Approx. Theory & its Appl. 18: 1, 2002, 51 – 67

# ROUGH OPERATORS AND COMMUTATORS ON HOMOGENEOUS WEIGHTED HERZ SPACES*

Jiang Yinsheng and Liu Mingju

(*Xinjiang University, China*)

Received Apr. 18, 2001 Revised Nov. 20, 2001

**Abstract**

*The authors establish the boundedness on homogeneous weighted Herz spaces for a large class of rough operators and their commutators with BMO functions. In particular, the Calderón-Zygmund singular integrals and the rough R. Fefferman singular integral operators and the rough Ricci-Stein oscillatory singular integrals and the corresponding commutators are considered.*

## 1 Introduction

Let $B_k=\{x\in \mathbb{R}^n: |x|\leqslant 2^k\}$ and $E_k=B_k\backslash B_{k-1}$ for any $k\in \mathbb{Z}$. Let $\chi_k=\chi_{E_k}$ for $k\in \mathbb{Z}$ be the characteristic function of the set $E_k$.

*Definition* 1.1. Let $\alpha\in \mathbb{R}, 0<p\leqslant\infty, 1<q<\infty, w_1$ and $w_2$ be non-negative weight functions. The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1,w_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(w_1,w_2)\equiv\{f\in L_{\rm loc}^q(\mathbb{R}^n\backslash\{0\},w_2): \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}<\infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}\equiv\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\|f\chi_k\|_{L^q(w_2)}^p\Big\}^{1/p}$$

with the usual modificiation made when $p=\infty$.

*Remark* 1.1. The weighted Herz space is introduced by Lu and Yang in [1] with $0<\alpha<\infty$. Now, we extend the range of $\alpha$ to $\mathbb{R}$.

---

* Project 19871071 supported by Natural Science Foundation of China.

For any power exponent $r\in[1,\infty]$, we denote the conjugate one $r/(r-1)$ by $r'$. For any $r\in[1,\infty]$, any nonnegative function $w$ and any Lebesgue measurable function $f$, we write

$$\| f \|_{L^r(w)} = \left\{\int_{R^n} |f(x)|^r w(x)\mathrm{d}x\right\}^{1/r}.$$

If $w\equiv w_1\equiv w_2\equiv 1$, we denote $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and $L^r(w)$, respectively, by $\dot{K}_q^{\alpha,p}(\mathrm{R}^n)$ and $L^r(\mathrm{R}^n)$. Let $T$ be a linear operator and $b\in BMO(\mathrm{R}^n)$. The commutator $[b,T]$ is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

The properties of the $BMO(\mathrm{R}^n)$ function $b$ are referred to [2].

Stein in [3] proved that if

$$Tf(x) = \mathrm{p.v.} \int_{R^n} k(x,y)f(y)\mathrm{d}y \tag{1.1}$$

is bounded on $L^q(\mathrm{R}^n)$ with some $q\in(1,\infty)$, and if $k(x,y)$ satisfies the standard size condition

$$|k(x,y)| \leqslant \frac{c}{|x-y|^n}, \qquad x\neq y, \tag{1.2}$$

for some absolute constant $c$, which is satisfied by a large class of operators in harmonic analysis, then $T$ is also bounded on $L^q_{|x|^a}(\mathrm{R}^n)$ for $|x|^a\in A_q(\mathrm{R}^n)$ (the Muckenhoupt weight class), or equivalently, $a\in(-n,n(q-1))$. Recently, Soria and Weiss in [4] extended the Stein's result to more general cases, where the power weight was replaced by more general weighted $w\in A_p$ satisfying

$$\sup_{2^{k-2}<|x|<2^{k+1}} w_2(x) \leqslant c_1 \inf_{2^{k-2}<|x|<2^{k+1}} w_2(x), \qquad k\in \mathrm{Z}, \tag{1.3}$$

where $c_1$ is independent of $k\in\mathrm{Z}$.

Note that $L^q_{|x|^a}(\mathrm{R}^n) = \dot{K}_q^{a/q,q}(\mathrm{R}^n)$, a special case of the general homogeneous Herz space. Lu-Yang [1] generalized Stein's result and Soria-Weiss's result to the case of sublinear operators on weighted Herz spaces.

A well-known result of Coifman, Rochberg and Weiss [5] states that if $T$ is a standard Claderón-Zygmund operator and $b\in BMO(\mathrm{R}^n)$, then $[b,T]$ is bounded on $L^p(\mathrm{R}^n)$ for $p\in(1,\infty)$. Lu and Yang in [6] generalized these results to the case of Herz spaces. In fact, motivated by [4] and [3], Lu and yang in [6] studied the commutators generated by $BMO(\mathrm{R}^n)$ functions and the linear operators with kernels $k$ satisfying (1.2). Recently, Jiang-Tang-Yang [7] obtained the boundedness on weighted Herz spaces with $0<\alpha<\infty$ for the commutators.

Let $\Omega\in L^r(S^{n-1})$ for some $r\in(1,\infty]$ be homogeneous of degree zero. If we replace (1.2) by the following "rough" size condition:

$$|k(x,y)| \leqslant c \frac{|\Omega(x-y)|}{|x-y|^n}, \qquad x \neq y, \tag{1.4}$$

what can be said about the commutator $[b,T]$ and the operator $T$ satisfying (1.4) in the set of Herz-type spaces? Hu-Lu-Yang [8] have investigated the boundedness of the operators satisfying (1.4) on homogenous Herz spaces. Tang-Jiang-Lu [9] obtained the boundedness of the operators satisfying (1.4) on weighted Herz spaces with $0<\alpha<\infty$. For their commutators Lu-Tang-Yang [10] have obtained the boundedness on homogeneous Herz spaces.

The main purpose of this paper is to generalize the above results to the case of the homogeneous wieghted Herz spaces with $\alpha \in \mathbb{R}$. It should be mentioned that the above results on homogeneous weighted Herz spaces with $0<\alpha<\infty$ are obtained via the central block decomposition characterization of $\dot{K}_q^{\alpha,p}$(see [1,7,9]). However, this characterization is not true when $\alpha \leqslant 0$. Therefore, the main results in this paper are deduced from the definitions of $\dot{K}_q^{\alpha,p}$.

We will consider the following operators and their commutators with *BMO* functions: the rought Hardy-Littlewood maximal operators $M_\Omega$ given by

$$M_\Omega f(x) = \sup_{\rho>0} \frac{1}{\rho^n} \int_{|y|<\rho} |\Omega(x-y) f(y)| \mathrm{d}y, \tag{1.5}$$

the R. Fefferman singular integral $T_\Omega$ with rough kernel given by

$$T_\Omega f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{h(x-y)\Omega(x-y)}{|x-y|^n} f(y) \mathrm{d}y, \tag{1.6}$$

and the Ricci-Stein oscillatory singular integral operator $T_{\Omega,P}$ with rough kernel given by

$$T_{\Omega,P} f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{h(x-y)\Omega(x-y)}{|x-y|^n} f(y) \mathrm{d}y, \tag{1.7}$$

where $\Omega \in L^r(S^{n-1})$ is homogeneous of degree zero, $1<r\leqslant\infty$, $h$ is radial and $P(x,y)$ is a real polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. However, the results for these operators and commutators are obtained as some corollaries of our main theorems.

