BIVARIATE REAL-VALUED ORTHOGONAL PERIODIC WAVELETS*

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Abstract

In this paper, we construct a kind of bivariate real-valued orthogonal periodic wavelets. The corresponding decomposition and reconstruction algorithms involve only 8 terms respectively which are very simple in practical computation. Moreover, the relation between periodic wavelets and Fourier series is also discussed.

Key words *periodic multiresolution analysis, two-scale dilation equation, periodic wavelet, discrete Fourier transform*

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1 Introduction and Notations

Wavelet analysis is a kind of well known and widely used mathematical method in such areas as signal processing and image processing etc.. It is well known that, in application areas, periodization method is the important approach to deal with periodic problems and boundary phenomena.

In [1], Y. Meyer studied periodic wavelets first by periodizing known wavelets. Since then, the theory of periodic wavelets has been developed rapidly (cf. e.g., [2-10]). In multidimension settings, Liang, Jin and Chen[11] constructed a class of bivariate orthogonal periodic wavelets associated with box spline functions. But, the wavelets are complex-valued. Goh, Lee, and Teo[12] studied the multidimensional periodic multiwavelets. Chen and Li[13] investigated the multidimensional biorthogonal periodic multiwavelets. In [7], a general construction method of univariate real-valued orthogonal periodic wavelets was given. The corresponding decomposition and reconstruction algorithms are simple.

In this paper, we are interested in generalizing the construction method in [7] to bivariate setting. That is, we shall constructed a class of bivariate orthogonal periodic wavelets and the corresponding decomposition and reconstruction algorithms are very simple. It should be pointed out that our construction method is not a simple generalization of the counterpart in univariate setting.

This paper is organized as follows. In sections 2, we will construct a bivariate periodic multiresolution analysis associated with a class of special functions. In section 3, we will construct the corresponding periodic scaling functions and the periodic wavelets which are orthogonal. There are only 8 terms in the two-scale dilation equations. In sections 4, we will establish the decomposition and reconstruction algorithms. Section 5 discusses the relation between periodic wavelets and Fourier series. We will give an example to illustrate our conclusions in section 6.

We will use the following notations.

Let $T = Kh$, where K is a positive even integer and h is a positive real number. Suppose that $K = 2N$. Let $N_j := 2^j N$, $K_j := 2^j K$, and $h_j := 2^{-j}h$. Denoted by $Z(K_j) = \{l \in Z^2;\ 0 \le l_i \le K_j - 1, i = 1, 2\}$. Let $TT = [0,T] \times [0,T]$. $L^2_*(TT)$ represents the set of all periodic, square-integrable functions defined on TT, equipped with the inner product $\langle f, g\rangle = \frac{1}{T^2}\int_{TT} f(x)\overline{g(x)}\mathrm{d}x$.

2 The Periodic Multiresolution Analysis

In this section, we will construct a so-called bivariate periodic multiresolution analysis. To this end, we give its definition first.

Definition 2.1. A bivariate periodic multiresolution analysis is a nested subspace sequence $\{V_j\}_{j\ge 0}$ satisfying

(1) $V_j \subset V_{j+1}$, for any $j \ge 0$.

(2) $\cup_{j\ge 0} V_j$ is dense in $L^2_*(TT)$.

(3) For any $j \ge 0$, there exists a function f_j in V_j such that the h_j-shifts of f_j:$\{f_j(\cdot - lh_j);\ l \in Z(K_j)\}$ produce V_j, i.e.

$$V_j = \mathrm{span}\{f_j(\cdot - lh_j);\ l \in Z(K_j)\}.$$

To construct a bivariate periodic multiresolution analysis, we suppose that a compactly supported real valued function $\omega(x) \in L^2(R^2)$ satisfies

(1) For some positive integer p, $2p \le N$, the support of $\omega(x)$: supp $\omega(x) \subset TT$.

(2) $\omega(x)$ is refinable, i.e. there exists $\{c_k\} \in l^2(Z^2)$ such that

$$\omega(x) = 4 \sum_{k \in Z^2} c_k \omega(2x - kh). \tag{2.1}$$

(3)

$$\int_{R^2} \omega(x) \mathrm{d}x \neq 0. \tag{2.2}$$

(4) $\{\omega(x - lh); 1 - p \leq l_i \leq K + p - 1, i = 1, 2\}$ are linearly independent on TT.

Remark 2.1. Note that the number of nonzero items in the right summation in the condition (2.1) is finite since $\omega(x)$ is compactly supported. Therefore the the number of the nonzero items of refinable sequence $\{c_k\}$ are finite.

Definition 2.2. For $j \geq 0$, $\alpha \in Z^2$, the T-periodization of $\omega(x)$ is defined as

$$\Omega_\alpha^j(x) := \sum_{\lambda \in Z^2} \omega(2^j(x - \lambda T - \alpha h_j))$$

From the above definition and formula (2.1), we have

Proposition 2.1. *For any $j \geq 0$, $\alpha \in Z^2$, the function Ω_α^j defined in Definition 2.1 has the following properties:*

(1) *$\Omega_\alpha^j(x + \lambda T) = \Omega_\alpha^j(x)$, for all $\lambda \in Z^2$.*

(2) *$\Omega_\alpha^j(x) = \Omega_\gamma^j(x)$, where $\gamma \in Z(K_j)$ satisfies: there exists $\eta \in Z^2$ such that $\alpha = \eta K_j + \gamma$.*

(3) *The function Ω_α^j satisfies the following scaling relation*

$$\Omega_\alpha^j(x) = 4 \sum_{k \in Z^2} c_k \Omega_{k+2\alpha}^{j+1}(x), \quad for\ x \in TT. \tag{2.3}$$

For $j \geq 0$, let $V_j = \mathrm{span}\{\Omega_\alpha^j(x);\ \alpha \in Z(K_j)\}$, then $V_j \subset V_{j+1}$ evidently. Furthermore, by generalizing the proofs of Lemma 2.2 and Lemma 2.3 in [7], we can get the following propositions.

