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A BIVARIATE EXTENSION OF BLEIMANN-BUTZER-HAHN OPERATOR

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Abstract

Let $C(R_+^2)$ be a class of continuous functions f on R_+^2. A bivariate extension $L_n(f,x,y)$ of Bleimann-Butzer-Hahn operator is defined and its standard convergence properties are given. Moreover, a local analogue of Voronovskaja theorem is also given for a subclass of $C(R_+^2)$.

1 Introduction

For $f\in C[0,\infty)$, Bleimann-Butzer-Hahn (BBH) [5] introduced a Bernstein-type operator defined by

$$L_n(f,x) = (1+x)^{-n}\sum_{j=0}^{n}\binom{n}{j}f\left(\frac{j}{n-j+1}\right)x^j. \tag{1}$$

They studied several interesting convergence/uniform convergence properties of this operator provided f is bounded. Motivated by their work, several authors have further studied this operator on various facets of local and global properties. Jayasri and Sitarman ([6], [7]) further study its properties for largest possible class of functions. Adell, de la Cal and Miguel [2] have extended BBH operator to a bivariate operator $B_n(f,x,y)$ for a continuous function f on a subset of R_+^2 defined by $\{(x,y):x\geqslant 0, y\geqslant 0, xy<1\}$. In the spirit of BBH operator we introduce an analogue of $L_n(f,x)$ for continuous functions on $R_+^2=\{(x,y):x\geqslant 0,y\geqslant 0\}$, and its standard properties are discussed. To define the bivariate extension we use the following notations. Let $\begin{bmatrix} n \\ k,j\end{bmatrix}=\frac{n!}{k!\ j!\ (n-k-j)!}$ and $f_n(k,j)=$

$f\left(\frac{k}{n-k-j+1},\frac{j}{n-k-j+1}\right)$, where $f(x,y)\in C(R_+^2)$. An extension of BBH operator is defined by

$$L_n(f,x,y)=(1+x+y)^{-n}\sum_{k=0}^{n}\sum_{j=0}^{n-k}f_n(k,j)\begin{bmatrix}n\\k,j\end{bmatrix}x^ky^j. \tag{2}$$

The purpose of this note is to provide some standard convergence properties of $L_n(f,x,y)$ under suitable conditions on f. A subcalss $\mathscr{B}$ of $C(R_+^2)$ is defined which is a bivariate version of the class introduced in [6] and [7], and it is shown that $L_n(f,x,y)\to f(x,y)$ whenever $f\in\mathscr{B}$. An estimate of $|L_n(f,x,y)-f(x,y)|$ in terms of the modulus of continuity $\omega(\delta)$ is given, and an analogue of Theorem 2 of [5] is also given by estimating $|L_n(f)-f|$ in terms of the second modulus of continuity $\omega_2(\delta)$ for functions defined on certain closed bounded subsets of R_+^2. A local version of Voronovskaja type theroem is also obtained for the subclass $\mathscr{B}$. To simplify arguments and notations we make extensive use of probability theory, and the required notations and certain well-known facts are given in Section 2. Section 3 provides the main convergence results and a local version of the Voronovskaja type theorem is given in Section 4.

2 Preliminaries

Let X_n be a Bernoulli random variable such that

$$P_{n,j}(x)=P(X_n=j)=\begin{bmatrix}n\\j\end{bmatrix}p^jq^{n-j},\qquad j=0,1,\cdots,n,$$

where $p=\frac{x}{1+x}(x>0)$ and $q=1-p=\frac{1}{1+x}$.

A sharp probability inequality which plays a crucial role throughout the paper is due to Alon and Spencer [3]. In fact, if $Y_n=\frac{X_n}{n}$ and $\delta>0$, it follows from their well-known result (cf. [3], p. 236, corollary A. 7) that

$$P(|Y_n-p|\geqslant\delta)=\sum_{j;|j-np|\geqslant n\delta}P_{n,j}(x)\leqslant 2\exp(-2n\delta^2). \tag{3}$$

