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SHAPE-PRESERVING BIVARIATE POLYNOMIAL APPROXIMATION IN $C([-1,1]\times[-1,1])$

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Abstract

In this paper we construct bivariate polynomials attached to a bivariate function, that approximate with Jackson-type rate involving a bivariate Ditzian-Totik ω_1^φ-modulus of smoothness and preserve some natural kinds of bivariate monotonicity and convexity of function.

The result extends that in univariate case of D. Leviatan in [5-6], improves that in bivariate case of the author in [3] and in some special cases, that in bivariate case of G. Anastassiou in [1].

1 Introduction

In a very recent paper [3] we proved the following Jackson-type result in bivariate monotone approximation, by using some natural kinds of bivariate monotonicities.

Theorem 1.1. *If* $f:[-1,1]\times[-1,1]\to\mathbb{R}$ *is continuous, then there exists a sequence of bivariate polynomials* $(P_{n_1,n_2}(f)(x,y))_{n_1,n_2\in\mathbb{N}}$, degree $(P_{n_1,n_2}(f))\leqslant n_1+n_2$, *such that*

$$|f(x,y)-P_{n_1,n_2}(f)(x,y)|\leqslant C\omega_1\left(f;\frac{1}{n_1},\frac{1}{n_2}\right),$$

$\forall\, n_1,n_2\in\mathbb{N}$, $x,y\in[-1,1]$, $C>0$ *independent of* f, n_1,n_2,x *and* y, *satisfying moreover the following shape-preserving properties:*

(i) *if* $f(x,y)$ *is increasing (decreasing) with respect to* x *on* $[-1,1]$, *then so is* $P_{n_1,n_2}(f)(x,y)$ *too;*

(ii) *if* $f(x,y)$ *is increasing (decreasing) with respect to* y *on* $[-1,1]$, *then so is*

$P_{n_1,n_2}(f)(x,y)$ *too*;

(iii) *if* $f(x,y)$ *is upper (lower) bidimensional monotone on* $[-1,1]\times[-1,1]$, *then so is* $P_{n_1,n_2}(f)(x,y)$ *too*;

(iv) *if* $f(x,y)$ *is totaly upper (lower) motonone on* $[-1,1]\times[-1,1]$, *then so is* $P_{n_1,n_2}(f)(x,y)$ *too.*

In this paper, in Section 3 we improve Theorem 1.1 in the sense that $\omega_1(f;\cdot,\cdot)$ can be replaced by a bivariate Ditzian-Totik modulus of smoothness $\omega_2^\varphi(f;\cdot,\cdot)$, and that the sequence $(P_{n_1,n_2}(f)(x,y))_{n_1,n_2\in N}$, in addition preserve some kinds of bivariate convexity too.

2 Preliminaries

In order to extend the results in monotone and convex approximation from univariate case to bivariate case, obviously we need suitable bivariate moduli of smoothenss and some suitable extensions of monotonicity and convexity to bivariate functions.

In this paper we consider the bivariate Ditzian-Totik modulus of smoothness (attached to a function $f:[-1,1]\times[-1,1]\to R$ given by (see [2,Chapter 12])

$$\omega_2^\varphi(f;\delta_1,\delta_2)=\sup\{|\Delta_{h_1\varphi(x),h_2\varphi(y)}f(x,y)| : 0\leqslant h_i\leqslant\delta_i,\ i=1,2 \text{ and } x,y\in[-1,1]\},$$

where $\varphi(t)=\sqrt{1-t^2}$,

$$\Delta^2_{h_1\varphi(x),h_2\varphi(y)}f(x,y)=\sum_{k=0}^{2}\binom{2}{k}(-1)^k f(x+(1-k)h_1\varphi(x),y+(1-k)h_2\varphi(y)),$$

if $(x\pm h_1\varphi(x),y\pm h_2\varphi(y))\in[-1,1]\times[-1,1]$, $\Delta^2_{h_1\varphi(x),h_2\varphi(y)}f(x,y)=0$ elsewhere, and the concepts of bivariate convexities of various orders introduced by T. Popoviciu.