Proposition 2.2. $\cup_{j \geq 0} V_j = L_*^2(TT)$.

Proposition 2.3. *Suppose $\Omega_\alpha^j(x)$, $\alpha \in Z(K_j)$ is defined by Definition 2.2, then $\{\Omega_\alpha^j(x);\ \alpha \in Z(K_j)\}$ is linearly independent in TT, hence* $\dim V_j = K_j^2$.

Therefore, we have

Theorem 2.1. *The space sequence $\{V_j\}_{j \geq 0}$ forms a bivariate periodic multiresolution analysis in $L_*^2(TT)$.*

3 The Periodic Scaling Functions and Wavelets

In this section, we will construct the bivariate orthogonal periodic scaling functions and wavelets. To this end, for $j \geq 0,\ \alpha \in Z^2$, denoted by

$$\alpha_{j+1}^0 = \alpha,\ \alpha_{j+1}^1 = (\alpha_1, K_j - \alpha_2),\ \alpha_{j+1}^2 = (K_j - \alpha_1, \alpha_2),\ \alpha_{j+1}^3 = (K_j - \alpha_1, K_j - \alpha_2).$$

For $j \geq 0,\ \alpha, \lambda \in Z^2$, let

$$\begin{aligned} CC_\lambda^{j;\alpha} &= \cos\frac{2\pi\lambda_1\alpha_1}{K_j}\cos\frac{2\pi\lambda_2\alpha_2}{K_j}, \qquad & CS_\lambda^{j;\alpha} &= \cos\frac{2\pi\lambda_1\alpha_1}{K_j}\sin\frac{2\pi\lambda_2\alpha_2}{K_j}, \\ SC_\lambda^{j;\alpha} &= \sin\frac{2\pi\lambda_1\alpha_1}{K_j}\cos\frac{2\pi\lambda_2\alpha_2}{K_j}, \qquad & SS_\lambda^{j;\alpha} &= \sin\frac{2\pi\lambda_1\alpha_1}{K_j}\sin\frac{2\pi\lambda_2\alpha_2}{K_j}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} CC\Omega_\alpha^j(x) &= \sum_{\lambda\in Z(K_j)} CC_\lambda^{j;\alpha}\Omega_\lambda^j(x), \qquad & CS\Omega_\alpha^j(x) &= \sum_{\lambda\in Z(K_j)} CS_\lambda^{j;\alpha}\Omega_\lambda^j(x), \\ SC\Omega_\alpha^j(x) &= \sum_{\lambda\in Z(K_j)} SC_\lambda^{j;\alpha}\Omega_\lambda^j(x), \qquad & SS\Omega_\alpha^j(x) &= \sum_{\lambda\in Z(K_j)} SS_\lambda^{j;\alpha}\Omega_\lambda^j(x). \end{aligned} \tag{3.2}$$

Then we can easily see that

Proposition 3.1. *For $j \geq 0$, $\alpha \in Z^2$, we have*

$$\begin{aligned} CC\Omega_\alpha^j(x) &= CC\Omega_{\alpha_{j+1}^1}^j(x) = CC\Omega_{\alpha_{j+1}^2}^j(x) = CC\Omega_{\alpha_{j+1}^3}^j(x), \\ CS\Omega_\alpha^j(x) &= -CS\Omega_{\alpha_{j+1}^1}^j(x) = CS\Omega_{\alpha_{j+1}^2}^j(x) = -CS\Omega_{\alpha_{j+1}^3}^j(x), \\ SC\Omega_\alpha^j(x) &= SC\Omega_{\alpha_{j+1}^1}^j(x) = -SC\Omega_{\alpha_{j+1}^2}^j(x) = -SC\Omega_{\alpha_{j+1}^3}^j(x), \\ SS\Omega_\alpha^j(x) &= -SS\Omega_{\alpha_{j+1}^1}^j(x) = -SS\Omega_{\alpha_{j+1}^2}^j(x) = SS\Omega_{\alpha_{j+1}^3}^j(x). \end{aligned}$$

Moreover, we can prove the following proposition.

