n,s}(x) = \sum_{\nu=0}^{n}(\nu - nx)^s\binom{n}{\nu}x^\nu(1-x)^{n-\nu}, \qquad s = 0,1,2\cdots.$$

The drawback of formula (2) is that the terms $T_{n,s}(x)$ contain the parameter n in a very implicit manner. The Bernstein polynomials B_n possess a complete asymptotic expansion of the form

$$B_n(f;\, x) \sim f(x) + \sum_{k=1}^{\infty}c_k(f;\, x)n^{-k}, \qquad n \to \infty, \tag{3}$$

provided f admits derivatives of sufficiently high order at $x \in [0,\, 1]$. All coefficients $c_k(f;\, x)$ are independent on n. Foumula (3) means that for all $m=1,2,\cdots$, there holds

$$B_n(f;\, x) = f(x) + \sum_{k=1}^{m}c_k(f;\, x)n^{-k} + o(n^{-m}), \qquad n \to \infty.$$

In a very recent paper[19] R. Zhang presented a pointwise asymptotic expansion for Bernstein Polynomials on a triangle in terms of the two-dimensional generalization of $T_{n,s}(x)$.

The aim of this paper is to derive a multi-dimensional version of Eq. (3) for multivariate Bernstein polynomials. All the coefficients of n^{-k} that only depend on f and x are calculated explicitly. It turns out that combinatorial numbers play an important role.

Let $S \subset \mathrm{R}^d (d \in \mathrm{N})$ be the simplex defined by

$$S = \{\mathbf{x} \in \mathrm{R}^d : x_i \geqslant 0 \ (i = 1,\cdots,d),\ 1 - |\mathbf{x}| \geqslant 0\}.$$

Throughout this paper, for $\mathbf{x}=(x_1,\cdots,x_d) \in \mathrm{R}^d$ and $\mathbf{k}=(k_1,\cdots,k_d) \in \mathrm{N}_0^d$, we denote as usual

$$|\mathbf{x}| = \sum_{i=1}^{d}x_i, \qquad \mathbf{x}^{\mathbf{k}} = \prod_{i=1}^{d}x_i^{k_i},$$

$$|\mathbf{k}| = \sum_{i=1}^{d}k_i, \qquad \mathbf{k}! = \prod_{i=1}^{d}k_i!$$

and, for $\mathbf{k}$, $\mathbf{m} \in \mathrm{R}^d$, we write $\mathbf{k} \leqslant \mathbf{m}$ iff there holds $k_i \leqslant m_i (i=1,\cdots,d)$. For $n \in \mathrm{N}_0$, $\mathbf{k} \in \mathrm{N}_0^d$, we put $\binom{n}{\mathbf{k}} = n^{\underline{|\mathbf{k}|}}/\mathbf{k}!$, where $x^{\underline{m}} = x(x-1)\cdots(x-m+1)$, $x^{\underline{0}}=1$ denotes the falling factorial.

For each $n \in \mathrm{N}_0$ and $f:\, \mathscr{S} \to \mathrm{R}$, the multivariate Bernstein polynomial on the simplex $\mathscr{S}$ is defined by

$$B_n(f;\, \mathbf{x}) = \sum_{|\mathbf{k}| \leqslant n}p_{n,\mathbf{k}}(\mathbf{x})f\left(\frac{\mathbf{k}}{n}\right), \qquad \mathbf{x} \in \mathscr{S}, \tag{4}$$

where $p_{n,k}(\mathbf{x}) = \binom{n}{\mathbf{k}}\mathbf{x}^{\mathbf{k}}(1-|\mathbf{x}|)^{n-\mathbf{k}}$ and $\alpha\mathbf{x} = (\alpha x_1,\cdots,\alpha x_d)$, for $\alpha \in \mathrm{R}$.

It is obvious that in the special case $d=1$ we obtain the well-known univariate Bernstein polynomials (1). The case $d=2$ are the Bernstein polynomials on a triangle which studied by Zhang[19].

We mention that analogous results for the Bernstein-Kantorovich operators, the Meyer-König and Zeller operators and the operators of Butzer, Bleimann and Hahn can be found in [4,1,3,2,5]. Similar results on a certain positive linear operator can be found in [10,7].