Throughout the paper we write E for the expectation operator associated with a random variable or equivalently its distribution. Clearly, $L_n(f,x)=Ef(\xi_n)$ where $L_n(f,x)$ is defined by (1) and $\xi_n=\frac{X_n}{n-X_n+1}=\frac{Y_n}{1-Y_n+\frac{1}{n}}$. Moreover, it is well-known (cf. [5]) that

$$L_n(t,x)=E\xi_n=x-xp^n=x+O(n^{-2}), \tag{4}$$

and

$$L_n((t-x)^2,x)=e_n(x)=E(\xi_n-x)^2=\frac{x(1+x)^2}{n+1}+O(n^{-2}). \tag{5}$$

Since the operator $L_n(f,x,y)$ defined by (2) is analogous to $L_n(f,x)$, in order to utilize probabilistic arguments we introduce a trinomial distribution as follows. Set $p_1=x/(1+x+y)$, $p_2=y/(1+x+y)$, and $p_3=1/(1+x+y)$ $(x,y>0)$, and let X_1 and X_2 be random variables such that

$$P_n(x_1,x_2)=P(X_1=x_1,\,X_2=x_2)=\begin{bmatrix} n \\ x_1,x_2 \end{bmatrix} p_1^{x_1}p_2^{x_2}p_3^{n-x_1-x_2}, \tag{6}$$

where $x_1,x_2=0,1,\cdots,n(x_1+x_2\leqslant n)$, and $\begin{bmatrix} n \\ x_1,x_2 \end{bmatrix}=n!\ /x_1!\ x_2!\ (n-x_1-x_2)!$. Define

$$\xi_1=\xi_1(n)=\frac{X_1}{n-X_1-X_2+1}\quad\text{and}\quad\xi_2=\xi_2(n)=\frac{X_2}{n-X_1-X_2+1}, \tag{7}$$

and note that the operator defined by (2) can be written as $L_n(f,x,y)=Ef(\xi_1,\xi_2)$. The properties of this operator are based on the following quantities needed in the sequel:

$$\sigma_n^2(x,y)=E(\xi_1-x)^2,\ \tau_n^2(x,y)=E(\xi_2-y)^2,\ c_n(x,y)=E(\xi_1-x)(\xi_2-y). \tag{8}$$

These quantities are crucial and their properties will be discussed in the next section.

3 Convergence proerties of $L_n(f,x,y)$

We frequently use the defintions ξ_1 and ξ_2 by (7) throughout the paper. The non-asymptotic properties of the preceding quantities related to ξ_1 and ξ_2 are given by the following lemma.

Lemma 1.

(i) $E\xi_1=x-x(p_1+p_2)^n$, $E\xi_2=y-y(p_1+p_2)^n$,

(ii) $\sigma_n^2(x,y)\leqslant\dfrac{(1+x+y)(9x^2+8x)}{n+1}$, $\tau_n^2(x,y)\leqslant\dfrac{(1+x+y)(9y^2+8y)}{n+1}$,

(iii) $|c_n(x,y)|\leqslant\sigma_n(x,y)\tau_n(x,y)\leqslant\dfrac{(1+x+y)\sqrt{(9x^2+8x)(9y^2+8y)}}{n+1}$.

Proof. In what follows $B(m,p)$ denotes a binomial distribution with m trials and success probability p. For the joint distribution in (6) it is well-known that X_1 is $B(n,p_1)$ and X_2 is $B(n,p_2)$ so that $EX_1=np_1$, $EX_2=np_2$. Moreover, the conditional distribution of X_1 given $X_2=x_2$ is $B(n-x_2,p_1/(1-p_2))$ and the conditional distribution of X_2 given $X_1=x_1$ is $B(n-x_1,p_2/(1-p_1))$. Since $\dfrac{p_1}{1-p_2}=p=\dfrac{x}{1+x}$, it follows from (4) that

$$E(\xi_1|X_2 = x_2) = x - xp^{n-x_2}.$$

Since X_2 is $B(n,p_2)$, we have

$$Ep^{n-X_2} = \sum_{j=0}^{n} p^{n-j}\binom{n}{j} p_2^j(1-p_2)^{n-j} = (p_2 + p(1-p_2))^n = (p_1 + p_2)^n.$$