For inhomgeneous weighted Herz spaces, we have similar result to all the theorems obtained in this paper. On the fractional case, we also obtain some theorems parallel to the non-fractional case. We omit the details here.

Throughtout this paper, $c$, $c_0$ and $c_1$ always mean constants independent of the main parameters involved, but whose values may vary from line to line.

## 2 Sublinear operators

We begin this section with the boundedness of the rough Hardy-Littlewood maximal operator $M_\Omega$ on homogeneous weighted Herz spaces $\dot{K}_q^{\alpha,p}(w_1,w_2)$. We need the following lemma.

**Lemma 2.1**[11]. *Let $\Omega\in L^r(S^{n-1})$ for some $r\in(1,\infty]$. If $a>0$, $0<d\leqslant r$ and $-n+(n-1)d/r<\beta<\infty$, then*

$$\left(\int_{|x|\leqslant a|y|}|\Omega(x-y)|^d|x|^\beta dx\right)^{1/d}\leqslant c|y|^{(\beta+n)/d}\|\Omega\|_{L^r(S^{n-1})}.$$

**Theorem 2.1.** *Let $0<p<\infty$, $1<q<\infty$, $w_1,w_2\in A_1$ and $w_2$ satisfy (1.3). Let $\Omega\in L^r(S^{n-1})$ be homogeneous of degree zero, $1<r\leqslant\infty$. If $M_\Omega$ is bounded on $L^q(w_2)$, then there exists a positive constant $\delta>0$ depends only on $w_2$ such that $M_\Omega$ is bounded on $\dot{K}_q^{\alpha,p}(w_1,w_2)$, if $\alpha$, $q$ and $r$ satisfy one of the following*

(i) $r>q$ *and* $\min\{n(1/r-\delta/q)-1/r,0\}<\alpha<n(1-1/q)$; *or*

(ii) $r'<q$ *and* $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$.

*Proof.* Let $f\in\dot{K}_q^{\alpha,p}(w_1,w_2)$. Write

$$f(x)=\sum_{j=-\infty}^{\infty}f(x)\chi_j(x)\equiv\sum_{j=-\infty}^{\infty}f_j(x).$$

Then,

$$\begin{aligned}\|M_\Omega(f)\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}&=\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\|\chi_kM_\Omega(f)\|_{L^q(w_2)}^p\Big\}^{1/p}\\&\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=-\infty}^{k-2}\|\chi_kM_\Omega(f_j)\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\\&\quad+c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k-1}^{k+1}\|\chi_kM_\Omega(f_j)\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\\&\quad+c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k+2}^{\infty}\|\chi_kM_\Omega(f_j)\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\\&\equiv I_1+I_2+I_3.\end{aligned}$$

For the estimate of $I_2$, by the $L^q(w_2)$-boundedness of $M_\Omega$, we have

$$\begin{aligned}I_2&=c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k-1}^{k+1}\|\chi_kM_\Omega(f_j)\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\\&\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k-1}^{k+1}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\end{aligned}$$

$$\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\|f_k\|^p_{L^q(w_2)}\Big\}^{1/p}$$

$$\equiv c\|f\|_{K_q^{\alpha,p}(w_1,w_2)},$$

which is what we want.

For $I_1$, since $x\in E_k, y\in E_j$ and $j\leqslant k-2$, we get $|x-y|\sim|x|$. So

$$M_\Omega(f_j)(x)\leqslant c2^{-kn}\int_{E_j}|\Omega(x-y)||f(y)|\mathrm{d}y.$$

Therefore, by (1.3),

$$\|\chi_k M_\Omega(f_j)\|_{L^q(w_2)}\leqslant c2^{-kn}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||f(y)|\mathrm{d}y\Big)^q w_2(x)\mathrm{d}x\Big\}^{1/q}$$

$$\leqslant c2^{-kn}[\sup_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||f(y)|\mathrm{d}y\Big)^q\mathrm{d}x\Big\}^{1/q}$$

$$\leqslant cc_1 2^{-kn}[\inf_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||f(y)|\mathrm{d}y\Big)^q\mathrm{d}x\Big\}^{1/q}.$$

If $r>q, j\leqslant k-2$ and $w_2\in A_1$ satsfying (1.3), then

$$\|\chi_k M_\Omega(f_j)\|_{L^q(w_2)}\leqslant c2^{-kn}[\inf_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_j}|f(y)|\Big[\int_{E_k}|\Omega(x-y)|^q\mathrm{d}x\Big]^{1/q}\mathrm{d}y\Big\}$$

$$\leqslant c2^{-kn}\Big[\frac{w_2(B_k)}{|B_k|}\Big]^{1/q}\int_{E_j}|f(y)|\mathrm{d}y\Big(\int_{2^{k-1}}^{2^k}s^{n-1}\mathrm{d}s\Big)^{1/q}\|\Omega\|_{L^q(S^{n-1})}$$

$$\leqslant c\Big[\frac{w_2(B_j)}{|B_j|}\Big]^{1/q}2^{-kn+kn/q+jn/q'}\|f_j\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c[\operatorname{essinf}_{x\in B_j}w_2(x)]^{1/q}2^{(j-k)n/q'}\|f_j\|_{L^q(\mathbb{R}^n)}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c2^{(j-k)n/q'}\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})},$$

that is,

$$\|\chi_k M_\Omega(f_j)\|_{L^q(w_2)}\leqslant c2^{(j-k)n/q'}\|f_j\|_{L^q(w_2)}. \tag{2.1}$$

Note that if $\alpha<0$ and $k>j$, then $[w_1(B_k)]^{\alpha p/n}\leqslant[w_1(B_j)]^{\alpha p/n}$. By substituting (2.1) into $I_1$ on the case of $r>q$ and $\alpha<0$, we obtain

$$I_1\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=-\infty}^{k-2}2^{(j-k)n/q'}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=-\infty}^{k-2}2^{(j-k)n/q'}[w_1(B_j)]^{\alpha/n}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$
\leqslant c\begin{cases}\Big\{\sum\limits_{k=-\infty}^{\infty}\Big[\sum\limits_{j=-\infty}^{k-2}2^{(j-k)np/(2q')}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big] \\ \qquad\qquad \times\Big[\sum\limits_{j=-\infty}^{k-2}2^{(j-k)np'/(2q')}\Big]^{p/p'}\Big\}^{1/p} & (p>1)\\ \Big\{\sum\limits_{k=-\infty}^{\infty}\Big[\sum\limits_{j=-\infty}^{k-2}2^{(j-k)np/q'}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big]\Big\}^{1/p} & (0<p\leqslant 1)\end{cases}
$$

$$
\leqslant c\Big\{\sum_{j=-\infty}^{\infty}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big\}^{1/p}
$$

$$
\equiv c\|f\|_{K_q^{\alpha,p}(w_1,w_2)},
$$

where for $p>1$, we have used Hölder's inequality and for $0<p\leqslant 1$ we have used the well-known inequality:

$$
\Big(\sum_{j=-\infty}^{\infty}|a_j|\Big)^p\leqslant\sum_{j=-\infty}^{\infty}|a_j|^p \tag{2.2}
$$

for any $a_j\in\mathbb{R}$.