2 Main Results

Let $q\in \mathbb{N}$. For a fixed $\mathbf{x}=(x_1,\cdots,x_d)\in\mathscr{S}$, let $K^{[q]}(\mathbf{x})$ be the class of all bounded functions $f:\mathscr{S}\to\mathbb{R}$ such that f and all its partial derivaties of order $\leqslant q$ are continuous in $\mathbf{x}$.

Theorem 1. (Complete asymptotic expansion for the operators B_n). *Let $q\in\mathbb{N}$, $\mathbf{x}\in\mathscr{S}$, and $f\in K^{[2q]}(\mathbf{x})$. The multivariate Bernstein polynomials on the simplex $\mathscr{S}$ satisfy the asymptotic relation*

$$B_n(f;\mathbf{x})=f(\mathbf{x})+\sum_{k=1}^{q}n^{-k}\sum_{\substack{\mathbf{s}\\ k\leqslant|\mathbf{s}|\leqslant 2k}}\frac{1}{\mathbf{s}!}\left(\frac{\partial^{|\mathbf{s}|}}{\partial x_1^{s_1}\cdots\partial x_d^{s_d}}f(\mathbf{x})\right)$$

$$\sum_{\substack{\boldsymbol{\nu}\\ \boldsymbol{\nu}\leqslant\mathbf{s}}}a(k,\mathbf{s},\boldsymbol{\nu})\mathbf{x}^{\mathbf{s}-\boldsymbol{\nu}}+o(n^{-q}),\qquad n\to\infty,$$

where the coefficients $a(k,\mathbf{s},\boldsymbol{\nu})$ are given by

$$a(k,\mathbf{s},\boldsymbol{\nu})=\sum_{\substack{\mathbf{r}\\ \boldsymbol{\nu}\leqslant\mathbf{r}\leqslant\mathbf{s},\,|\mathbf{r}|\geqslant k}}(-1)^{|\mathbf{s}|-|\mathbf{r}|}S(|\mathbf{r}-\boldsymbol{\nu}|,|\mathbf{r}|-k)\prod_{i=1}^{d}\left(\binom{s_i}{r_i}\sigma(r_i,r_i-\nu_i)\right).\tag{5}$$

Remark 1. If $f\in K^{[\infty]}(\mathbf{x})=\bigcap_{q=1}^{\infty}K^{[q]}(\mathbf{x})$, the multivariate Bernstein polynomials on the simplex $\mathscr{S}$ possess the complete asymptotic expansion

$$B_n(f;\mathbf{x})\sim f(\mathbf{x})+\sum_{k=1}^{\infty}c_k(f;\mathbf{x})n^{-k},\qquad n\to\infty,$$

where

$$c_k(f;\mathbf{x})=\sum_{\substack{\mathbf{s}\\ k\leqslant|\mathbf{s}|\leqslant 2k}}\frac{1}{\mathbf{s}!}\left(\frac{\partial^{|\mathbf{s}|}}{\partial x_1^{s_1}\cdots\partial x_d^{s_d}}f(\mathbf{x})\right)\sum_{\substack{\boldsymbol{\nu}\\ \boldsymbol{\nu}\leqslant\mathbf{s}}}a(k,\mathbf{s},\boldsymbol{\nu})\mathbf{x}^{\mathbf{s}-\boldsymbol{\nu}},$$

and $a(k,\mathbf{s},\boldsymbol{\nu})$ is as defined in (5).

The quantities $S(n,k)$ and $\sigma(n,k)$ denote the Stirling numbers of the first and second kind, respectively. Recall that the Stirling numbers are defined by the equations

$$x^{\underline{n}}=\sum_{k=0}^{n}S(n,k)x^k\quad\text{resp.}\quad x^n=\sum_{k=0}^{n}\sigma(n,k)x^{\underline{k}},\quad n=0,1,\cdots.\tag{6}$$

Furthermore we put $S(n,k)=\sigma(n,k)=0$ if $k>n$.

Remark 2. In the univariate case $d=1$ we obtain the well-known formula

$$B_n(f;\ x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{s=k}^{2k} \frac{1}{s!} f^{(s)}(x) \sum_{\nu=0}^{s} a(k,s,\nu) x^{s-\nu} + o(n^{-q}).$$

as $n\to\infty$, where

$$a(k,s,\nu) = \sum_{r=\max(\nu,k)}^{s} (-1)^{s-r} \binom{s}{r} S(r-\nu,\ r-k)\sigma(r,\ r-\nu),$$

provided f is bounded on $[0,1]$ and admits a derivative of order $2q$ at $x\in[0,1]$ (cf. [2, Lemma 1]).