Proposition 3.2. *For $j \geq 0$, $\alpha \in Z(K_j)$, then we have*

$$\Omega_\alpha^j(x) = \frac{1}{K_j^2}\sum_{\lambda\in Z(K_j)} (CC_\lambda^{j;\alpha}CC\Omega_\lambda^j(x) + CS_\lambda^{j;\alpha}CS\Omega_\lambda^j(x) + SC_\lambda^{j;\alpha}SC\Omega_\lambda^j(x) + SS_\lambda^{j;\alpha}SS\Omega_\lambda^j(x)).$$

Proof. Recall the following trigonometric identities

$$\sum_{l=0}^{2n-1}\cos\frac{2\pi lp}{2n}\cos\frac{2\pi lq}{2n} = \begin{cases} n, & \text{for } p+q=0 \text{ (mod 2n) and } p-q\neq 0 \text{ (mod 2n)} \\ & \text{or } p-q=0 \text{ (mod 2n) and } p+q\neq 0 \text{ (mod 2n)}, \\ 2n, & \text{for } p=q=0 \text{ (mod n)}, \\ 0, & \text{otherwise.} \end{cases}$$

For $\alpha = (\alpha_1, \alpha_2) \in Z(K_j),\ \alpha_1, \alpha_2 \notin \{0, N_j\}$, we have

$$\begin{aligned} \sum_{\lambda\in Z(K_j)} CC_\lambda^{j;\alpha}CC\Omega_\lambda^j(x) &= \sum_{\lambda\in Z(K_j)} CC_\lambda^{j;\alpha}\sum_{\mu\in Z(K_j)} CC_\mu^{j;\lambda}\Omega_\mu^j(x) \\ &= \sum_{\mu\in Z(K_j)}\Omega_\mu^j(x)\sum_{\lambda\in Z(K_j)} CC_\mu^{j;\lambda}CC_\lambda^{j;\alpha} \\ &= N_j^2(\Omega_\alpha^j(x) + \Omega_{\alpha_{j+1}^1}^j(x) + \Omega_{\alpha_{j+1}^2}^j(x) + \Omega_{\alpha_{j+1}^3}^j(x)). \end{aligned}$$

Similarly, we have

$$\sum_{\lambda\in Z(K_j)} CS_\lambda^{j;\alpha} CS\Omega_\lambda^j(x) = N_j^2(\Omega_\alpha^j(x) - \Omega_{\alpha_{j+1}^1}^j(x) + \Omega_{\alpha_{j+1}^2}^j(x) - \Omega_{\alpha_{j+1}^3}^j(x)),$$
$$\sum_{\lambda\in Z(K_j)} SC_\lambda^{j;\alpha} SC\Omega_\lambda^j(x) = N_j^2(\Omega_\alpha^j(x) + \Omega_{\alpha_{j+1}^1}^j(x) - \Omega_{\alpha_{j+1}^2}^j(x) - \Omega_{\alpha_{j+1}^3}^j(x)),$$
$$\sum_{\lambda\in Z(K_j)} SS_\lambda^{j;\alpha} SS\Omega_\lambda^j(x) = N_j^2(\Omega_\alpha^j(x) - \Omega_{\alpha_{j+1}^1}^j(x) - \Omega_{\alpha_{j+1}^2}^j(x) + \Omega_{\alpha_{j+1}^3}^j(x)).$$

By Proposition 2.1, it is the special case as above that the case α_1 or α_2 takes 0 or N_j. Therefore, we can obtain the proposition.

For $j \geq 0$, denoted by

$$\begin{aligned} I_j^0 &= \{\alpha = (\alpha_1,\alpha_2) \in Z^2; 0 \leq \alpha_1,\alpha_2 \leq N_j\},\\ I_j^1 &= \{\alpha = (\alpha_1,\alpha_2) \in Z^2; 0 \leq \alpha_1 \leq N_j, 1 \leq \alpha_2 \leq N_j - 1\},\\ I_j^2 &= \{\alpha = (\alpha_1,\alpha_2) \in Z^2; 1 \leq \alpha_1 \leq N_j - 1, 0 \leq \alpha_2 \leq N_j\},\\ I_j^3 &= \{\alpha = (\alpha_1,\alpha_2) \in Z^2; 1 \leq \alpha_1,\alpha_2 \leq N_j - 1\}, \end{aligned} \tag{3.3}$$

then by Proposition 2.3 and Proposition 3.1, we have

Lemma 3.1. *For $j \geq 0$, suppose $CC\Omega_\alpha^j(x)$, $CS\Omega_\alpha^j(x)$, $SC\Omega_\alpha^j(x)$ and $SS\Omega_\alpha^j(x)$ $(\alpha \in Z^2)$ are defined by* (3.2). *Let*

$$CC^j = \{CC\Omega_\alpha^j(x), \alpha \in I_j^0\}, \quad CS^j = \{CS\Omega_\alpha^j(x), \alpha \in I_j^1\},$$
$$SC^j = \{SC\Omega_\alpha^j(x), \alpha \in I_j^2\}, \quad SS^j = \{SS\Omega_\alpha^j(x)\}, \alpha \in I_j^3\}.$$

Then $CC^j \cup CS^j \cup SC^j \cup SS^j$ forms a basis for V_j, and $CC^j \cup CS^j$, $CC^j \cup SC^j$, $SS^j \cup CS^j$, $SS^j \cup SC^j$ are all orthogonal systems. Furthermore, if the functions are in $CC^j \cup CS^j \cup SC^j \cup SS^j$, then for any α, $\|CC\Omega_\alpha^j\| = \|CS\Omega_\alpha^j\| = \|SC\Omega_\alpha^j\| = \|SS\Omega_\alpha^j\|$, $\langle CC\Omega_\alpha^j, SS\Omega_\alpha^j\rangle = -\langle CS\Omega_\alpha^j, SC\Omega_\alpha^j\rangle$, $\langle CC\Omega_\alpha^j, SS\Omega_\beta^j\rangle = \langle CS\Omega_\alpha^j, SC\Omega_\beta^j\rangle = 0$, for $\alpha \neq \beta$.