For $r>q$ and $0\leqslant\alpha<n(1-1/q)$, we also get

$$
I_1\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{(j-k)pn/q'}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big[\frac{w_1(B_k)}{w_1(B_j)}\Big]^{\alpha p/n}\Big\}^{1/p}
$$

$$
\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{(j-k)pn/q'}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big[\frac{|B_k|}{|B_j|}\Big]^{\alpha p/n}\Big\}^{1/p}
$$

$$
\leqslant c\begin{cases}\Big\{\sum\limits_{k=-\infty}^{\infty}\Big[\sum\limits_{j=-\infty}^{k-2}2^{(k-j)(\alpha-n/q')p/2}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big] \\ \qquad\qquad \times\Big[\sum\limits_{j=-\infty}^{\infty}2^{(k-j)(\alpha-n/q')p'/2}\Big]^{p/p'}\Big\}^{1/p} & (p>1)\\ \Big\{\sum\limits_{k=-\infty}^{\infty}\Big[\sum\limits_{j=-\infty}^{k-2}2^{(k-j)(\alpha-n/q')p}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big]\Big\}^{1/p} & (0<p\leqslant 1)\end{cases}
$$

$$
\leqslant c\Big\{\sum_{j=-\infty}^{\infty}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big\}^{1/p}
$$

$$
\equiv c\|f\|_{K_q^{\alpha,p}(w_1,w_2)}.
$$

Now if $r'<q$, $j\leqslant k-2$ and $w_2\in A_1$ satisfying (1.3), then

$$
\|\chi_k M_\Omega(f_j)\|_{L^q(w_2)}
$$

$$
\leqslant c2^{-kn}[\sup_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||f_j(y)|\mathrm{d}y\Big)^q\mathrm{d}x\Big\}^{1/q}
$$

$$
\leqslant c2^{-kn}[\inf_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)|^r\mathrm{d}y\Big]^{q/r}\Big[\int_{E_j}|f_j(y)|^{r'}\mathrm{d}y\Big]^{q/r'}\mathrm{d}y\Big\}^{1/q}
$$

$$\leqslant c2^{-kn}[\operatorname{essinf}_{x\in B_j} w_2(x)]^{1/q}\Big[\int_{E_k}|x|^{nq/r}dx\Big]^{1/q}\Big[\int_{E_j}|f_j(y)|^q dy\Big]^{1/q}|E_j|^{1/r'-1/q}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c2^{-kn+jn/r'-jn/q+kn/r+kn/q}\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c2^{(k-j)n(1/q-1/r')}\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}.$$

Therefore, on the case of $r'<q$ and $\alpha<0$ we obtain

$$I_1\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=-\infty}^{k-2}2^{(k-j)n(1/q-1/r')}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=-\infty}^{k-2}2^{(k-j)n(1/q-1/r')}[w_1(B_j)]^{\alpha/n}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{j=-\infty}^{\infty}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big\}^{1/p}$$

$$\equiv c\|f\|_{K_q^{\alpha,p}(w_1,w_2)}.$$

For $r'<q$ and $0\leqslant\alpha<n(1/r'-1/q)+1/r$, choose $\beta$ such that $\alpha<-\beta+n/r'-n/q<n(1/r'-1/q)+1/r$, by Lemma 2.1 and (1.3), we obtain

$$\|\chi_k M_\Omega(f_j)\|_{L^q(w_2)}$$

$$\leqslant c2^{-kn}[\sup_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||f_j(y)|dy\Big)^q dx\Big\}^{1/q}$$

$$\leqslant cc_1 2^{-kn}[\inf_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)|^{q'}dy\Big]^{q/q'}dx\Big\}^{1/q}\|f_j\|_{L^q(R^n)}$$

$$\leqslant c2^{-kn}\Big[\frac{w_2(B_j)}{|B_j|}\Big]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)|^r dy\Big]^{q/r}dx\Big\}^{1/q}|E|^{(1/q'-1/r)}\|f_j\|_{L^q(R^n)}$$

$$\leqslant c2^{-kn+jn/q'-jn/r-j\beta}\Big[\frac{w_2(B_j)}{|B_j|}\Big]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)|^r|y|^{r\beta}dy\Big]^{q/r}dx\Big\}^{1/q}\|f_j\|_{L^q(R^n)}$$

$$\leqslant c2^{-kn+jn/q'-jn/r-j\beta)}[\operatorname{essinf}_{x\in B_j}w_2(x)]^{1/q}\Big\{\int_{E_k}|x|^{(n/r+\beta)q}dx\Big\}^{1/q}\|f_j\|_{L^q(R^n)}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c2^{(k-j)(\beta+n/r-n/q')}\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}.$$

On this case we obtain

$$I_1\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=-\infty}^{k-2}2^{(k-j)(\beta+n/r-n/q')}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha+\beta+n/r-n/q')}[w_1(B_j)]^{\alpha/n}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{j=-\infty}^{\infty}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big\}^{1/p}$$

$$\equiv c\|f\|_{K_q^{\alpha,p}(w_1,w_2)}.$$

Now we turn to the estimate for $I_3$. If $\alpha>0$, since $j\geqslant k+2$, by the $L^q(w_2)$-boundedness of $M_\Omega$, we have

$$
\begin{aligned}
I_3 &\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k+2}^{\infty}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\\
&\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=k+2}^{\infty}[w_1(B_j)]^{\alpha/n}\Big[\frac{w_1(B_k)}{w_1(B_j)}\Big]^{\alpha/n}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\\
&\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=k+2}^{\infty}[w_1(B_j)]^{\alpha/n}\Big[\frac{|B_k|}{|B_j|}\Big]^{\delta_1\alpha/n}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\\
&\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=k+2}^{\infty}[w_1(B_k)]^{\alpha/n}2^{(k-j)\alpha\delta_1}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}\\
&\leqslant c\Big\{\sum_{j=-\infty}^{\infty}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big\}^{1/p}\\
&\equiv c\|f\|_{K_q^{\alpha,p}(w_1,w_2)},
\end{aligned}
$$

where $\delta_1>0$ such that $w_1(B_k)/w_1(B_j)\leqslant c[|B_k|/|B_j|]^{\delta_1}$ with $c$ independent of $k<j$; see [12,page 401].