Hence

$$E\xi_1 = E(E(\xi_1|X_2)) = x - x(p_1+p_2)^n,$$

and a similar argument for $E\xi_2$ proves (i). As expected, $E\xi_1$ and $E\xi_2$ obviously converge to x and y as $n\to\infty$. To prove (ii) we proceed as

$$\sigma_n^2(x,y) = E(\xi_1 - x)^2 = E(\xi_1 - x)^2 I\{X_2 = n\} + E(\xi_1 - x)^2 I\{X_2 \leqslant n-1\}$$

$$= x^2p_2^n + \sum_{j=0}^{n-1} E((\xi_1 - x)^2|X_2 = j)P(X_2 = j).$$

Since the conditional distribution of X_1 given $X_2=j$ is $B(n-j, p=\frac{x}{1+x})$, letting $q_2=1-p_2$, it follows from Khan [9] that

$$\sigma_n^2(x,y) \leqslant x^2p_2^n + \sum_{j=0}^{n-1}\frac{4x(1+x)^2}{(n-j)}\binom{n}{j}p_2^j(1-p_2)^{n-j}$$

$$\leqslant x^2p_2^n + 4nx(1+x)^2q_2\sum_{k=0}^{n-1}\frac{1}{(k+1)^2}\binom{n-1}{k}q_2^kp_2^{n-1-k}.$$

Clearly,

$$\sum_{k=0}^{n-1}\frac{1}{(k+1)^2}\binom{n-1}{k}q_2^kp_2^{n-1-k} \leqslant 2\sum_{j=0}^{n-1}\frac{1}{(j+1)(j+2)}\binom{n-1}{j}q_2^jp_2^{n-1-j}$$

$$\leqslant \frac{2}{n(n+1)q_2^2}\sum_{k=2}^{n+1}\binom{n+1}{k}q_2^kp_2^{n+1-k} \leqslant \frac{2}{n(n+1)q_2^2}.$$

Thus

$$\sigma_n^2(x,y) \leqslant x^2p_2^n + \frac{8x(1+x)^2}{(n+1)q_2}.$$

Since $p_2=\frac{y}{1+x+y}$, $q_2=\frac{1+x}{1+x+y}$, and $p_2^n\leqslant\frac{1+x+y}{n+1}$, we have

$$\sigma_n^2(x,y) \leqslant \frac{(1+x+y)x^2}{n+1} + \frac{8x(1+x)(1+x+y)}{n+1} = \frac{(1+x+y)(9x^2+8x)}{n+1}.$$

The inequality for $\tau_n^2(x,y)$ follows by symmetry and (iii) is obvious by Schwarz inequality.

Lemma 1 entails a Shisha-Mond type of estimate for $|L_n(f)-f|$ in terms of the modulus of continuity of f. Let $u=(u_1,u_2)$, $v=(v_1,v_2)$, and $\| u \| = \sqrt{u_1^2+u_2^2}$. Then the modulus of continuity of f is defined by

$$\omega(f,\delta) = \omega(\delta) = \sup\{|f(u) - f(v)| : \| u - v \| < \delta\}, \qquad \delta > 0.$$

An estimate of $|L_n(f,x,y)-f(x,y)|$ in terms of $\omega(f,\delta)$ is given by the following.

Theorem 1. *Let* $f\in C(R_+^2)$, *where* $R_+^2=\{(x,y):x\geqslant 0,y\geqslant 0\}$. *Then*

$$|L_n(f,x,y) - f(x,y)| \leqslant \left(1 + \sqrt{(1+x+y)(9(x^2+y^2)+8(x+y))}\right)\cdot \omega\left(f,\frac{1}{\sqrt{n+1}}\right).$$

Proof. Let $\alpha=(x_1,y_1)$, $\beta=(x_2,y_2)$, and set $\lambda=\left[\frac{\|\alpha-\beta\|}{\delta}\right]$, where $[z]$ denotes the greatest integer $\leqslant z$. Clearly,

$$|f(\alpha) - f(\beta)| \leqslant (\lambda + 1)\omega(\delta).$$

Let $X=(\xi_1,\xi_2)$, $t=(x,y)$, and $\lambda=\left[\frac{\|X-t\|}{\delta}\right](\delta>0)$, where ξ_1 and ξ_2 are defined by (7). By the preceding inequality we have

$$|f(\xi_1,\xi_2) - f(x,y)| \leqslant (1 + \lambda)\omega(f,\delta),$$

so that

$$|L_n(f,x,y) - f(x,y)| \leqslant E|f(\xi_1,\xi_2) - f(x,y)| \leqslant (1 + E\lambda)\omega(f,\delta). \tag{9}$$

Since $\lambda\leqslant\delta^{-1}\sqrt{(\xi_1-x)^2+(\xi_2-y)^2}$, Jensen's inequality [10, p. 50] combined with Lemma 1 gives

$$\begin{aligned} E\lambda &\leqslant \delta^{-1}\sqrt{E(\xi_1-x)^2+E(\xi_2-y)^2} \\ &\leqslant \delta^{-1}\sqrt{\frac{(1+x+y)(9(x^2+y^2)+8(x+y))}{n+1}}. \end{aligned}$$

Hence the theorem follows from (9) by choosing $\delta=(n+1)^{-\frac{1}{2}}$.