Proof. First, by formula (3.2), we can see that $CC^j \cup CS^j \cup SC^j \cup SS^j \subset V_j$. On the other hand, Proposition 3.1 and Proposition 3.2 imply $CC^j \cup CS^j \cup SC^j \cup SS^j$ can span the space V_j. In addition, the number of the functions in $CC^j \cup CS^j \cup SC^j \cup SS^j$ is equal to $\dim V_j$. Therefore, $CC^j \cup CS^j \cup SC^j \cup SS^j$ forms a basis for V_j.

By the definition of $CC\Omega^j_\alpha$, for any $CC\Omega^j_\alpha, CC\Omega^j_\beta \in CC^j$, we have

$$
\begin{aligned}
&\langle CC\Omega^j_\alpha, CC\Omega^j_\beta\rangle \\
= &\sum_{\lambda\in Z(K_j)}\sum_{\mu\in Z(K_j)} CC^{j;\alpha}_\lambda CC^{j;\beta}_\mu \langle \Omega^j_\lambda, \Omega^j_\mu\rangle \\
= &\sum_{\lambda\in Z(K_j)}\sum_{\mu\in Z(K_j)} CC^{j;\alpha}_\lambda CC^{j;\beta}_\mu \langle \Omega^j_0, \Omega^j_{\mu-\lambda}\rangle \\
= &\sum_{\lambda\in Z(K_j)}\sum_{\mu_1=-\lambda_1}^{K_j-1-\lambda_1}\sum_{\mu_2=-\lambda_2}^{K_j-1-\lambda_2} CC^{j;\alpha}_\lambda CC^{j;\beta}_{\mu+\lambda} \langle \Omega^j_0, \Omega^j_\mu\rangle \\
= &\sum_{\lambda\in Z(K_j)}\sum_{\mu\in Z(K_j)} CC^{j;\alpha}_\lambda CC^{j;\beta}_{\mu+\lambda} \langle \Omega^j_0, \Omega^j_\mu\rangle \\
= &\sum_{\mu\in Z(K_j)} CC^{j;\beta}_\mu \langle \Omega^j_0, \Omega^j_\mu\rangle \Bigg(CC^{j;\beta}_\mu \sum_{\lambda\in Z(K_j)} CC^{j;\alpha}_\lambda CC^{j;\beta}_\lambda - CS^{j;\beta}_\mu \sum_{\lambda\in Z(K_j)} CC^{j;\alpha}_\lambda CS^{j;\beta}_\lambda \\
&\qquad - SC^{j;\beta}_\mu \sum_{\lambda\in Z(K_j)} CC^{j;\alpha}_\lambda SC^{j;\beta}_\lambda + SS^{j;\beta}_\mu \sum_{\lambda\in Z(K_j)} CC^{j;\alpha}_\lambda SS^{j;\beta}_\lambda \Bigg) \\
= &\sum_{\mu\in Z(K_j)} CC^{j;\beta}_\mu \langle \Omega^j_0, \Omega^j_\mu\rangle \sum_{\lambda\in Z(K_j)} CC^{j;\alpha}_\lambda CC^{j;\beta}_\lambda \\
= &\sum_{\mu\in Z(K_j)} CC^{j;\beta}_\mu \langle \Omega^j_0, \Omega^j_\mu\rangle \sum_{\lambda_1=0}^{K_j-1} \cos\frac{2\pi\lambda_1\alpha_1}{K_j}\cos\frac{2\pi\lambda_1\beta_1}{K_j}\sum_{\lambda_2=0}^{K_j-1}\cos\frac{2\pi\lambda_2\alpha_2}{K_j}\cos\frac{2\pi\lambda_2\beta_2}{K_j}.
\end{aligned}
$$

For the case $\alpha,\beta\in I^0_j$, $\alpha\neq\beta$, then $\langle CC\Omega^j_\alpha, CC\Omega^j_\beta\rangle = 0$. While for $\alpha,\beta\in I^0_j$, $\alpha=\beta$, we have

$$
\begin{aligned}
\|CC\Omega^j_\alpha\|^2 &= N_j^2 \sum_{\mu\in Z(K_j)} CC^{j;\alpha}_\mu \langle \Omega^j_0, \Omega^j_\mu\rangle \\
&= N_j^2 \langle \Omega^j_0, \sum_{\mu\in Z(K_j)} CC^{j;\alpha}_\mu \Omega^j_\mu\rangle \\
&= N_j^2 \langle \Omega^j_0, CC\Omega^j_\alpha\rangle.
\end{aligned}
$$

The other cases are can be similarly proved and the theorem follows.