Since $w_2\in A_1$, for $j>k$, we see that for some $\delta>0$,

$$\inf_{x\in E_k}w_2(x)\leqslant c\frac{w_2(B_k)}{|B_k|}\leqslant c\frac{w_2(B_j)}{|B_j|}\Big(\frac{|B_k|}{|B_j|}\Big)^{\delta-1}\leqslant c2^{(j-k)n(1-\delta)}\operatorname*{essinf}_{x\in B_j}w_2(x). \qquad (2.3)$$

If $r>q$ and $n(1/r-\delta/q)-1/r<\alpha\leqslant 0$, for $x\in E_k, y\in E_j$ and $j>k$, we have $|x-y|\sim|y|$. Choose $b$ such that $\alpha>b+n/r-n\delta/q>(n-1)/r-n\delta/q$, by Lemma 2.1 and (1.3) and (2.3), we have

$$
\begin{aligned}
&\|\chi_kM_\Omega(f_j)\|_{L^p(w_2)}\\
&\leqslant c2^{-jn}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)f(y)|\mathrm{d}y\Big]^q w_2(x)\mathrm{d}x\Big\}^{1/q}\\
&\leqslant c2^{-jn}[\sup_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)f(y)|\mathrm{d}y\Big]^q\mathrm{d}x\Big\}^{1/q}\\
&\leqslant c2^{-jn}[\inf_{x\in E_k}w_2(x)]^{1/q}\int_{E_j}\Big[\int_{E_k}|\Omega(x-y)|^q\mathrm{d}x\Big]^{1/q}|f(y)|\mathrm{d}y\\
&\leqslant c2^{(j-k)n(1-\delta)/q-jn+kn/q-kn/r}[\operatorname*{essinf}_{x\in B_j}w_2(x)]^{1/q}\int_{E_j}\Big[\int_{E_k}|\Omega(x-y)|^r\mathrm{d}x\Big]^{1/r}|f(y)|\mathrm{d}y\\
&\leqslant c2^{(j-k)n(1-\delta)/q-jn+kn/q-kn/r-kb}[\operatorname*{essinf}_{x\in B_j}w_2(x)]^{1/q}\int_{E_j}\Big[\int_{E_k}|\Omega(x-y)|^r|x|^{br}\mathrm{d}x\Big]^{1/r}|f(y)|\mathrm{d}y\\
&\leqslant c2^{(j-k)n(1-\delta)/q-jn+kn/q-kn/r-kb}[\operatorname*{essinf}_{x\in B_j}w_2(x)]^{1/q}\Big\{\int_{E_j}|f(y)||y|^{b+n/r}\mathrm{d}y\Big\}\|\Omega\|_{L^r(S^{n-1})}\\
&\leqslant c2^{(k-j)(n\delta/q-b-n/r)}\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}.
\end{aligned}
$$

From this, we obtain

$$I_3 \leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta/q-b-n/r)}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta/q-b-n/r)}[w_1(B_j)]^{\alpha/n}\Big[\frac{w_1(B_k)}{w_1(B_j)}\Big]^{\alpha/n}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha-b-n/r+n\delta/q)}[w_1(B_j)]^{\alpha/n}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{j=-\infty}^{\infty}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^q(w_2)}\Big\}^{1/p}$$

$$\equiv c\|f\|_{K_q^{\alpha,p}(w_1,w_2)}.$$

If $r'<q$ and $-n\delta<\alpha\leqslant 0$, by (1.3) and (2.3), we have

$$\|\chi_k M_\Omega(f_j)\|_{L^p(w_2)}$$

$$\leqslant c2^{-jn}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)||f(y)|\mathrm{d}y\Big]^q w_2(x)\mathrm{d}x\Big\}^{1/q}$$

$$\leqslant c2^{-jn}[\sup_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)|^{q'}\mathrm{d}y\Big]^{q/q'}\mathrm{d}x\Big\}^{1/q}\|f_j\|_{L^q(R^n)}$$

$$\leqslant c2^{-jn}[\inf_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)|^{q'}\mathrm{d}y\Big]^{q/q'}\Big\}^{1/q}\|f_j\|_{L^q(R^n)}$$

$$\leqslant c2^{-jn+(j-k)n(1-\delta)/q}[\operatorname*{essinf}_{x\in B_j}w_2(x)]^{1/q}\|f_j\|_{L^q(R^n)}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)|^{q'}\mathrm{d}y\Big]^{q/q'}\mathrm{d}x\Big\}^{1/q}$$

$$\leqslant c2^{-jn+(j-k)n/q+(k-j)n\delta/q}\|f_j\|_{L^p(w_2)}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)|^{r}\mathrm{d}y\Big]^{q/r}\mathrm{d}x\Big\}^{1/q}|E_j|^{1/q'-1/r}$$

$$\leqslant c2^{-jn+(j-k)n/q+(k-j)n\delta/q+jn/q'-jn/r+kn/q}\|f_j\|_{L^p(w_2)}\|\Omega\|_{L^r(S^{n-1})}\Big[\int_{2^{j-1}}^{2^j}s^{n-1}\mathrm{d}s\Big]^{1/r}$$

$$\leqslant c2^{(k-j)n\delta/q}\|\Omega\|_{L^r(S^{n-1})}\|f_j\|_{L^p(w_2)}.$$

Thereofre,

$$I_3\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k+2}^{\infty}2^{(k-j)n\delta/q}\|f_j\|_{L^q(w_2)}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{k=-\infty}^{\infty}\Big[\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha+n\delta/q)}\|f_j\|_{L^q(w_2)}[w_1(B_j)]^{\alpha/n}\Big]^p\Big\}^{1/p}$$

$$\leqslant c\Big\{\sum_{j=-\infty}^{\infty}[w_1(B_j)]^{\alpha p/n}\|f_j\|^p_{L^p(w_2)}\Big\}^{1/p}$$

$$\equiv c\|f\|_{K_q^{\alpha,p}(w_1,w_2)}.$$

This finishes the proof of Theorem 2.1.

Now, let us consider some sublinear operators with rough kernel.

**Theorem 2.2.** *Let $\Omega\in L^r(S^{n-1})$ be homogeneous of degree zero, $1<r\leqslant\infty$, let the weight functions $w_1, w_2\in A_1$ and $w_2$ satisfy* (1.3). *Assume that $T$ is a linear operator satisfying*

$$|Tf(x)|\leqslant c|x|^{-n}\int_{\mathbb{R}^n}|\Omega(x-y)f(x)|\mathrm{d}y$$

*for $f\in L^1(\mathbb{R}^n)$, $\operatorname{supp}f\subseteq E_k$ and $|x|\geqslant 2^{k+1}$ with $k\in\mathbb{Z}$, and*

$$|Tf(x)|\leqslant c2^{-kn}\int_{\mathbb{R}^n}|\Omega(x-y)f(x)|\mathrm{d}y$$

*for $f\in L^1(\mathbb{R}^n)$, $\operatorname{supp}f\subseteq E_k$ and $|x|\leqslant 2^{k-2}$ with $k\in\mathbb{Z}$. If $0<p<\infty$ and $T$ is bounded on $L^q(w_2)$ for some $1<q<\infty$, then, $T$ is also bounded on $\dot{K}_q^{\alpha,p}(w_1,w_2)$, provided that $\alpha, q$ and $r$ satisfy one of the following*

(i) *$r>q$ and $\min\{n(1/r-\delta/q)-1/r, 0\}<\alpha<n(1-1/q)$; or*

(ii) *$r'<q$ and $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$.*

The proof of Theorem 2.2 is similar to that of Theorem 2.1 and we omit it. From Theorem 2.2 we can obtain the following simple corollary which is convenient for applications.