In the special case $q=1$ Theorem 1 reveals the following Voronovskaja-type result.

Corollary 2. (Voronovskaja theorem for operators B_n). *Let* $\mathbf{x}\in\mathscr{S}$, *and* $f\in K^{[2]}(\mathbf{x})$. *The multivariate Bernstein polynomials on the simplex* $\mathscr{S}$ *satisfy the asymptotic relation*

$$\lim_{n\to\infty} n(B_n(f;\ \mathbf{x}) - f(\mathbf{x})) = \frac{1}{2}\left(\sum_{i=1}^{d} x_i \frac{\partial^2 f(\mathbf{x})}{\partial x_i^2} - \sum_{i,j=1}^{d} x_i x_j \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \right).$$

The classical theorem of Voronovskaja[18] is obtained in the special case $d=1$.

For the convenience of the reader we list the explicit expressions for the initial terms of the asymptotic expansion. In order to simplify the notation we restrict ourself to the case $d=2$.

Let $(x,y)\in\mathscr{S}$, and $f\in K^{[6]}(x,\ y)$. The bivariate Bernstein polynomials on the simplex $\mathscr{S}\subset\mathbb{R}^2$ satisfy the asymptotic relation:

$$\begin{aligned}
B_n f = {} & f + \frac{1}{2n}(x(1-x)f_{xx} - 2xyf_{xy} + y(1-y)f_{yy}) \\
& + \frac{1}{6n^2}\begin{bmatrix} x(1-x)(1-2x)f_{xxx} + 3xy(2x-1)f_{xxy} \\ + 3xy(2y-1)f_{xyy} + y(1-y)(1-2y)f_{yyy} \end{bmatrix} \\
& + \frac{1}{8n^2}\begin{bmatrix} x^2(1-x)^2 f_{x^4} - 4x^2y(1-x)f_{xy^3} \\ + 2xy(1-x-y+3xy)f_{x^2y^2} \\ - 4xy^2(1-y)f_{x^3y} + y^2(1-y)^2 f_{y^4} \end{bmatrix} \\
& + \frac{1}{24n^3}\begin{bmatrix} x(1-x)(1-6x-6x^2)f_{x^4} - 4xy(1-6x-6x^2)f_{x^3y} \\ - 6xy(1-2x-2y+6xy)f_{x^2y^2} \\ - 4xy(1-6y-6y^2)f_{xy^3} + y(1-y)(1-6y-6y^2)f_{y^4} \end{bmatrix} \\
& + \frac{1}{12n^3}\begin{bmatrix} x^2(1-x)^2(1-2x)f_{x^5} - 5x^2y(1-x)(1-2x)f_{x^4y} \\ + xy(1-6x-y+15xy+5x^2-20x^2y)f_{x^3y^2} \\ + xy(1-x-6y+15xy+5y^2-20xy^2)f_{x^2y^3} \\ - 5xy^2(1-y)(1-2y)f_{xy^4} + y^2(1-y)^2(1-2y)f_{y^5} \end{bmatrix}
\end{aligned}$$

$$+\frac{1}{48n^3}\left\{\begin{aligned}&x^3(1-x)^3f_{x^6}-6x^3y(1-x)^2f_{x^5y}\\&\quad+3x^2y(1-x)(1-x-y+5xy)f_{x^4y^2}\\&\quad-4x^2y^2(3-3x-3y+5xy)f_{x^3y^3}\\&\quad+3xy^2(1-y)(1-x-y+5xy)f_{x^2y^4}\\&\quad-6xy^3(1-y)^2f_{xy^5}+y^3(1-y)^3f_{y^6}\end{aligned}\right\}$$

$$+o(n^{-3})\qquad n\to\infty.$$

The corollary contains a result due to D. D. Stancu[17](cf. [12, Eq. (5.87), P. 68]).

3 Auxiliary results

For each multi-index $\mathbf{r}=(r_1,\cdots,r_d)\in\mathbb{N}_0^d$, we put $e_{\mathbf{r}}(\mathbf{x})=\mathbf{x}^{\mathbf{r}}$.