Lemma 3.1 gives a basis for V_j, but it is not orthogonal generally. To construct the orthogonal basis for V_j, for all $j\geq 0$, $\alpha\in Z^2$, we define

$$
\begin{aligned}
\tilde\varphi^{j,0}_\alpha(x) &= CC\Omega^j_\alpha(x) + SS\Omega^j_\alpha(x), & \tilde\varphi^{j,1}_\alpha(x) &= CS\Omega^j_\alpha(x) + SC\Omega^j_\alpha(x), \\
\tilde\varphi^{j,2}_\alpha(x) &= CS\Omega^j_\alpha(x) - SC\Omega^j_\alpha(x), & \tilde\varphi^{j,3}_\alpha(x) &= CC\Omega^j_\alpha(x) - SS\Omega^j_\alpha(x).
\end{aligned} \tag{3.4}
$$

Then, from Lemma 3.1, we can easily prove the following theorem.

Theorem 3.1. *For $j\geq 0$, $\alpha\in Z^2$, suppose that $\tilde\varphi^{j,i}_\alpha(x)$, $i=0,1,2,3$ are defined by* (3.4). *Let*

$$S^{j,i} = \{\tilde\varphi^{j,i}_\alpha(x), \alpha\in I^i_j\},\ i=0,1,2,3,\ S^j = \bigcup_{i=0}^{3} S^{j,i}.$$

Then S^j is an orthogonal basis for V_j. In addition, $\|\widetilde{\varphi}_\alpha^{j,2}\| = \|\widetilde{\varphi}_\alpha^{j,0}\|$, for $\alpha \in I_j^2$; $\|\widetilde{\varphi}_\alpha^{j,3}\| = \|\widetilde{\varphi}_\alpha^{j,1}\|$, for $\alpha \in I_j^3$.

Now, we turn to give the scaling relations of the orthogonal basis S^j.

Lemma 3.2. *For $j \geq 0$, $\alpha \in Z^2$, suppose $CC_\lambda^{j;\alpha}$, $CS_\lambda^{j;\alpha}$, $SC_\lambda^{j;\alpha}$ and $SS_\lambda^{j;\alpha}$; $CC\Omega_\alpha^j(x)$, $CS\Omega_\alpha^j(x)$, $SC\Omega_\alpha^j(x)$ and $SS\Omega_\alpha^j(x)$ are defined by (3.1) and (3.2). Let*

$$\begin{aligned} \sigma\sigma_\alpha^j &= \sum_{k\in Z^2} c_k CC_k^{j;\alpha}, & \sigma\delta_\alpha^j &= \sum_{k\in Z^2} c_k CS_k^{j;\alpha}, \\ \delta\sigma_\alpha^j &= \sum_{k\in Z^2} c_k SC_k^{j;\alpha}, & \delta\delta_\alpha^j &= \sum_{k\in Z^2} c_k SS_k^{j;\alpha}. \end{aligned} \tag{3.5}$$

Then, we have the following refinable equations

$$\Gamma_\alpha^j(x) = \sum_{i=0}^{3} A_\alpha^{j+1,i}\Gamma_{\alpha_{j+1}^i}^{j+1}(x), \tag{3.6}$$

where

$$\begin{aligned} &\Gamma_\alpha^j(x) = \left[CC\Omega_\alpha^j(x)\ \ CS\Omega_\alpha^j(x)\ \ SC\Omega_\alpha^j(x)\ \ SS\Omega_\alpha^j(x)\right]^t, \\ &A_\alpha^{j,0} = \left[\Lambda_\alpha^{j,0}\ \ \Lambda_\alpha^{j,1}\ \ \Lambda_\alpha^{j,2}\ \ \Lambda_\alpha^{j,3}\right]^t, A_\alpha^{j,1} = \left[\Lambda_{\alpha_j^1}^{j,0}\ \ -\Lambda_{\alpha_j^1}^{j,1}\ \ \Lambda_{\alpha_j^1}^{j,2}\ \ -\Lambda_{\alpha_j^1}^{j,3}\right]^t, \\ &A_\alpha^{j,2} = \left[\Lambda_{\alpha_j^2}^{j,0}\ \ \Lambda_{\alpha_j^2}^{j,1}\ \ -\Lambda_{\alpha_j^2}^{j,2}\ \ -\Lambda_{\alpha_j^2}^{j,3}\right]^t, A_\alpha^{j,3} = \left[\Lambda_{\alpha_j^3}^{j,0}\ \ -\Lambda_{\alpha_j^3}^{j,1}\ \ -\Lambda_{\alpha_j^3}^{j,2}\ \ \Lambda_{\alpha_j^3}^{j,3}\right]^t. \end{aligned}$$

and

$$\begin{aligned} &\Lambda_\alpha^{j,0} = [\sigma\sigma_\alpha^j\ \ \sigma\delta_\alpha^j\ \ \delta\sigma_\alpha^j\ \ \delta\delta_\alpha^j]^t, \qquad \Lambda_\alpha^{j,1} = [-\sigma\delta_\alpha^j\ \ \sigma\sigma_\alpha^j\ \ -\delta\delta_\alpha^j\ \ \delta\sigma_\alpha^j]^t, \\ &\Lambda_\alpha^{j,2} = [-\delta\sigma_\alpha^j\ \ -\delta\delta_\alpha^j\ \ \sigma\sigma_\alpha^j\ \ \sigma\delta_\alpha^j]^t, \ \ \Lambda_\alpha^{j,3} = [\delta\delta_\alpha^j\ \ -\delta\sigma_\alpha^j\ \ -\sigma\delta_\alpha^j\ \ \sigma\sigma_\alpha^j]^t. \end{aligned}$$

Proof. We need only to prove the first equation in (3.5) and the proofs of the other equations are similar.