Definition 2.1 ([10,*p.* 78]). The function $f:[-1,1]\times[-1,1]\to R$ is called convex of order (n,m) (where $n,m\in\{-1,0,1,\cdots,\}$) if for any $n+2$ distinct points $x_1<x_2<\cdots<x_{n+2}$ and any $m+2$ dictint points $y_1<y_2<\cdots<y_{m+2}$ in $[-1,1]$, we have

$$\begin{bmatrix} x_1,x_2,\cdots,x_{n+2} \\ y_1,y_2,\cdots,y_{m+2} \end{bmatrix}; f\Bigg]\geqslant 0,$$

where the symbol above represents the divided difference of a bivariate function and it is defined iteratively (by means of the divided difference of univariate functions) as (see [10, p. 64−65])

$$[x_1,\cdots,x_{n+2};[y_1,\cdots,y_{m+2};f(x,\cdot)]_y]_x=[y_1,\cdots,y_{m+2};[x_1,\cdots,x_{m+2};f(\cdot,y)]_x]_y$$

here

$$[x_1,\cdots,x_p;g(\cdot)]=\sum_{i=1}^{p}\frac{g(x_i)}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_p)}$$

represents the usual divided difference of univariate function g, $[g;x_1]=g(x_1)$.

Remarks. 1) It is obvious that convexities of orders $(-1,0)$ and $(0,-1)$ means in fact that $f(x,y)$ is increasing on $[-1,1]$ with respect to y (for any fixed $x\in[-1,1]$) and increasing with respect to x (for any fixed $y\in[-1,1]$), respectively. Also, convexity of order $(0,0)$ one reduces to upper bidimensional monotony introduced in [7, p. 33], simultaneously convexities of order $(-1,0)$, $(0,-1)$ and $(0,0)$ means the totally upper monotony introduced in [8], convexity of order $(-1,1)$ means in fact that $f(x,y)$ is convex on $[-1,1]$ with respect to y (for any fixed x), and so on.

2) Suppose f is of C^{n+m+2} class on $[-1,1]\times[-1,1]$.

By the mean value theorem we get that if $\frac{\partial^{n+m+2}f(x,y)}{\partial x^{n+1}\partial y^{m+1}}\geqslant 0$, $\forall\ (x,y)\in[-1,1]\times[-1,1]$, then $f(x,y)$ is convex of order (n,m) on $[-1,1]\times[-1,1]$.

3) We also need to introduce the following concept similar to that of totally upper monotony:

$f(x,y)$ will be called totally convex on $[-1,1]\times[-1,1]$ if $f(x,y)$ is simultaneously convex of the orders $(-1,1)$, $(1,-1)$, $(0,1)$, $(1,0)$ and $(1,1)$.

4) If f is continuous on $[-1,1]\times[-1,1]$ then the uniform norm of f is defined by

$$\|f\|=\sup\{|f(x,y)|;x,y\in[-1,1]\}.$$

3 Main result

Our main result can be stated as follows.

Theorem 3.1. *If $f:[-1,1]\times[-1,1]\to\mathbb{R}$ is continuous, then there exists a sequence of bivariate polynomials $(P_{n_1,n_2}(f)(x,y))_{n_1,n_2\in\mathbb{N}}$,* degree $(P_{n_1,n_2}(f)(x,y))\leqslant n_1+n_2$, *such that*

$$\|f-P_{n_1,n_2}\|\leqslant C\omega_2^{\varphi}\left(f;\frac{1}{n_1},\frac{1}{n_2}\right),\qquad \forall\ n_1,n_2\in\mathbb{N},$$

where $C>0$ is independent of f, n_1 and n_2, satisfying moreover the following shape-preserving properties:

(i) *if f is convex of order $(0,0)$ on $[-1,1]\times[-1,1]$ (i.e. according to Remark 1 in Section 2 f is upper bidimensional monotone) then so is $P_{n_1,n_2}(f)$ too;*