Lemma 3. *For all $\mathbf{r}=(r^1,\cdots,r_d)\in\mathbb{N}_0^d$, the monents of the Bernstein polynomials possess the representation*

$$B_n(e_{\mathbf{r}};\mathbf{x})=\sum_{k=1}^{|\mathbf{r}|}n^{-k}\sum_{\substack{\boldsymbol{\nu}\\ \boldsymbol{\nu}\leqslant\mathbf{r},\ |\boldsymbol{\nu}|\geqslant|\mathbf{r}|-k}}\mathbf{x}^{\boldsymbol{\nu}}S(|\boldsymbol{\nu}|,|\mathbf{r}|-k)\prod_{i=1}^{d}\sigma(r_i,\nu_i),\quad \mathbf{x}\in S.$$

Proposition 4. *For all $\mathbf{s}=(s_1,\cdots,s_d)\in\mathbb{N}_0^d$, the central monents of the Bernstein polynomials possess the representation*

$$B_n((\cdot-\mathbf{x})^{\mathbf{s}};\mathbf{x})=\sum_{k=0}^{|\mathbf{s}|}n^{-k}\sum_{\substack{\boldsymbol{\nu}\\ \boldsymbol{\nu}\leqslant\mathbf{s}}}a(k,\mathbf{s},\boldsymbol{\nu})\mathbf{x}^{\mathbf{s}-\boldsymbol{\nu}},\quad \mathbf{x}\in S,\tag{7}$$

where the coefficients $a(k,\mathbf{s},\nu)$ are given by Eq. (5).

In order to show Theorem 1 we use a general approximation theorem for positive linear operators[6].

Lemma 5. *Let $q\in\mathbb{N}$ and $\mathbf{x}\in\mathscr{S}$. Moreover, let L_n: $K^{[2q]}(\mathbf{x})\to C(\mathscr{S})$ be a sequence of positive linear operators. If, for $k=2q$ and $k=2q+2$,*

$$L_n(\|\cdot-\mathbf{x}\|_2^k;\mathbf{x})=\mathscr{O}(n^{-[(k+1)/2]}),\quad n\to\infty,\tag{8}$$

then we have, for each $f\in K^{[2q]}(\mathbf{x})$,

$$L_n(f;\mathbf{x})=\sum_{\substack{\mathbf{s}\\ |\mathbf{s}|\leqslant 2q}}\frac{1}{\mathbf{s}!}\left(\frac{\partial^{|\mathbf{s}|}f(\mathbf{x})}{\partial x_1^{s_1}\cdots\partial x_d^{s_d}}\right)L_n((\cdot-\mathbf{x})^{\mathbf{s}};\mathbf{x})+o(n^{-q}),\quad n\to\infty.$$

The special case $d=1$ is due to Sikkema cf. [15, Theorems 1 and 2].

Remark 3. For $r=0,1,2,\cdots$, we have

$$\|\mathbf{t}-\mathbf{x}\|_2^{2r}=\Big(\sum_{i=1}^{d}(t_i-x_i)^2\Big)^r=\sum_{|\mathbf{s}|=r}\binom{r}{\mathbf{s}}(\mathbf{t}-\mathbf{x})^{2\mathbf{s}}.$$

In order to apply Lemma 5, we have to check whether the Bernstein polynomials satisfy condition (8). By Remark 3, we have to show that

$$L_n((\cdot-\mathbf{x})^{2\mathbf{s}};\ \mathbf{x})=\mathcal{O}(n^{-|\mathbf{s}|}),\quad n\to\infty,$$

for all $\mathbf{s}\in\mathbb{N}_0^d$. In the following Lemma we shall prove a slightly more general result.

Lemma 6. *For each* $\mathbf{x}\in\mathcal{S}$ *and all* $\mathbf{s}\in\mathbb{N}_0^d$, *the central moments of the Berstein polynomials satisfy the estimation*

$$B_n((\cdot-\mathbf{x})^{\mathbf{s}};\ \mathbf{x})=\mathcal{O}(n^{-[(|\mathbf{s}|+1)/2]}),\quad n\to\infty.$$

For the proof we shall make use of the following well-known representations for the Stirling numbers of the first kind

$$S(n,\,n-k)=\sum_{j=k}^{2k}\binom{n}{j}S_2(j,j-k),\quad k=0,1,\cdots n;\ n=0,1,2,\cdots \tag{9}$$

resp. second kind

$$\sigma(n,\,n-k)=\sum_{j=k}^{2k}\binom{n}{j}\sigma_2(j,j-k),\quad k=0,1,\cdots n;\ n=0,1,2,\cdots \tag{10}$$

(see [9, p. 227], cf. [11, pp. 149—153]). The quantities $S_2(n,k)$, $\sigma_2(n,k)$ denote the associated Stirling numbers of the first resp. second kind. Recall that the associated Stirling numbers are defined by their double generating function

$$\sum_{n,k\geqslant 0}S_2(n,k)t^nu^k/n!=e^{-tu}(1+t)^u$$

resp.

$$\sum_{n,k\geqslant 0}\sigma_2(n,k)t^nu^k/n!=\exp(u(e^t-1-t)).$$

(see [9. p. 295 and p. 222]).