$$\begin{aligned} CC\Omega_\alpha^j(x) &= \sum_{\lambda\in Z(K_j)} CC_\lambda^{j;\alpha}\Omega_\lambda^j(x) \\ &= \sum_{\lambda\in Z(K_j)} CC_\lambda^{j;\alpha} 4\sum_{k\in Z^2} c_k\Omega_{k+2\lambda}^{j+1}(x) \\ &= \sum_{\lambda\in Z(K_j)} CC_\lambda^{j;\alpha} 4\sum_{k\in Z^2} c_k \frac{1}{K_{j+1}^2}\sum_{\mu\in Z(K_{j+1})} [CC_\mu^{j+1;k+2\lambda}CC\Omega_\mu^{j+1}(x) \\ &\quad + CS_\mu^{j+1;k+2\lambda}CS\Omega_\mu^{j+1}(x) + SC_\mu^{j+1;k+2\lambda}SC\Omega_\mu^{j+1}(x) + SS_\mu^{j+1;k+2\lambda}SS\Omega_\mu^{j+1}(x)] \\ &= \frac{4}{K_{j+1}^2}\sum_{k\in Z^2} c_k \sum_{\mu\in Z(K_{j+1})} [CC_\mu^{j+1;k}CC\Omega_\mu^{j+1}(x) + CS_\mu^{j+1;k}CS\Omega_\mu^{j+1}(x) \\ &\quad + SC_\mu^{j+1;k}SC\Omega_\mu^{j+1}(x) + SS_\mu^{j+1;k}SS\Omega_\mu^{j+1}(x)] \sum_{\lambda\in Z(K_j)} CC_\lambda^{j;\alpha}CC_\lambda^{j;\mu}. \end{aligned}$$

By Proposition 3.1, we can get

$$CC\Omega_\alpha^j(x) = \gamma_1^t \sum_{i=0}^{3} A_\alpha^{j+1,i}\Gamma_{\alpha_{j+1}^i}^{j+1}(x),$$

where $\gamma_1 = [1,0,0,0]^t$. The proof is finished.

By the formula (3.3) and Lemma 3.2, we can give the relations between the orthogonal basis S^j of V_j given in Theorem 3.1 and the orthogonal basis S^{j+1} of V_{j+1}.

Theorem 3.2. *The orthogonal basis S^j of V_j given in Theorem 3.1 and the orthogonal basis S^{j+1} of V_{j+1} have the following scaling relations*

$$\widetilde{\Phi}_\alpha^j(x) = \sum_{i=0}^{3} \widetilde{H}_\alpha^{j+1,i}\widetilde{\Phi}_{\alpha_{j+1}^i}^{j+1}(x), \tag{3.7}$$

where

$$\widetilde{\Phi}_\alpha^j(x) = \left[\widetilde{\varphi}_\alpha^{j,0}(x) \;\; \widetilde{\varphi}_\alpha^{j,1}(x) \;\; \widetilde{\varphi}_\alpha^{j,2}(x) \;\; \widetilde{\varphi}_\alpha^{j,3}(x)\right]^t,$$

$$\widetilde{H}_\alpha^{j,0} = \left[\widetilde{\Delta}_\alpha^{j,0} \;\; \widetilde{\Delta}_\alpha^{j,1} \;\; \widetilde{\Delta}_\alpha^{j,2} \;\; \widetilde{\Delta}_\alpha^{j,3}\right]^t, \quad \widetilde{H}_\alpha^{j,1} = \left[\widetilde{\Delta}_{\alpha_j^1}^{j,3} \;\; -\widetilde{\Delta}_{\alpha_j^1}^{j,2} \;\; -\widetilde{\Delta}_{\alpha_j^1}^{j,1} \;\; \widetilde{\Delta}_{\alpha_j^1}^{j,0}\right]^t,$$

$$\widetilde{H}_\alpha^{j,2} = \left[\widetilde{\Delta}_{\alpha_j^2}^{j,3} \;\; \widetilde{\Delta}_{\alpha_j^2}^{j,2} \;\; \widetilde{\Delta}_{\alpha_j^2}^{j,1} \;\; \widetilde{\Delta}_{\alpha_j^2}^{j,0}\right]^t, \quad \widetilde{H}_\alpha^{j,3} = \left[\widetilde{\Delta}_{\alpha_j^3}^{j,0} \;\; -\widetilde{\Delta}_{\alpha_j^3}^{j,1} \;\; -\widetilde{\Delta}_{\alpha_j^3}^{j,2} \;\; \widetilde{\Delta}_{\alpha_j^3}^{j,3}\right]^t.$$

with

$$\widetilde{\Delta}_\alpha^{j,0} = [\sigma\sigma_\alpha^j + \delta\delta_\alpha^j \;\; 0 \;\; \sigma\delta_\alpha^j - \delta\sigma_\alpha^j \;\; 0]^t, \qquad \widetilde{\Delta}_\alpha^{j,1} = [0 \;\; \sigma\sigma_\alpha^j - \delta\delta_\alpha^j \;\; 0 \;\; -\sigma\delta_\alpha^j - \delta\sigma_\alpha^j]^t,$$

$$\widetilde{\Delta}_\alpha^{j,2} = [-\sigma\delta_\alpha^j + \delta\sigma_\alpha^j \;\; 0 \;\; \delta\delta_\alpha^j + \sigma\sigma_\alpha^j \;\; 0]^t, \quad \widetilde{\Delta}_\alpha^{j,3} = [0 \;\; \sigma\delta_\alpha^j + \delta\sigma_\alpha^j \;\; 0 \;\; \sigma\sigma_\alpha^j - \delta\delta_\alpha^j]^t.$$