(ii) *if f is simultaneously convex of the orders $(-1,0)$, $(0,-1)$ and $(0,0)$ (i.e. according to Remark 1 of Section 2 $f(x,y)$ is totally upper monotone) then so is $P_{n_1,n_2}(f)$ too;*

(iii) *if* f *is convex of order* $(1,1)$ *on* $[-1,1]\times[-1,1]$, *then so is* $P_{n_1,n_2}(f)$ *too*;

(iv) *if* f *is totally convex on* $[-1,1]\times[-1,1]$, *then so is* $P_{n_1,n_2}(f)$ *too.*

Proof. If $g:[-1,1]\to \mathbb{R}$, then according to [5, relation (5)], the approximation polynomials are given by

$$P_n(g)(x) = g(-1) + \sum_{j=0}^{n-1} s_{j,n}(R_{j,n}(x) - R_{j+1,n}(x)),$$

where $s_{j,n}=\dfrac{g(\xi_{j+1,n})-g(\xi_{j,n})}{\xi_{j+1,n}-\xi_{j,n}}$, $\{\xi_{j,n}, j=\overline{0,n}\}$ suitable nodes in $[-1,1]$ and $R_{j,n}(x)$ suitable choosen polynomials of degree $\leqslant n$.

According to [5, Theorem 1], we have

$$\| g - P_n(g) \| \leqslant C\omega_2^{\varphi}\left(g;\frac{1}{n}\right), \qquad \forall\, n \in \mathbb{N},$$

where $\omega_2^{\varphi}(g;\delta)$ is the usual Ditzian-Totik modulus of smoothness.

We will construct the polynomials $P_{n_1,n_2}(f)(x,y)$ by applying the tensor product method (see e. g. , [9, p. 195—196]).

We obtain

$$\begin{aligned} P_{n_1,n_2}(f)(x,y) &= f(-1,-1) \\ &+ \sum_{i=0}^{n_2-1} \frac{(f-1,\eta_{i+1,n_2}) - f(-1,\eta_{i,n_2})}{\eta_{i+1,n_2} - \eta_{i,n_2}}[R_{i,n_2}(y) - R_{i+1,n_2}(y)] \\ &+ \sum_{j=0}^{n_1-1} \frac{f(\xi_{j+1,n_1},-1) - f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1} - \xi_{j,n_1}}[R_{j,n_1}(x) - R_{j+1,n_1}(x)] \\ &+ \sum_{j=0}^{n_1-1}\sum_{i=0}^{n_2-1} S_{i,j}^{*}[R_{i,n_2}(y) - R_{i+1,n_2}(y)][R_{j,n_1}(x) - R_{j+1,n_1}(x)], \end{aligned}$$

where

$$\begin{aligned} S_{i,j}^{*} &= \frac{f(\xi_{j+1,n_1},\eta_{i+1,n_2}) - f(\xi_{j,n_1},\eta_{i+1,n_2}) - f(\xi_{j+1,n_1},\eta_{i,n_2}) + f(\xi_{j,n_1},\eta_{i,n_2})}{(\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})} \\ &= \begin{bmatrix} \xi_{j,n_1},\xi_{j+1,n_1} & \\ & ;f \\ \eta_{i,n_2},\eta_{i+1,n_2} & \end{bmatrix}, \end{aligned}$$

and $\{\xi_{j,n_1}\}, R_{j,n_1}(x), j=\overline{0,n_1}, \{\eta_{i,n_2}\}, R_{i,n_2}(y), i=\overline{0,n_2}$, are constructed as in univariate case in [5].

Obviously, degree $(P_{n_1,n_2}(f))\leqslant n_1+n_2$.

Firstly we prove the estimate in Theorem 3. 1.

For any univariate function g, we have

$$\| P_n(g) \| \leqslant \| g \| + \| P_n(g) - g \| \leqslant \| g \| + C\omega_2^{\varphi}\left(g;\frac{1}{n}\right)\leqslant (1 + 2C) \| g \|,$$

that is passing to supremum with $\| g \| \leqslant 1$, we obtain (for the linear operator P_n)

$$|||P_n||| \leqslant (1+2C), \qquad \forall\, n \in \mathrm{N},$$

where $C>0$ is independent of n.