4 Proofs

Proof of Lemma 3. By definition (4) and (6), we have

$$\begin{aligned}
&B_n(e_{\mathbf{r}};\mathbf{x})\\
&=n^{-|\mathbf{r}|}\sum_{\substack{|\mathbf{k}|\\ \mathbf{k}\leqslant n}}p_{n,\mathbf{k}}(\mathbf{x})\mathbf{k}^{\mathbf{r}}=n^{-|\mathbf{r}|}\sum_{\substack{|\mathbf{k}|\\ \mathbf{k}\leqslant n}}p_{n,\mathbf{k}}(\mathbf{x})\prod_{i=1}^{d}\Big(\sum_{\nu_i=0}^{r_i}\sigma(r_i,\nu_i)k_i^{\underline{\nu_i}}\Big)\\
&=n^{-|\mathbf{r}|}\sum_{\substack{|\mathbf{k}|\\ \mathbf{k}\leqslant n}}\binom{n}{\mathbf{k}}\mathbf{x}^{\mathbf{\nu}}(1-|\mathbf{x}|)^{n-|\mathbf{k}|}\sum_{\substack{\mathbf{\nu}\\ \mathbf{\nu}\leqslant \mathbf{r}}}\prod_{i=1}^{d}\Big(\sigma(r_i,\nu_i)\big(\frac{\partial}{\partial x_i}\big)^{\nu_i}x_i^{k_i}\Big)\\
&=n^{-|\mathbf{r}|}\sum_{\substack{\mathbf{\nu}\\ \mathbf{\nu}\leqslant \mathbf{r}}}\mathbf{x}^{\mathbf{\nu}}\prod_{i=1}^{d}\Big(\sigma(r_i,\nu_i)\big(\frac{\partial}{\partial x_i}\big)^{\nu_i}\Big)\sum_{\substack{\mathbf{k}\\ |\mathbf{k}|\leqslant n}}\binom{n}{\mathbf{k}}\mathbf{x}^{\mathbf{k}}(1-|\mathbf{y}|)^{n-|\mathbf{k}|}\Big|_{\mathbf{y}=\mathbf{x}}
\end{aligned}$$

Application of the binomial formula leads to

$$B_n(e_{\mathbf{r}};\ \mathbf{x})=n^{-|\mathbf{r}|}\sum_{\substack{\mathbf{\nu}\\ \mathbf{\nu}\leqslant \mathbf{r}}}n^{\underline{|\mathbf{\nu}|}}\mathbf{x}^{\mathbf{\nu}}\prod_{i=1}^{d}\sigma(r_i,\nu_i)$$

and the assertion follows since, by Eq. (6),

$$n^{\underline{|\mathbf{v}|}} = \sum_{k=1}^{|\mathbf{v}|} S(|\mathbf{v}|,k)n^k.$$

Proof of Proposition 4. There holds

$$B_n((\cdot - \mathbf{x})^{\mathbf{s}};\ \mathbf{x}) = \sum_{\substack{\mathbf{r} \\ \mathbf{r}\leqslant\mathbf{s}}} B_n(e_{\mathbf{r}};\ \mathbf{x}) \prod_{i=1}^{d} \binom{s_i}{r_i} (-x_i)^{s_i - r_i},$$