Remark. Under suitable conditions $\omega(f,\delta)\to 0$ as $\delta\to 0$, and $L_n(f,x,y)\to f(x,y)$, and the convergence is uniform for every compact subset of R_+^2. Of course, if $f(x,y)$ is bounded, the defintion $L_n(f,x,y)=Ef(\xi_1,\xi_2)$ and the bounded convergence theorem implies that $L_n(f,x,y)\to f(x,y)$.

It is interesting to note that an estimate in terms of the second modulus of continuity $\omega_2(f,\delta)$ can also be obtained in the present case. To this end, let C_B be a subclass of $C(R_+^2)$ where B is a closed bounded subset of R_+^2 with a certain cone property (cf. [8], pp. 122—123), and let $\| f \|$ be the usual sup-norm in C_B. Let $u=(u_1,u_2)$, $v=(v_1,v_2)$, $h=u-v$, and set $\Delta_h^2=f(v+2h)-2f(v+h)+f(v)$. Here u and v are in R_+^2 such that $v+2h\geqslant(0,0)$, and let A be this set. Then $\omega_2(f,\delta)$ is defined by

$$\omega_2(f,\delta) = \sup\{|\Delta_h^2|: h = u - v,\ u,v \in A,\ \|h\| \leqslant \delta\}.$$

Moreover, let C_B^2 be a subclass of functions g in C_B having partial derivatives up to the second order. Let $\|g_d\|$ be the maximum of the sup-norms of $g_x, g_y, g_{xx}, g_{yy}, g_{xy}$ in C_B^2. Set $h=\xi_1-x, k=\xi_2-y$, where ξ_1 and ξ_2 are defined by (7). Then for any $g\in C_B^2$, it is easily seen from the second order Taylor representation of $g(\xi_1,\xi_2)$ that

$$|L_n(g) - g| \leqslant \|g_d\| (|Eh| + |Ek| + \frac{1}{2}(Eh^2 + 2|Ehk| + Ek^2)).$$

Observe from Lemma 1 that $\frac{|Eh|}{x}=\frac{|Ek|}{y}=(p_1+p_2)^n\leqslant\frac{(1+x+y)}{n+1}$. Moreover, setting $b(x,y)=(1+x+y)(9x^2+8x)$ it follows from above and Lemma 1 that

$$|L_n(g) - g| \leqslant \|g_d\| (\gamma(x,y)/(n+1)),$$

where

$$\gamma(x,y) = (x+y)(1+x+y) + \frac{b(x,y)+b(y,x)}{2} + \sqrt{b(x,y)b(y,x)}. \qquad (10)$$

Consequently,

$$|L_n(f) - f| \leqslant 2\|f-g\| + \|g_d\| \frac{\gamma(x,y)}{n+1},$$

and

$$|L_n(f) - f| \leqslant \inf_{g\in C_B^2}\{2\|f-g\| + \|g_d\| \frac{\gamma(x,y)}{n+1}\}. \qquad (11)$$

Then using (11) and the K-functional inequality in [4] (p. 360) one obtains the following analogue of Theorem 2 in [5].

Theorem 2. *Let $f\in C_B$, and let $N(x,y)$ be the least integer $\geqslant\frac{1}{2}\gamma(x,y)-1$, where $\gamma(x,y)$ is defined by* (10). *Then for $n\geqslant N(x,y)$ we have*

$$|L_n(f) - f| \leqslant 2M\left[\omega_2(f,\sqrt{\frac{\gamma(x,y)}{2(n+1)}}) + \|f\| \frac{\gamma(x,y)}{2(n+1)}\right],$$

where M is a constant.

Now we will show the convergence of $L_n(f,x,y)$ for a large class of functions which is analogous to the class $\mathscr{F}$ introduced in [6] and [7]. Let

$$\mathscr{B} = \{f\in C(R_+^2);\ \text{for each } A>0 \text{ and each } B>0,\ f(x,y) = O(1)\exp(Ax+By)\}.$$

The ensuing convergence of $L_n(f)$ requires the following localization lemma. In the lemma below and elsewhere I denotes the usual indicator (characteristic) function.