Applying now Theorem 5 in [4], we immediately get

$$\| f - P_{n_1,n_2}(f) \| \leqslant C\left[\omega_{2,x}^{\varphi}\left(f;\frac{1}{n_1}\right) + \omega_{2,y}^{\varphi}\left(f;\frac{1}{n_2}\right)\right],$$

where $\omega_{2,x}^{\varphi}$ and $\omega_{2,y}^{\varphi}$ are the partial moduli defined in [2 Chapter 12]. Taking into account that obviously

$$\omega_{2,x}^{\varphi}\left(f;\frac{1}{n_1}\right) + \omega_{2,y}^{\varphi}\left(f;\frac{1}{n_2}\right) \leqslant 2\omega_2^{\varphi}\left(f;\frac{1}{n_1},\frac{1}{n_2}\right),$$

we obtain the desired estimate.

In what follows we will prove the shape-preserving properties.

(i) Suppose f is convex of order $(0,0)$. According to Remark 2 in Section 2, we have to prove that

$$\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial x \partial y} \geqslant 0, \qquad \forall\, x,y \in [-1,1].$$

We have

$$\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial x \partial y} = \sum_{j=0}^{n_1-1}\sum_{i=0}^{n_2-1} S_{i,j}^{*}(R'_{i,n_2}(y) - R'_{i+1,n_2}(y))(R'_{j,n_1}(x) - R'_{j+1,n_1}(x)) \geqslant 0,$$

because from univariate case, each from $R_{i,n_2}(y) - R_{i+1,n_2}(y)$ and $R_{j,n_1}(x) - R_{j+1,n_1}(x)$ are increasing with respect to y and x, respectively, and $S_{i,j}^{*} \geqslant 0$, $\forall\, i=\overline{0,n_2}, j=\overline{0,n_1}$.

(ii) Suppose f is simultaneously convex of the orders $(-1,0)$, $(0,-1)$ and $(0,0)$. It follows

$$\frac{f(-1,\eta_{i+1,n_2}) - f(-1,\eta_{i,n_2})}{\eta_{i+1,n_2} - \eta_{i,n_2}} \geqslant 0,$$

$$\frac{f(\xi_{j+1,n_1},-1) - f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1} - \xi_{j,n_1}} \geqslant 0,$$

and $S_{i,j}^{*} \geqslant 0, \forall\, i=\overline{0,n_2}, j=\overline{0,n_1}$.

Also, $R_{j,n_1}(-1) = R_{i,n_2}(-1) = 0$, $\forall\, j=\overline{0,n_1}$, $i=\overline{0,n_2}$, which implies

$$R_{j,n_1}(x) - R_{j+1,n_1}(x) \geqslant 0, \qquad R_{i,n_2}(y) - R_{i+1,n_2}(y) \geqslant 0,$$

$\forall\, j=\overline{0,n_1}, i=\overline{0,n_2}$.

Therefore we obtain

$$\frac{\partial P_{n_1,n_2}(f)(x,y)}{\partial x} = \sum_{j=0}^{n_1-1} \frac{f(\xi_{j+1,n_1},-1) - f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1} - \xi_{j,n_1}}(R'_{j,n_1}(x) - R'_{j+1,n_1}(x))$$

$$+\sum_{j=0}^{n_1-1}\sum_{i=0}^{n_2-1}\begin{bmatrix}\xi_{j+1,n_1},\xi_{j,n_1}\\ \eta_{i+1,n_2},\eta_{i,n_2}\end{bmatrix};f\Bigg](R'_{j,n_1}(x)-R'_{j+1,n_1}(x))(R_{i,n_2}(y)-R_{i+1,n_2}(y))\geqslant 0,$$

$\forall\ (x,y)\in[-1,1]\times[-1,1]$.