and, by Lemma 3, we have

$$\begin{aligned}
&B_n((\cdot - \mathbf{x})^{\mathbf{s}};\mathbf{x}) \\
&= \sum_{k=0}^{|\mathbf{s}|} n^{-k} \sum_{\substack{\mathbf{r} \\ \mathbf{r}\leqslant\mathbf{s},\,|\mathbf{r}|\geqslant k}} \left(\prod_{i=1}^{d} \binom{s_i}{r_i} (-x_i)^{s_i - r_i} \right) \\
&\quad \times \sum_{\substack{\mathbf{v} \\ \mathbf{v}\leqslant\mathbf{r},\,|\mathbf{v}|\geqslant r-k}} x^{\mathbf{v}} S(|\mathbf{v}|, |\mathbf{r}| - k) \prod_{i=1}^{d} \sigma(r_i, \nu_i) \\
&= \sum_{k=0}^{|\mathbf{s}|} n^{-k} \sum_{\substack{\mathbf{r} \\ \mathbf{r}\leqslant\mathbf{s},\,|\mathbf{r}|\geqslant k}} \left(\prod_{i=1}^{d} \binom{s_i}{r_i} \right) (-\mathbf{x})^{\mathbf{s}-\mathbf{r}} \\
&\quad \times \sum_{\substack{\mathbf{v} \\ \mathbf{v}\leqslant\mathbf{r},\,|\mathbf{v}|\leqslant k}} x^{\mathbf{r}-\mathbf{v}} S(|\mathbf{r}| - |\mathbf{v}|, |\mathbf{r}| - k) \prod_{i=1}^{d} \sigma(r_i, r_i - \nu_i) \\
&= \sum_{k=0}^{|\mathbf{s}|} n^{-k} \sum_{\substack{\mathbf{v} \\ \mathbf{v}\leqslant\mathbf{s}}} \mathbf{X}^{\mathbf{s}-\mathbf{v}} \sum_{\substack{\mathbf{r} \\ \mathbf{v}\leqslant\mathbf{r}\leqslant\mathbf{s}\leqslant,\,|\mathbf{r}|\geqslant k}} (-1)^{|\mathbf{s}|-|\mathbf{r}|} S(|\mathbf{r}| - |\mathbf{v}|, |\mathbf{r}| - k) \\
&\quad \times \prod_{i=1}^{d} \left(\binom{s_i}{r_i} \sigma(r_i, r_i - \nu_i) \right)
\end{aligned}$$

which completes the proof of Proposition 4.

Proof of Lemma 6. According to Proposition 4 it is sufficient to show that $a(k,\mathbf{s},\mathbf{v}) = 0$ if $0 \leqslant k < [(|s|+1)/2]$. An index-shift replacing $\mathbf{r}$ by $\mathbf{r}+\mathbf{v}$ yields

$$\begin{aligned}
a(k,\mathbf{s},\mathbf{v}) = &\sum_{\substack{r \\ r\leqslant s-\nu,\,|r|\geqslant k-|\mathbf{v}|}} (-1)^{|\mathbf{s}|-|\mathbf{v}|-|\mathbf{r}|} S(|\mathbf{r}|, |\mathbf{r}| + |\mathbf{v}| - k) \\
&\times \prod_{i=1}^{k} \left(\binom{s_i}{r_i + \nu_i} \sigma(r_i + \nu_i, r_i) \right).
\end{aligned}$$

Using Formula (10), we obtain, for $i = 1, \cdots, d$,

$$\begin{aligned}
\binom{s_i}{r_i + \nu_i} \sigma(r_i + \nu_i, r_i) &= \binom{s_i}{r_i + \nu_i} \sum_{j_i=0}^{\nu_i} \binom{r_i + \nu_i}{j_i + \nu_i} \sigma_2(j_i + \nu_i, j_i) \\
&= s_i^{\underline{\nu_i}} \binom{s_i - \nu_i}{r_i} \sum_{j_i=0}^{\nu_i} \frac{\sigma_2(j_i + \nu_i, j_i)}{(j_i + \nu_i)!} r_i^{\underline{j_i}}.
\end{aligned}$$

Inserting the formulae and the use of Eq. (9) yields

$$a(k,\mathbf{s},\mathbf{v}) = \sum_{l=0}^{k-|\mathbf{v}|} \frac{S_2(l+k-|\mathbf{v}|,l)}{(l+k-|\mathbf{v}|)!} \sum_{\substack{\mathbf{j} \\ \mathbf{j}\leqslant \mathbf{v}}} b(k,\mathbf{s},\mathbf{v},l,\mathbf{j}) \prod_{i=1}^{d} \left(\frac{\sigma_2(j_i+\nu_i,j_i)}{(j_i+\nu_i)!} s_i^{\nu_i} \right)$$

with

$$b(k,\mathbf{s},\mathbf{v},l,\mathbf{j}) = \sum_{\substack{\mathbf{r} \\ \mathbf{r}\leqslant \mathbf{s}-\mathbf{v},|\mathbf{r}|\geqslant k-|\mathbf{v}|}} (-1)^{|\mathbf{s}|-|\mathbf{v}|-|\mathbf{r}|} |\mathbf{r}|^{\underline{l+k-|\mathbf{v}|}} \prod_{i=1}^{d} \left(\binom{s_i-\nu_i}{r_i} r_i^{\underline{j_i}} \right).$$