Lemma 2. *Let $\psi(x,y)$ be a positive function in $\mathscr{B}$. Set $h=\xi_1-x, k=\xi_2-y$, where ξ_1 and ξ_2 are defined by* (7), *and let $\delta_1,\delta_2>0$. Then for some positive α independent of n,*

$$E\psi(\xi_1,\xi_2)I\{|h|\geqslant\delta_1 \text{ or } |k|\geqslant\delta_2\}=O(\exp(-an)).$$

Proof. Let

$$E=\{|h|\geqslant\delta_1 \text{ or } |k|\geqslant\delta_2\}=\{(i,j):|\frac{i}{n-i-j+1}-x|$$

$$\geqslant\delta_1 \text{ or } |\frac{j}{n-i-j+1}-y|\geqslant\delta_2\}.$$

Since $\psi\in\mathscr{B}$, for large $n,n\geqslant n_0$, say, we have

$$E\psi(\xi_1,\xi_2)I\{|h|\geqslant\delta_1 \text{ or } |k|\geqslant\delta_2\}\leqslant O(1)\exp(An+Bn)P(E). \tag{12}$$

Set $Y_i=X_i/n(i=1,2)$ (cf. (7)) and let $0<\eta_2<1$. Clearly,

$$P(|h|\geqslant\delta_1)\leqslant P(|h|\geqslant\delta_1,\ |Y_2-p_2|\leqslant\eta_2)+P(|Y_2-p_2|>\eta_2). \tag{13}$$

Since $X_1\sim B(n,p_1)$, it can be seen from the definition of ξ_1 that there exists $\varepsilon_1>0$ such that for suitably large $n\geqslant n_0$ we obtain from (3) that

$$P(|h|\geqslant\delta_1,\ |Y_2-p_2|\leqslant\eta_2)\leqslant P(|Y_1-p_1|\geqslant\varepsilon_1)\leqslant 2\exp(-2n\varepsilon_1^2),$$

and

$$P(|Y_2-p_2|>\eta_2)\leqslant 2\exp(-2n\eta_2^2).$$

Consequently, for $n\geqslant n_0$, (13) gives

$$P(|h|\geqslant\delta_1)\leqslant 2(\exp(-2n\varepsilon_1^2)+\exp(-2n\eta_2^2)).$$

Similarly, for $0<\eta_1<1$ and large $n\geqslant n_0$, there exists $\varepsilon_2>0$ such that

$$P(|k|\geqslant\delta_2)\leqslant 2(\exp(-2n\varepsilon_2^2)+\exp(-2n\eta_1^2)).$$

Hence letting $\varepsilon=\min(\varepsilon_1,\varepsilon_2,\eta_1,\eta_2)$ we have

$$P(E)=P(|h|\geqslant\delta_1 \text{ or } |k|\geqslant\delta_2)\leqslant P(|h|\geqslant\delta_1)+P(|k|\geqslant\delta_2)\leqslant 8\exp(-2n\varepsilon^2),$$

and the lemma follows from (12) by choosing $(A+B)<2\varepsilon^2$.

The preceding lemma implies the following convergence property of $L_n(f,x,y)$.

Theorem 3. *$L_n(f,x,y)\to f(x,y)$ as $n\to\infty$ $\forall$ $f\in\mathscr{B}$ and for each $(x,y)\in R_+^2$.*

Proof. Let $Q=f(\xi_1,\xi_2)-f(x,y)$. Given $\varepsilon>0$, choose positive δ_1 and δ_2 such that $|Q|<\varepsilon$ whenever $|h|<\delta_1$ and $|k|<\delta_2$, where h and k have been defined in Lemma 2. Clearly,

$$EQ=EQI\{|h|<\delta_1,\ |k|<\delta_2\}+EQI\{|h|\geqslant\delta_1 \text{ or } |k|\geqslant\delta_2\},$$

and

$$|EQ|\leqslant E|Q|I\{|h|<\delta_1,\ |k|<\delta_2\}+E|Q|I\{|h|\geqslant\delta_1 \text{ or } |k|\geqslant\delta_2\}.$$

Hence if $f\in\mathscr{B}$,

$$|L_n(f,x,y)-f(x,y)|\leqslant\varepsilon+O(1)\exp(An+Bn)P(|h|\geqslant\delta_1 \text{ or } |k|\geqslant\delta_2),$$

and since ε is arbitrary, the conclusion follows from Lemma 2.