**Theorem 2.3.** *Let $0<p<\infty$, $1<q<\infty$, $w_1, w_2\in A_1$ and $w_2$ satisfy* (1.3). *Let $\Omega\in L^r(S^{n-1})$ be homogeneous of degree zero, $1<r\leqslant\infty$. Then sublinear operat to $T$ is bounded on $\dot{K}_q^{\alpha,p}(w_1,w_2)$ if $T$ is bounded on $L^q(w_2)$ and there is a constant $c_0$ independent of $f$ such that*

$$|Tf(x)|\leqslant c_0\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\mathrm{d}y \tag{2.4}$$

*for any $f\in L^1(\mathbb{R}^n)$ with compact support and $x\neq y$, and provided that $\alpha, q$ and $r$ satisfy one of the following*

(i) *$r>q$ and $\min\{n(1/r-\delta/q)-1/r, 0\}<\alpha<n(1-1/q)$; or*

(ii) *$r'<q$ and $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$.*

## 3 Commutators

We begin this section with the boundedness of the rough Hardy-Littlewood maximal commutators $M_{b,\Omega}$ relative to $b\in BMO(\mathbb{R}^n)$ and $\Omega\in L^r(S^{n-1})$ defined by

$$M_{b,\Omega}f(x)=\sup_{\rho>0}\frac{1}{\rho^n}\int_{|y|<\rho}|\Omega(x-y)||b(x)-b(y)||f(y)|\mathrm{d}y. \tag{3.1}$$

**Theorem 3.1.** *Let $0<p<\infty$, $1<q<\infty$, $w_1, w_2\in A_1$ and $w_2$ satisfy* (1.3). *Let $\Omega\in$*

*$L^r(S^{n-1})$ be homogeneous of degree zero*, $1<r\leqslant\infty, b\in BMO(\mathbb{R}^n)$. *If the maximal commutator $M_{b,\Omega}$ is bounded on $L^q(w_2)$, then the maximal commutator $M_{b,\Omega}$ is bounded on $K_q^{\alpha,p}(w_1, w_2)$ if $\alpha, q$ and $r$ satisfy one of the following*

(i) $r>q$ *and* $\min\{n(1/r-\delta/q)-1/r, 0\}<\alpha<n(1-1/q)$; *or*

(ii) $r'<q$ *and* $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$.

*Proof.* Write $f$ as in the proof of Theorem 2.1 and

$$\begin{aligned}\| M_{b,\Omega}(f)\|_{K_q^{\alpha,p}(w_1,w_2)} &= \Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\|\chi_k M_{b,\Omega}(f)\|_{L^q(w_2)}^p\Big\}^{1/p} \\ &\leqslant c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=-\infty}^{k-2}\|\chi_k M_{b,\Omega}(f_j)\|_{L^q(w_2)}\Big]^p\Big\}^{1/p} \\ &\quad + c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k-1}^{k+1}\|\chi_k M_{b,\Omega}(f_j)\|_{L^q(w_2)}\Big]^p\Big\}^{1/p} \\ &\quad + c\Big\{\sum_{k=-\infty}^{\infty}[w_1(B_k)]^{\alpha p/n}\Big[\sum_{j=k+2}^{\infty}\|\chi_k M_{b,\Omega}(f_j)\|_{L^q(w_2)}\Big]^p\Big\}^{1/p} \\ &\equiv H_1+H_2+H_3.\end{aligned}$$

Similar to the estimate of $I_2$ in the proof in Theorem 2.1, the estimate of $H_2$ is easy to get by the $L^q(w_2)$-boundedness of $M_{b,\Omega}$, thus we omit it.

For $H_1$, since $x\in E_k, y\in E_j$ and $j\leqslant k-3$, we obtian $|x-y|\sim|x|$. Then

$$M_{b,\Omega}(f_j)(x)\leqslant c2^{-kn}\int_{E_j}|\Omega(x-y)||b(x)-b(y)||f(y)|\mathrm{d}y$$

and

$$\begin{aligned}\|\chi_k M_{b,\Omega}(f_j)\|_{L^q(w_2)} &\leqslant c2^{-kn}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||b(x)-b(y)||f(y)|\mathrm{d}y\Big)^q w_2(x)\mathrm{d}x\Big\}^{1/q} \\ &\leqslant c2^{-kn}[\sup_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||b(x)-b(y)||f(y)|\mathrm{d}y\Big)^q\mathrm{d}x\Big\}^{1/q} \\ &\leqslant cc_1 2^{-kn}[\inf_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||b(x)-b_j||f(y)|\mathrm{d}y\Big)^q\mathrm{d}x\Big\}^{1/q} \\ &\quad + cc_1 2^{-kn}[\inf_{x\in E_k}w_2(x)]^{1/q}\Big\{\int_{E_k}\Big(\int_{E_j}|\Omega(x-y)||b(y)-b_j||f(y)|\mathrm{d}y\Big)^q\mathrm{d}x\Big\}^{1/q},\end{aligned}$$

where and in what follows

$$b_j=\frac{1}{|B(0,2^j)|}\int_{B(0,2^j)}b(y)\mathrm{d}y.$$

Now if $r>q$, for $\alpha<n(1-1/q)$ and $w_2\in A_1$ satisfying (1.3), then

$$\|\chi_k M_{b,\Omega}(f_j)\|_{L^q(w_2)}$$

$$\leqslant c2^{-kn}[\inf_{x\in E_k} w_2(x)]^{1/q}\int_{E_j}|f(y)|\Big[\int_{E_k}(|\Omega(x-y)||b(x)-b_j|)^q dx\Big]^{1/q}dy$$

$$+c2^{-kn}[\inf_{x\in E_k} w_2(x)]^{1/q}\Big\{\int_{E_j}|f(y)||b(y)-b_j|\Big[\int_{E_k}|\Omega(x-y)|^q dx\Big]^{1/q}dy$$

$$\leqslant c2^{-kn}\Big[\frac{w_2(B_k)}{|B_k|}\Big]^{1/q}\int_{E_j}|f(y)|\Big[\int_{E_k}|\Omega(x-y)|^r dx\Big]^{1/r}\Big[\int_{E_k}|b(x)-b_j|^{rq/(r-q)}dx\Big]^{1/q-1/r}dy$$

$$+c2^{-kn}\Big[\frac{w_2(B_k)}{|B_k|}\Big]^{1/q}\int_{E_j}|f(y)||b(y)-b_j|\Big[\int_{E_k}|\Omega(x-y)|^r dx\Big]^{1/r}dy\Big\}|E_k|^{1/q-1/r}$$

$$\leqslant c2^{-kn+kn/q}(k-j)\Big[\frac{w_2(B_j)}{|B_j|}\Big]^{1/q}\Big\{\int_{E_j}|f(y)|dy\Big\}\|b\|_{BMO(R^n)}\|\Omega\|_{L^r(S^{n-1})}$$

$$+c2^{-kn+kn/q}\Big[\frac{w_2(B_j)}{|B_j|}\Big]^{1/q}\Big\{\int_{E_j}|f(y)||b(y)-b_j|dy\Big\}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c[\operatorname*{essinf}_{x\in B_j} w_2(x)]^{1/q}2^{-kn+kn/q+jn/q'}(k-j)\|f_j\|_{L^q(R^n)}\|b\|_{BMO(R^n)}\|\Omega\|_{L^r(S^{n-1})}$$

$$+c[\operatorname*{essinf}_{x\in B_j} w_2(x)]^{1/q}2^{-kn+kn/q+jn/q'}\|f_j\|_{L^q(R^n)}\|b\|_{BMO(R^n)}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c2^{(j-k)n/q'}(k-j)\|f_j\|_{L^q(w_2)}\|b\|_{BMO(R^n)}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c2^{(j-k)n(1-1/q)}(k-j)\|f_j\|_{L^q(w_2)}.$$

From this, it is easy to deduce a desirable estimate for $H_1$ in the case of $r>q$ and $\alpha<n(1-1/q)$.