Similarly we get

$$\frac{\partial P_{n_1,n_2}(f)(x,y)}{\partial y}\geqslant 0,\qquad \forall\ x,y\in[-1,1],$$

and finally from the previous point (i), we have

$$\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial x\partial y}\geqslant 0,\qquad \forall\ x,y\in[-1,1].$$

(iii) Applying relation (5) in [5, p. 473], we obtain the following form

$$P_{n_1,n_2}(f)(x,y)=f(-1,-1)+\frac{f(-1,\eta_{1,n_2})-f(-1,-1)}{\eta_{1,n_2}+1}(1+y)$$

$$+\sum_{i=1}^{n_2-1}\left[\frac{f(-1,\eta_{i+1,n_2})-f(-1,\eta_{i,n_2})}{\eta_{i+1,n_2}-\eta_{i,n_2}}-\frac{f(-1,\eta_{i,n_2})-f(-1,\eta_{i-1,n_2})}{\eta_{i,n_2}-\eta_{i-1,n_2}}\right]R_{i,n_2}(y)$$

$$+\frac{f(\xi_{1,n_1}-1)-f(-1,-1)}{\xi_{1,n_1}+1}(1+x)$$

$$+\sum_{j=1}^{n_1-1}\left[\frac{f(\xi_{j+1,n_1},-1)-f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1}-\xi_{j,n_1}}-\frac{f(\xi_{j,n_1},-1)-f(\xi_{j-1,n_1},-1)}{\xi_{j,n_1}-\xi_{j-1,n_1}}\right]R_{j,n_1}(x)$$

$$+(1+x)(1+y)S^*_{0,0}+(1+x)\sum_{i=1}^{n_2-1}(S^*_{i,0}-S^*_{i-1,0})R_{i,n_2}(y)$$

$$+(1+y)\sum_{j=1}^{n_1-1}(S^*_{0,j}-S^*_{0,j-1})R_{j,n_1}(x)$$

$$+\sum_{j=1}^{n_1-1}\sum_{i=1}^{n_2-1}(S^*_{i,j}-S^*_{i,j-1}-S^*_{i-1,j}+S^*_{i-1,j-1})R_{i,n_2}(y)R_{j,n_1}(x),$$

where

$$S^*_{i,j}=\frac{f(\xi_{j+1,n_1},\eta_{i+1,n_2})-f(\xi_{j,n_1},\eta_{i+1,n_2})-f(\xi_{j+1,n_1},\eta_{i,n_2})+f(\xi_{j,n_1},\eta_{i,n_2})}{(\xi_{j+1,n_1}-\xi_{j,n_1})(\eta_{i+1,n_2}-\eta_{i,n_2})}.$$

Let us remark that

$$\frac{f(-1,\eta_{i+1,n_2})-f(-1,\eta_{i,n_2})}{\eta_{i+1,n_2}-\eta_{i,n_2}}-\frac{f(-1,\eta_{i,n_2})-f(-1,\eta_{i-1,n_2})}{\eta_{i,n_2}-\eta_{i-1,n_2}}$$

$$=(\eta_{i+1,n_2}-\eta_{i-1,n_2})\begin{bmatrix}-1\\ \eta_{i-1,n_2},\eta_{i,n_2},\eta_{i+1,n_2}\end{bmatrix};f\Bigg];$$

$$\frac{f(\xi_{j+1,n_1},-1)-f(\xi_{j,n_1},-1)}{\xi_{j+1,n_1}-\xi_{j,n_1}}-\frac{f(\xi_{j,n_1},-1)-f(\xi_{j-1,n_1},-1)}{\xi_{j,n_1}-\xi_{j-1,n_1}}$$

$$= (\xi_{j+1,n_1} - \xi_{j-1,n_1})\begin{bmatrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ -1 \end{bmatrix}; f\Bigg];$$