4 Voronovskaja theorem for $L_n(f,x,y)$

Before stating and proving the intended theorem we need the following lemma.

Lemma 3. *Let $\sigma_n^2(x,y)$, $\tau_n^2(x,y)$ and $c_n(x,y)$ be defined by (8). Then*

(i) $\sigma_n^2(x,y)=\frac{x(1+x)(1+x+y)}{n+1}+O(n^{-2})$,

(ii) $\tau_n^2(x,y)=\frac{y(1+y)(1+x+y)}{n+1}+O(n^{-2})$,

(iii) $c_n(x,y)=\frac{xy(1+x+y)}{n+1}+O(n^{-2})$.

Proof. We first observe two simple facts. Let X have a binomial distribution $B(n,p)$ so that $Y=n-X$ is $B(n,q=1-p)$. Then

$$E\frac{1}{(n-X+1)}=E\frac{1}{Y+1}=\sum_{j=0}^{n}\frac{1}{j+1}\binom{n}{j}q^jp^{n-j}=\frac{1-p^{n+1}}{(n+1)q}. \tag{14}$$

Next we have

$$\begin{aligned} E\frac{1}{(n-X+1)^2}&=E\frac{1}{(Y+1)^2}\leqslant 2E\frac{1}{(Y+1)(Y+2)} \\ &=2\sum_{j=0}^{n}\frac{n!}{(j+2)!(n-j)!}q^jp^{n-j} \\ &\leqslant\frac{2}{q^2(n+1)(n+2)}\sum_{k=2}^{n+2}\binom{n+2}{k}q^kp^{n+2-k} \\ &\leqslant\frac{2}{q^2(n+1)(n+2)}=O(n^{-2}). \end{aligned} \tag{15}$$

To prove (i) we recall (7) and that the conditional distribution of X_1 given $X_2=j$ is $B(n-j,\frac{p_1}{1-p_2}=\frac{x}{1+x})$ while the marginal distribution of X_2 is $B(n,p_2=\frac{y}{1+x+y})$ with $q_2=1-p_2=\frac{1+x}{1+x+y}$. Hence it follows from (5) that

$$\begin{aligned} \sigma_n^2(x,y)&=\sum_{j=0}^{n}E((\xi_1-x)^2|X_2=j)P(X_2=j) \\ &=\sum_{j=0}^{n}\frac{x(1+x)^2}{(n-j+1)}P(X_2=j)+\sum_{j=0}^{n}O((n-j+1)^{-2})P(X_2=j). \end{aligned}$$

Thus (14) and (15) give

$$\begin{aligned} \sigma_n^2(x,y)&=\frac{x(1+x)^2}{(n+1)q_2}(1-p_2^{n+2})+O(n^{-2}) \\ &=\frac{x(1+x)(1+x+y)}{n+1}+O(n^{-2}). \end{aligned}$$

Of course, (ii) is obvious by symmetry. To see (iii), note that $\xi_1+\xi_2=\dfrac{X_1+X_2}{n-X_1-X_2+1}$ and X_1+X_2 is $B(n,p=p_1+p_2=\dfrac{x+y}{1+x+y})$. Hence using (5) again we obtain that

$$E(\xi_1+\xi_2-x-y)^2=\frac{(x+y)(1+x+y)^2}{n+1}+O(n^{-2}).$$

Since

$$E(\xi_1+\xi_2-x-y)^2=\sigma_n^2(x,y)+\tau_n^2(x,y)+2c_n(x,y),$$

hence it follows from (i), (ii) and above that

$$\begin{aligned}2c_n(x,y)=&\frac{(x+y)(1+x+y)^2-x(1+x)(1+x+y)-y(1+y)(1+x+y)}{n+1}\\&+O(n^{-2})\\=&\frac{2xy(1+x+y)}{n+1}+O(n^{-2}),\end{aligned}$$

and the lemma is proved.

We can now prove a local version of Voronovskaja theorem. In what follows $f_x=\dfrac{\partial f}{\partial x}$ at (x,y), etc., and $N=\{(\alpha,\beta):|\alpha-x|<\delta_1,|\beta-y|<\delta_2\}$ denotes a neighborhood of (x,y).