Let us now consider the case of $r'<q$ and $\alpha<n(1/r'-1/q)+1/r$. Choose $\eta$ such that $\alpha<-\eta+n/r'-n/q<n(1/r'-1/q)+1/r$, by the properties of $BMO(R^n)$ functions and Lemma 2.1, for $j\leqslant k-3$, we have

$$\|\chi_k M_{b,\Omega}(f_j)\|_{L^q(w_2)}$$

$$\leqslant c2^{-kn}[\sup_{x\in E_k} w_2(x)]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_k}|\Omega(x-y)||b(x)-b(y)||f(y)|dy\Big]^q dy\Big\}^{1/q}$$

$$\leqslant cc_1 2^{-kn}[\inf_{x\in E_k} w_2(x)]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)||b(x)-b_j||f(y)|dy\Big]^q dx\Big\}^{1/q}$$

$$+cc_1 2^{-kn}[\inf_{x\in E_k} w_2(x)]^{1/q}\Big\{\int_{E_k}\Big[\int_{E_j}|\Omega(x-y)||b(y)-b_j||f(y)|dy\Big]^q dx\Big\}^{1/q}$$

$$\leqslant c2^{-kn}[\operatorname*{essinf}_{x\in B_j} w_2(x)]^{1/q}\|f_j\|_{L^q(R^n)}\Big\{\int_{E_k}|b(x)-b_j|^q\Big[\int_{E_j}|\Omega(x-y)|^{q'}dy\Big]^{q/q'}dx\Big\}^{1/q}$$

$$+c2^{-kn}[\operatorname*{essinf}_{x\in B_j} w_2(x)]^{1/q}\|f_j\|_{L^q(R^n)}\Big\{\int_{E_k}\Big[\int_{E_j}(|\Omega(x-y)||b(y)-b_j|)^{q'}dy\Big]^{q/q'}dx\Big\}^{1/q}$$

$$\leqslant c2^{-kn}\|f_j\|_{L^q(w_2)}\left\{\int_{E_k}|b(x)-b_j|^q\left[\int_{E_j}|\Omega(x-y)|^r\mathrm{d}y\right]^{q/r}\mathrm{d}x\right\}^{1/q}|E_j|^{(1/q'-1/r)}$$

$$+c2^{-kn+jn-jn/r-jn/q}\|f_j\|_{L^q(w_2)}\left\{\int_{E_k}\left[\int_{E_j}|\Omega(x-y)|^r\mathrm{d}y\right]^{q/r}\mathrm{d}x\right\}^{1/q}\|b\|_{BMO(\mathbb{R})^n}$$

$$\leqslant c2^{-kn+jn/q'-jn/r-j\eta}\|f_j\|_{L^q(w_2)}\left\{\int_{E_k}|b(x)-b_j|^q\left[\int_{E_j}|\Omega(x-y)|^r|y|^{r\eta}\mathrm{d}y\right]^{q/r}\mathrm{d}x\right\}^{1/q}$$

$$+c2^{-kn+jn-jn/r-jn/q-j\eta}\|f_j\|_{L^q(w_2)}\|b\|_{BMO(\mathbb{R}^n)}\left\{\int_{E_k}\left[\int_{E_j}|\Omega(x-y)|^r|y|^{r\eta}\mathrm{d}y\right]^{q/r}\mathrm{d}x\right\}^{1/q}$$

$$\leqslant c2^{-kn+jn/q'-jn/r-j\eta}\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}\left\{\int_{E_k}|b(x)-b_j|^q|x|^{(\eta+n/r)q}\mathrm{d}x\right\}^{1/q}$$

$$+c2^{-kn+jn/q'-jn/r-j\eta}\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}\|b\|_{BMO(\mathbb{R}^n)}\left\{\int_{E_k}|x|^{(\eta+n/r)q}\mathrm{d}x\right\}^{1/q}$$

$$\leqslant c2^{-kn+jn/q'-jn/r-j\eta+k\eta+kn/r+kn/q}(k-j)\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}\|b\|_{BMO(\mathbb{R}^n)}$$

$$+c2^{-kn+jn/q'-jn/r-j\eta+k\eta+kn/r+kn/q}\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}\|b\|_{BMO(\mathbb{R}^n)}$$

$$\leqslant c2^{(k-j)(\eta+n/r-n/q')}(k-j)\|f_j\|_{L^q(w_2)}\|\Omega\|_{L^r(S^{n-1})}\|b\|_{BMO(\mathbb{R}^n)}.$$

By this, we can easily obtain another desirable estimate for $H_1$.

For $H_3$, the case of $\alpha>0$ is all the same to the equivalent part of $I_3$, we omit it. Now, let us consider $H_3$ on the case of $\alpha\leqslant 0$. The proof of this part is similar to the equivalent part of $I_3$, so we just give out an outline.

If $r>q$ and $n(1/r-\delta.q)-1/r<\alpha\leqslant 0$, for $w_1\in A_1$, $x\in E_k$, $y\in E_j$ and $j\geqslant k+3$, choose $b$ such that $\alpha>b+n/r-n\delta/q>(n-1)/r-n\delta/q$, by Lemma 2.1, we have

$$\|\chi_k M_{b,\Omega}(f_j)\|_{L^q(w_2)}$$

$$\leqslant c2^{-jn}[\sup_{x\in E_k}w_2(x)]^{1/q}\left\{\int_{E_k}\left[\int_{E_j}|\Omega(x-y)||b(x)-b(y)||f(y)|\mathrm{d}y\right]^q\mathrm{d}x\right\}^{1/q}$$

$$\leqslant cc_1 2^{-jn}[\inf_{x\in E_k}w_2(x)]^{1/q}\left\{\int_{E_k}\left[\int_{E_j}|\Omega(x-y)||b(x)-b_j||f(y)|\mathrm{d}y\right]^q\mathrm{d}x\right\}^{1/q}$$

$$+cc_1 2^{-jn}[\inf_{x\in E_k}w_2(x)]^{1/q}\left\{\int_{E_k}\left[\int_{E_j}|\Omega(x-y)||b(y)-b_j||f(y)|\mathrm{d}y\right]^q\mathrm{d}x\right\}^{1/q}$$

$$\leqslant c2^{-jn}[\inf_{x\in E_k}w_2(x)]^{1/q}\int_{E_j}|f(y)|\left[\int_{E_k}(|\Omega(x-y)||b(x)-b_j|)^q\mathrm{d}x\right]^{1/q}\mathrm{d}y$$

$$+c2^{-jn}[\inf_{x\in E_k}w_2(x)]^{1/q}\int_{E_j}|f(y)||b(x)-b_j|\left[\int_{E_k}|\Omega(x-y)|^q\mathrm{d}x\right]^{1/q}\mathrm{d}y$$

$$\leqslant c2^{-jn}[\inf_{x\in E_k}w_2(x)]^{1/q}\int_{E_j}|f(y)|\left[\int_{E_k}|\Omega(x-y)|^r\mathrm{d}x\right]^{1/r}\left[\int_{E_k}|b(x)-b_j|^{rq/(r-q)}\mathrm{d}x\right]^{1/q-1/r}\mathrm{d}y$$