Theorem 3.2 establishes the relations between the orthogonal basis for V_j and V_{j+1}. Now we define W_j as the orthogonal complement of V_j in V_{j+1}, that is, $W_j \perp V_j$ and $V_{j+1} = V_j + W_j$. Denoted by $V_{j+1} = V_j \oplus W_j$, the orthogonal sum of V_{j+1}. We can conclude that $W_j \perp W_r$ for $j \neq r$, and $L_*^2(TT) = V_0 \oplus (\oplus_{j\geq 0} W_j)$.

Now, by Theorem 3.2, we can construct an orthogonal basis for each W_j. To this end, note that $\|\widetilde{\varphi}_\alpha^{j,2}\| = \|\widetilde{\varphi}_\alpha^{j,0}\|$, $\|\widetilde{\varphi}_\alpha^{j,1}\| = \|\widetilde{\varphi}_\alpha^{j,3}\|$ if they are in S^j. Let

$$\begin{aligned} \Phi_\alpha^j(x) &= \mathrm{diag}[\|\widetilde{\varphi}_\alpha^{j,0}\|^{-1}, \|\widetilde{\varphi}_\alpha^{j,3}\|^{-1}, \|\widetilde{\varphi}_\alpha^{j,0}\|^{-1}, \|\widetilde{\varphi}_\alpha^{j,3}\|]^{-1}]\widetilde{\Phi}_\alpha^j(x), \\ R\Phi_\alpha^j(x) &= [(\Phi_\alpha^j(x))^t, (\Phi_{\alpha_j^1}^j(x))^t, (\Phi_{\alpha_j^2}^j(x))^t, (\Phi_{\alpha_j^3}^j(x))^t]^t. \end{aligned} \tag{3.8}$$

Thus, we can rewrite the scaling equation (3.7) as the following form:

$$\Phi_\alpha^j(x) = \widetilde{M}_\alpha^{j+1} R\Phi_\alpha^{j+1}(x), \tag{3.9}$$

where

$$\widetilde{M}_\alpha^j = \mathrm{diag}[\|\widetilde{\varphi}_\alpha^{j,0}\|^{-1}, \|\widetilde{\varphi}_\alpha^{j,3}\|^{-1}, \|\widetilde{\varphi}_\alpha^{j,0}\|^{-1}, \|\widetilde{\varphi}_\alpha^{j,3}\|]^{-1}][H_\alpha^{j,0}, \cdots, H_\alpha^{j,3}],$$

$$H_\alpha^{j,0} = \left[\Delta_\alpha^{j,0}\ \ \Delta_\alpha^{j,1}\ \ \Delta_\alpha^{j,2}\ \ \Delta_\alpha^{j,3}\right]^t, \quad H_\alpha^{j,1} = \left[\Delta_{\alpha_j^1}^{j,3}\ \ -\Delta_{\alpha_j^1}^{j,2}\ \ -\Delta_{\alpha_j^1}^{j,1}\ \ \Delta_{\alpha_j^1}^{j,0}\right]^t,$$

$$H_\alpha^{j,2} = \left[\Delta_{\alpha_j^2}^{j,3}\ \ \Delta_{\alpha_j^2}^{j,2}\ \ \Delta_{\alpha_j^2}^{j,1}\ \ \Delta_{\alpha_j^2}^{j,0}\right]^t, \quad H_\alpha^{j,3} = \left[\Delta_{\alpha_j^3}^{j,0}\ \ -\Delta_{\alpha_j^3}^{j,1}\ \ -\Delta_{\alpha_j^3}^{j,2}\ \ \Delta_{\alpha_j^3}^{j,3}\right]^t$$

with

$$\Delta_\alpha^{j,0} = \|\widetilde{\varphi}_\alpha^{j,0}\|\widetilde{\Delta}_\alpha^{j,0} := [h_\alpha^{j,0}\ \ 0\ \ h_\alpha^{j,2}\ \ 0]^t, \qquad \Delta_\alpha^{j,1} = \|\widetilde{\varphi}_\alpha^{j,3}\|\widetilde{\Delta}_\alpha^{j,1} := [0\ \ h_\alpha^{j,3}\ \ 0\ \ -h_\alpha^{j,1}]^t,$$

$$\Delta_\alpha^{j,2} = \|\widetilde{\varphi}_\alpha^{j,0}\|\widetilde{\Delta}_\alpha^{j,2} := [-h_\alpha^{j,2}\ \ 0\ \ h_\alpha^{j,0}\ \ 0]^t, \quad \Delta_\alpha^{j,3} = \|\widetilde{\varphi}_\alpha^{j,3}\|\widetilde{\Delta}_\alpha^{j,3} := [0\ \ h_\alpha^{j,1}\ \ 0\ \ h_\alpha^{j,3}]^t.$$

We can get a new 2×8 matrix $\widetilde{M_\alpha^{j,0}}$ by selecting the nonzero elements in the first and the third row of matrix $\widetilde{M_\alpha^{j}}$ following their original order, i.e.