Theorem 4. *Let $f\in\mathscr{B}$ and suppose that the first two partial derivatives of f exist and are continuous in N. Let $L_n(f,x,y)$ be defined by* (2). *Then*

$$\begin{aligned}\lim_{n\to\infty}n(L_n(f,x,y)-f(x,y))=&\frac{(1+x+y)}{2}[x(1+x)f_{xx}\\&+2xyf_{xy}+y(1+y)f_{yy}].\end{aligned}$$

Proof. Let $u=s-x, v=t-y$, and set

$$P(s,t)=f(x,y)+uf_x+vf_y+\frac{1}{2}(u^2f_{xx}+2uvf_{xy}+v^2f_{yy}).$$

Now consider the Taylor expansion of $f(s,t)$ in N as given by

$$f(s,t)=P(s,t)+R(s,t),$$

where $R(s,t)$ denotes the remainder. Letting $0<\theta<1$, $R(s,t)$ can be written as

$$R(s,t)=\frac{1}{2}(u^2Q_{xx}(s,t)+2uvQ_{xy}(s,t)+v^2Q_{yy}(s,t)),$$

where

$$Q_{xx}(s,t)=f_{xx}(x+\theta u,y+\theta v)-f_{xx}(x,y),$$

$$Q_{xy}(s,t)=f_{xy}(x+\theta u,y+\theta v)-f_{xy}(x,y),$$

$$Q_{yy}(s,t)=f_{yy}(x+\theta u,y+\theta v)-f_{yy}(x,y).$$

Recalling that $h=\xi_1-x$ and $k=\xi_2-y$, define $\omega=\{|h|<\delta_1, |k|<\delta_2\}$. Clearly, by Lemma 2 we have

$$L_n(f,x,y)=Ef(\xi_1,\xi_2)I(\omega)+Ef(\xi_1,\xi_2)I(\bar{\omega})$$
$$=Ef(\xi_1,\xi_2)I(\omega)+o(n^{-2}). \tag{16}$$

It is obvious that

$$Ef(\xi_1,\xi_2)I(\omega)=EP(\xi_1,\xi_2)I(\omega)+ER(\xi_1,\xi_2)I(\omega)$$
$$=EP(\xi_1,\xi_2)-EP(\xi_1,\xi_2)I(\bar{\omega})+ER(\xi_1,\xi_2)I(\omega). \tag{17}$$

Since $P(s,t)\in\mathscr{B}$, using Lemma 2 again we have

$$EP(\xi_1,\xi_2)I(\bar{\omega})=o(n^{-2}). \tag{18}$$

Moreover, since

$$EP(\xi_1,\xi_2)=f(x,y)+f_xEh+f_yEk+\frac{1}{2}(f_{xx}Eh^2+2f_{xy}Ehk+f_{yy}Ek^2),$$

it follows from Lemma 1 (i) and Lemma 3 that

$$EP(\xi_1,\xi_2)=f(x,y)+\frac{(1+x+y)}{2(n+1)}\varphi(x,y)+O(n^{-2}),$$

where $\varphi(x,y)=x(1+x)f_{xx}+2xyf_{xy}+y(1+y)f_{yy}$.

This combined with (17) and (18) gives

$$Ef(\xi_1,\xi_2)I(\omega)=f(x,y)+\frac{(1+x+y)}{2(n+1)}\varphi(x,y)+O(n^{-2})+ER(\xi_1,\xi_2)I(\omega). \tag{19}$$

Now we will show that

$$\lim_{n\to\infty} nER(\xi_1,\xi_2)I(\omega)=0. \tag{20}$$

But this is tantamount to showing that $nEh^2|Q_{xx}(\xi_1,\xi_2)|\to 0$, $nEk^2|Q_{yy}(\xi_1,\xi_2)|\to 0$, and $n|EhkQ_{xy}(\xi_1,\xi_2)|\to 0$ as $n\to\infty$. Since $h=\xi_1(n)-x\to 0$ a. s., and $k=\xi_2(n)-y\to 0$ a. s. (as $n\to\infty$), choose n sufficiently large ($n\geqslant n_0$, say) such that for $\varepsilon>0$, $|Q_{xx}(\xi_1,\xi_2)|<\varepsilon$. Thus for $n\geqslant n_0$, by Lemma 3 we have

$$Eh^2|Q_{xx}(\xi_1,\xi_2)|\leqslant \varepsilon Eh^2\leqslant\frac{\varepsilon x(1+x)(1+x+y)}{n+1}+O(n^{-2}).$$

Since ε is arbitrary, the desired conclusion follows. Similar arguments apply to the other two limits, and (20) holds. Hence the theorem follows from (16), (19) and (20).

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