$$+ c2^{-jn+kn/q-kn/r}[\inf_{x\in E_k} w_2(x)]^{1/q}\int_{E_j}|f(y)||b(x)-b_j|\left[\int_{E_k}|\Omega(x-y)|^r dx\right]^{1/r}dy$$

$$\leqslant c2^{-jn-kb+(j-k)n(1-\delta)/q+kn/q-kn/r}(j-k)[\operatorname{essinf}_{x\in B_j} w_2(x)]^{1/q}\|b\|_{BMO(\mathbb{R}^n)}$$

$$\times\int_{E_j}|f(y)|\left[\int_{E_k}|\Omega(x-y)|^r|x|^{rb}dx\right]^{1/r}dy$$

$$+ c2^{-jn-kb+(j-k)n(1-\delta)/q+kn/q-kn/r}[\operatorname{essinf}_{x\in B_j} w_2(x)]^{1/q}$$

$$\times\int_{E_j}|f(y)||b(x)-b_j|\left[\int_{E_k}|\Omega(x-y)|^r|x|^{rb}dx\right]^{1/r}dy$$

$$\leqslant c2^{-jn-kb+(j-k)n(1-\delta)/q+kn/q-kn/r}(j-k)$$

$$\times[\operatorname{essinf}_{x\in B_j} w_2(x)]^{1/q}\|b\|_{BMO(\mathbb{R}^n)}\|\Omega\|_{L^r(S^{n-1})}\int_{E_j}|f(y)||y|^{b+n/r}dy$$

$$+ c2^{-jn-kb+(j-k)n(1-\delta)/q+kn/q-kn/r}[\operatorname{essinf}_{x\in B_j} w_2(x)]^{1/q}\int_{E_j}|f(y)||b(x)-b_j||y|^{b+n/r}dy$$

$$\leqslant c2^{(k-j)(n\delta/q-b-n/r)}(j-k)\|f_j\|_{L^q(w_2)}\|b\|_{BMO(\mathbb{R}^n)}\|\Omega\|_{L^r(S^{n-1})}.$$

From this, it is easy to deduce a desirable estimate for $H_3$. Now we consider the case of $r'<q$ and $-n\delta/q<\alpha\leqslant 0$. For $j\geqslant k+3$, by the properties of $BMO(\mathbb{R}^n)$ functions and Hölder's inequality, we have

$$\|\chi_k M_{b,\Omega}(f_j)\|_{L^q(w_2)}$$

$$\leqslant c2^{-jn}[\inf_{x\in E_k} w_2(x)]^{1/q}\left\{\int_{E_k}\left[\int_{E_j}|\Omega(x-y)||b(x)-b_j||f(y)|dy\right]^q dx\right\}^{1/q}$$

$$+ c2^{-jn}[\inf_{x\in E_k} w_2(x)]^{1/q}\left\{\int_{E_k}\left[\int_{E_j}|\Omega(x-y)||b(y)-b_j||f(y)|dy\right]^q dx\right\}^{1/q}$$

$$\leqslant c2^{-jn+(j-k)n(1-\delta)/q}[\operatorname{essinf}_{x\in B_j} w_2(x)]^{1/q}\|f_j\|_{L^q(\mathbb{R}^n)}$$

$$\times\left\{\int_{E_k}|b(x)-b_j|^q\left[\int_{E_j}|\Omega(x-y)|^{q'}dy\right]^{q/q'}dx\right\}^{1/q}$$

$$+ c2^{-jn+(j-k)n(1-\delta)/q}[\operatorname{essinf}_{x\in B_j} w_2(x)]^{1/q}\|f_j\|_{L^q(\mathbb{R}^n)}$$

$$\times\left\{\int_{E_k}\left[\int_{E_j}|\Omega(x-y)|^r dy\right]^{q/r}\left[\int_{E_j}|b(y)-b_j|^{rq/(rq-r-q)}dy\right]^{(1-1/q-1/r)q}dx\right\}^{1/q}$$

$$\leqslant c2^{-jn+(j-k)n(1-\delta)/q}\|f_j\|_{L^q(w_2)}\left\{\int_{E_k}|b(x)-b_j|^q dx\right\}^{1/q}\left(\int_{2^{j-1}}^{2^j}s^{n-1}ds\right)^{1/q'}\|\Omega\|_{L^{q'}(S^{n-1})}$$

$$+ c2^{-jn+(j-k)n(1-\delta)/q+jn(1-1/q-1/r)+kn/q}\|f_j\|_{L^q(w_2)}\left(\int_{2^{j-1}}^{2^j}s^{n-1}ds\right)^{1/r}\|b\|_{BMO(\mathbb{R}^n)}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leqslant c2^{(k-j)n\delta/q}(j-k)\,\|f_j\|_{L^q(w_2)}\,\|b\|_{BMO(\mathbb{R}^n)}\,\|\Omega\|_{L^r(S^{n-1})}.$$

From this, it is easy to deduce another esirable estimate for $H_3$.

This finishes the proof of Theorem 3.1.

Now, we state one of our main theorems on the commutators generated by some rough linear operators with $BMO$ functions.

**Theorem 3.2.** *Let $b\in BMO(\mathbb{R}^n)$ and $\Omega\in L^r(S^{n-1})$ be homogeneous of degree zero, $1<r\leqslant\infty$. Let the weight functions $w_1,w_2\in A_1$ and $w_2$ satisfy (1.3). Assume that $T$ is a linear operator satisfying the size condition in Theorem 2.2. If $0<p<\infty$ and the commutator $[b,T]$ is bounded on $L^q(w_2)$ for some $1<q<\infty$, then, $[b,T]$ is bounded on $\dot{K}_q^{\alpha,p}(w_1,w_2)$, provided that $\alpha,q$ and $r$ satisfy one of the following*

(i) $r>q$ *and* $\min\{n(/r-\delta/q)-1/r,0\}<\alpha<n(1-1/q)$; *or*

(ii) $r'<q$ *and* $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$.

The proof of This theorem is similar to that of Theorem 3.1, we omit the details here. The following result which is convenient for applications is directly deduced from Theorem 3.2.

**Theorem 3.3.** *Let $b\in BMO(\mathbb{R}^n)$ and $\Omega\in L^r(S^{n-1})$ be homogeneous of degree zero, $1<r\leqslant\infty$. Let the weight functions $w_1,w_2\in A_1$ and $w_2$ satisfy (1.3). Assume that $T$ is a linear operator satisfying the size condition (2.4). If $0<p<\infty$ and the commutator $[b,T]$ is bounded on $L^q(w_2)$ for some $1<q<\infty$, then, $[b,T]$ is bounded on $\dot{K}_q^{\alpha,p}(w_1,w_2)$, provided that $\alpha,q$ and $r$ satisfy one of the following*

(i) $r>q$ *and* $\min\{n(/r-\delta/q)-1/r,0\}<\alpha<n(1-1/q)$; *or*

(ii) $r'<q$ *and* $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$.

## 4 Some corollaries

In this section, we will give some corollaries of our main results. We state the corollaries only for the case of the commutators; and for the case of the sublinear operators the same results are hold, we omit them.