$$S^*_{i,0} - S^*_{i-1,0} = \begin{bmatrix} -1, \xi_{1,n_1} \\ \eta_{i,n_2}, \eta_{i+1,n_2} \end{bmatrix}; f\Bigg] - \begin{bmatrix} -1, \xi_{1,n_1} \\ \eta_{i-1,n_2}, \eta_{i,n_2} \end{bmatrix}; f\Bigg]$$

$$= (\eta_{i+1,n_2} - \eta_{i-1,n_2})\begin{bmatrix} -1, \xi_{1,n_1} \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i+1,n_2} \end{bmatrix}; f\Bigg];$$

$$S^*_{0,j} - S^*_{0,j-1} = (\xi_{j+1,n_1} - \xi_{j-1,n_1})\begin{bmatrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ -1, \eta_{1,n_2} \end{bmatrix}; f\Bigg];$$

$$S^*_{i,j} - S^*_{i,j-1} - S^*_{i-1,j} + S^*_{i-1,j-1} = S^*_{i,j} - S^*_{i,j-1} - (S^*_{i-1,j} - S^*_{i-1,j-1})$$

$$= (\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})\begin{bmatrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ \eta_{i-1,n_2}, \eta_{i,n_2}, \eta_{i+1,n_2} \end{bmatrix}; f\Bigg].$$

We have

$$\frac{\partial^4 P_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2} = \sum_{j=1}^{n_1-1}\sum_{i=1}^{n_2-1}(\xi_{j+1,n_1} - \xi_{j,n_1})(\eta_{i+1,n_2} - \eta_{i,n_2})$$

$$\cdot \begin{bmatrix} \xi_{j-1,n_1}, \xi_{j,n_1}, \xi_{j+1,n_1} \\ \eta_{i-1,n_2}, \eta_{i+1,n_2}, \eta_{i,n_2} \end{bmatrix}; f\Bigg] R''_{i,n_2}(y) R''_{j,n_1}(x) \geqslant 0,$$

by hypothesis on f and by construction $R''_{i,n_2}(y) \geqslant 0$, $R''_{j,n_1}(x) \geqslant 0, \forall\, x, y \in [-1,1]$ (see [5]).

(iv) Firstly, let us remark that by construction we have (see [5])

$$R'_{j,n_1}(x) \geqslant 0, \quad R_{j,n_1}(x) \geqslant 0, \quad R''_{j,n_1}(x) \geqslant 0, \quad \forall\, x \in [-1,1], j = \overline{0,n_1},$$

$$R'_{i,n_2}(y) \geqslant 0, \quad R_{i,n_2}(y) \geqslant 0, \quad R''_{i,n_2}(y) \geqslant 0, \quad \forall\, y \in [-1,1], i = \overline{0,n_2}.$$

These inequalities combined with the hypothesis on f, immediately give

$$\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial x^2} \geqslant 0, \quad \frac{\partial^3 P_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y} \geqslant 0, \quad \frac{\partial^4 P_{n_1,n_2}(f)(x,y)}{\partial x^2 \partial y^2} \geqslant 0,$$

$$\frac{\partial^2 P_{n_1,n_2}(f)(x,y)}{\partial y^2} \geqslant 0, \quad \frac{\partial^3 P_{n_1,n_2}(f)(xy)}{\partial y^2 \partial x} \geqslant 0, \quad \forall\, x, y \in [-1,1],$$

which proves the theorem.

Remarks. 1) The polynomials constructed by Theorem 3. 1 do not preserve the monotonicity with respect to each variable but still preserve the totally monotonicity, that is the most natural concept of bivariate monotonicity, because by [8], a totally monotone function has at most countable numbers of points of discontinuity.

2) Even if the polynomials in Theorem 3. 1 preserve some kinds of bivariate convexity of higher order, however do not preserve the usual convexity of a bivariate functions, that

is

$$f(\alpha X + (1-\alpha)Y) \leqslant \alpha f(X) + (1-\alpha)f(Y), \qquad \forall\, \alpha \in [0,1],$$

$\forall\, X=(x_1,x_2),\ Y=(y_1,y_2)\in[-1,1]\times[-1,1]$.

This question will be considered by another paper.

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