Note that, if $|\mathbf{r}|<k-|\mathbf{v}|$ and $l\geqslant 0$, then $|\mathbf{r}|^{\underline{e+k-|\mathbf{v}|}}=0$. Therefore, $b(k,\mathbf{s},\mathbf{v},l,\mathbf{j})$ can also be written in the form

$$b(k,\mathbf{s},\mathbf{v},l,\mathbf{j}) = \sum_{\substack{\mathbf{r} \\ \mathbf{r}\leqslant \mathbf{s}-\mathbf{v}}} (-1)^{|\mathbf{s}|-|\mathbf{v}|-|\mathbf{r}|} \left(\left(\frac{\partial}{\partial z}\right)^{l+k-|\mathbf{v}|} z^{|\mathbf{r}|} \right)_{z=1}$$

$$\times \prod_{i=1}^{d} \left(\binom{s_i-\nu_i}{r_i} \left(\left(\frac{\partial}{\partial y_i}\right)^{j_i} y_i^{r_i} \right) \Big|_{y_i=1} \right)$$

$$= \left(\frac{\partial}{\partial z}\right)^{l+k-\mathbf{v}|} \frac{\partial^{|\mathbf{j}|}}{\partial y_1^{j_1}\cdots \partial y_d^{j_d}} \prod_{i=1}^{d} (zy_i-1)^{s_i-\nu_i} \Big|_{\mathbf{y}=(1,\cdots,1),z=1}$$

$$= \left(\prod_{i=1}^{k} (s_i-\nu_i)^{\underline{j_i}} \right) \left(\frac{\partial}{\partial z}\right)^{l+k-|\mathbf{v}|} (z^{|\mathbf{j}|}(z-1)^{|\mathbf{s}|-|\mathbf{v}|-|\mathbf{j}|}) \Big|_{z=1}.$$

One can immediately see that $b(k,\mathbf{s},\mathbf{v},l,\mathbf{j})=0$ if $k<|\mathbf{s}|-l-|\mathbf{j}|$. Since $l+|\mathbf{j}|\leqslant k$, it follows $b(k,\mathbf{s},\mathbf{v},l,\mathbf{j})=0$ if $2k\leqslant|\mathbf{s}|$. This completes the proof of Lemma 6.

Proof of Theorem 1. Combining Lemma 5 with Prop. 4 and Lemma 6, we obtain

$$B_n(f;\mathbf{x})$$

$$= \sum_{\substack{\mathbf{s} \\ |\mathbf{s}|\leqslant 2q}} \frac{1}{\mathbf{s}!} \left(\frac{\partial^{|\mathbf{s}|} f(\mathbf{x})}{\partial x_1^{s_1}\cdots \partial x_d^{s_d}} \right) \sum_{k=[(|\mathbf{s}|+1)/2]}^{|\mathbf{s}|} n^{-k} \sum_{\substack{\mathbf{v} \\ \mathbf{v}\leqslant \mathbf{s}}} a(k,\mathbf{s},\mathbf{v}) \mathbf{x}^{\mathbf{s}-\mathbf{v}} + o(n^{-q})$$

$$= \sum_{k=0}^{q} n^{-k} \sum_{\substack{\mathbf{s} \\ k\leqslant|\mathbf{s}|\leqslant 2k}} \frac{1}{\mathbf{s}!} \left(\frac{\partial^{|\mathbf{s}|} f(\mathbf{x})}{\partial x_1^{s_1}\cdots \partial x_d^{s_d}} \right) \sum_{\substack{\mathbf{v} \\ \mathbf{v}\leqslant \mathbf{s}}} a(k,\mathbf{s},\mathbf{v}) \mathbf{x}^{\mathbf{s}-\mathbf{v}} + o(n^{-q})$$

as $n\to\infty$. A close look into formula (5) reveals that $a(k,\mathbf{s},\mathbf{v})=0$ if $|\mathbf{s}|=k$ and $k\in\mathbb{N}$. This completes the proof of Theorem 1.

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