For the rough Hardy-Littlewood maximal commutator $M_{b,\Omega}$, by [13] and Theorem 3.1, we can easily obtain the following corollary.

**Corollary 4.1.** *Lt $0<p<\infty,1<q<\infty,w_1,w_2\in A_1$ and $w_2$ satisfy (1.3). Let $\Omega\in L^r(S^{n-1})$ be homogeneous of degree zero, $1<r\leqslant\infty$. $r'<q$ and $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$, then the maximal commutator $M_{b,\Omega}$ is bounded on $\dot{K}_q^{\alpha,p}(w_1,w_2)$.*

We point out that the size condition (1.4) is saitsfied by many operators in harmonic analysis, such as Calderón-Zygmund operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stain's oscillatory singular integrals, the Bochner-Riesz operators at critical index, and so on. For example, as the special cases of Theorem 3.3, we have the following corollaries.

**Corollary 4.2.** *Let* $0<p<\infty, 1<q<\infty$ *and* $-n\delta/q<\alpha<n(1-1/q)$. *Suppose that* $b \in BMO(\mathbb{R}^n)$ *and* $T$ *be the Calderón-Zygmund operators. Moreover,* $w_1, w_2 \in A_1$ *and* $w_2$ *satisfies* (1.3). *Then* $[b,T]$ *is bounded on the Herz spaces* $\dot{K}_q^{\alpha,p}(w_1,w_2)$.

**Corollary 4.3.** *Let* $0<p<\infty, 1<q<\infty$ *and* $-n\delta/q<\alpha<n(1-1/q)$. *Suppose that* $b \in BMO(\mathbb{R}^n)$, $k$ *be a standard Calderón-Zygmund kernel, and* $I$ *be defined by*

$$If(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} k(x-y) f(y) \mathrm{d}y,$$

*where* $P(x,y)$ *is a real polynomial on* $\mathbb{R}^n \times \mathbb{R}^n$. *Moreover, that* $w_1, w_2 \in A_1$ *and* $w_2$ *satisfies* (1.3). *Then,* $[b,I]$ *is bounded on* $\dot{K}_q^{\alpha,p}(w_1,w_2)$ *with the operator norm independent of the coefficients of* $P$.

**Corollary 4.4.** *Let* $0<p<\infty, 1<q<\infty, w_1, w_2 \in A_1$ *and* $w_2$ *satisfy* (1.3). *Let* $\Omega \in L^r(S^{n-1})$ *be homogeneous of degree zero,* $1<r\leqslant\infty$, *and* $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. *Assume that* $T_\Omega$ *is given by* (1.6) *and* $h \in L^\infty(\mathbb{R}_+)$. *If* $r'<q$ *and* $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$, *then the commutator* $[b,T_\Omega]$ *is bounded on* $\dot{K}_q^{\alpha,p}(w_1,w_2)$.

**Corollary 4.5.** *Let* $0<p<\infty, 1<q<\infty, w_1, w_2 \in A_1$ *and* $w_2$ *satisfy* (1.3). *Let* $\Omega \in L^r(S^{n-1})$ *be homogeneous of degree zero,* $1<r\leqslant\infty$, *assume that* $T_{\Omega,P}$ *is given by* (1.7) *and* $h$ *is a bounded variation function on* $\mathbb{R}_+$. *If* $r'<q$ *and* $-n\delta/q<\alpha<n(1/r'-1/q)+1/r$, *then the commutator* $[b,T_{\Omega,P}]$ *is bounded on* $\dot{K}_q^{\alpha,p}(w_1,w_2)$.

*Remark* 4.1. If the linear operaotr $T$ is bounded on $L^q(w_2)$, by Theorem 2.13 in [14], we know that the commutator $[b,T]$ is bounded on $L^q(w_2)$. Then, by Theorem 3.3, Corollaries 4.2 and 4.3 are obvious; by Theorem 3.3 and Theorem 1 in [15], we can easily deduce Corollary 4.4; by Theorem 3.3 and Theorem 6 in [16], we obtain Corollary 4.5.

*Remark* 4.2. If we choose $w_1 \equiv w_2 \equiv 1$ in Theorem 3.1 and Theorem 3.2, then we get Theorem 2 and Theorem 4 in [10].

*Acknowledgement.* We would like to thank an anonymous referee for the helpful comment which improved the presentation of this paper.

## References

[1] Lu, S.Z. and Yang, D., The Decomposition of Weighted Herz Space on $R^n$ and its Applica-

tions, Science in China, Ser. A, 38(1995), 147—158.

[2] Stein, E. M., Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Univ. Press, Princeton, N. J., 1993.

[3] Stein, E. M., Note on Singular Integrals, Proc. Amer. Math. Soc., 8(1957), 250—254.

[4] Soria, F. and Weiss, G., A Remark on Singular Integrals and Power Weights, Indiana Univ. Math., J., 43(1994), 187—204.

[5] Coifman, R. R., Rochberg, R. and Weiss, G., Fractorization Theorems for Hardy Spaces in Several Variable, Ann. of Math., 103(1976), 611—635.

[6] Lu, S. Z. & Yang, D., The Continuity of Commutators on Herz-Type Spaces, Michigan Math. J., 44(1997), 255—281.

[7] Jiang, Y. S., Tang, L. & Yang, D., Continuity of Commutators on Weighted Herz-Type Spaces, Kyushu J. Math., 53(1999), 245—263.

[8] Hu, G., Lu, S. Z. & Yang, D., Boundedness of Rough Singular Integral Operators on Herz Spaces, J. Austral. Math. Soc. (Series A), 66(1999), 201—223.

[9] Tang, L., Jiang Y. & Lu, S. Z., Sublinear Operators on Weighted Herz Spaces, J. Beijing Normal Univ. (Natural Science), 35(1999), 167—173.

[10] Lu, S. Z., Tang, L., Yang, D., Boundedness of Commutators on Homogeneous Herz Spaces, Science in China, Ser. A, 41(1998), 1023—1033.

[11] Muckenhoupt B. & Wheeden, R. L., Weighted Norm Inequalities for Singular and Fractional Integrals, Trans. Amer. Math. Soc., 161(1971), 249—258.

[12] Garcia-Cuerva J. & Rubio de Francia, J. L., Weighted Norm Inequalities and Related Topics, North-Holland Math. Studies, Vol,16, North-Holland, Amsterdam, 1985.

[13] Hu, G., Weighted Norm Inequalities for Commutators of Homogeneous Singular Integrals, Acta Math. Sinica (New Ser.), 11(1995), 77—88.

[14] Alvarez, J., Babgy, R. J., Kurtz, D. S. & Pérez, C., Weighted Estimate for Commutators of Linear Operators, Studia Math., 104(1993), 195—209.

[15] Watson, D., Weighted Estimates for Singular Integrals Via Fourier Transform Estimates, Duke Math. J., 60(1990), 389—399.

[16] Jiang, Y. S. and Lu, S. Z., Oscillatory Singular Integral with Rough Kernel, Harmonic Analysis in China, Kluwer Academic Publishers, the Netherlands, 1995, 135—145.

Department of Mathematics
Xinjiang University
Urumqi 830046
PR China
E-mail: ysjiang@xju.edu.cn