$$\widetilde{M_\alpha^{j,0}} = \begin{bmatrix} h_{\alpha_j^0}^{j,0} & h_{\alpha_j^0}^{j,2} & h_{\alpha_j^1}^{j,1} & h_{\alpha_j^1}^{j,3} & h_{\alpha_j^2}^{j,1} & h_{\alpha_j^2}^{j,3} & h_{\alpha_j^3}^{j,0} & h_{\alpha_j^3}^{j,2} \\ -h_{\alpha_j^0}^{j,2} & h_{\alpha_j^0}^{j,0} & -h_{\alpha_j^1}^{j,3} & h_{\alpha_j^1}^{j,1} & h_{\alpha_j^2}^{j,3} & -h_{\alpha_j^2}^{j,1} & h_{\alpha_j^3}^{j,2} & -h_{\alpha_j^3}^{j,0} \end{bmatrix}. \tag{3.10}$$

We see that the two rows in matrix $\widetilde{M_\alpha^{j,0}}$ are orthogonal. Furthermore, the elements in each row are the same except for their order and sign. Let

$$M_\alpha^{j,0} = \begin{bmatrix} h_{\alpha_j^0}^{j,0} & h_{\alpha_j^0}^{j,2} & h_{\alpha_j^1}^{j,1} & h_{\alpha_j^1}^{j,3} & h_{\alpha_j^2}^{j,1} & h_{\alpha_j^2}^{j,3} & h_{\alpha_j^3}^{j,0} & h_{\alpha_j^3}^{j,2} \\ -h_{\alpha_j^0}^{j,2} & h_{\alpha_j^0}^{j,0} & -h_{\alpha_j^1}^{j,3} & h_{\alpha_j^1}^{j,1} & h_{\alpha_j^2}^{j,3} & -h_{\alpha_j^2}^{j,1} & h_{\alpha_j^3}^{j,2} & -h_{\alpha_j^3}^{j,0} \\ -h_{\alpha_j^1}^{j,1} & h_{\alpha_j^1}^{j,3} & h_{\alpha_j^0}^{j,0} & -h_{\alpha_j^0}^{j,2} & h_{\alpha_j^3}^{j,0} & -h_{\alpha_j^3}^{j,2} & -h_{\alpha_j^2}^{j,1} & h_{\alpha_j^2}^{j,3} \\ -h_{\alpha_j^1}^{j,3} & -h_{\alpha_j^1}^{j,1} & h_{\alpha_j^0}^{j,2} & h_{\alpha_j^0}^{j,0} & h_{\alpha_j^3}^{j,2} & h_{\alpha_j^3}^{j,0} & -h_{\alpha_j^2}^{j,3} & -h_{\alpha_j^2}^{j,1} \\ -h_{\alpha_j^2}^{j,1} & -h_{\alpha_j^2}^{j,3} & -h_{\alpha_j^3}^{j,0} & -h_{\alpha_j^3}^{j,2} & h_{\alpha_j^0}^{j,0} & h_{\alpha_j^0}^{j,2} & h_{\alpha_j^1}^{j,1} & h_{\alpha_j^1}^{j,3} \\ -h_{\alpha_j^2}^{j,3} & h_{\alpha_j^2}^{j,1} & h_{\alpha_j^3}^{j,2} & -h_{\alpha_j^3}^{j,0} & -h_{\alpha_j^0}^{j,2} & h_{\alpha_j^0}^{j,0} & h_{\alpha_j^1}^{j,3} & -h_{\alpha_j^1}^{j,1} \\ -\tilde{h}_{\alpha_j^3}^{j,0} & -h_{\alpha_j^3}^{j,2} & h_{\alpha_j^2}^{j,1} & h_{\alpha_j^2}^{j,3} & -h_{\alpha_j^1}^{j,1} & -h_{\alpha_j^1}^{j,3} & h_{\alpha_j^0}^{j,0} & h_{\alpha_j^0}^{j,2} \\ -h_{\alpha_j^3}^{j,2} & h_{\alpha_j^3}^{j,0} & -h_{\alpha_j^2}^{j,3} & h_{\alpha_j^2}^{j,1} & -h_{\alpha_j^1}^{j,3} & h_{\alpha_j^1}^{j,1} & -h_{\alpha_j^0}^{j,2} & h_{\alpha_j^0}^{j,0} \end{bmatrix} \tag{3.11}$$

Then we can prove the following lemma.

Lemma 3.3 *The 8×8 matrix $M_\alpha^{j,0}$ defined by* (3.31) *satisfies*

(1) *The first two rows in the matrix $M_\alpha^{j,0}$ are the two rows of matrix $\widetilde{M_\alpha^{j,0}}$ defined in* (3.10).

(2) *any pair of rows of $M_\alpha^{j,0}$ is orthogonal.*

(3) *The elements in every row of $M_\alpha^{j,0}$ are the same except for their order and sign.*

Similarly, for the second row and the forth row of $\widetilde{M_\alpha^{j}}$, we can get another matrix $M_\alpha^{j,1}$ with the similar properties as in Lemma 3.3. Let

$$\begin{aligned} L^j_\alpha &= \left[\begin{array}{c} \zeta^{16}_1, \zeta^{16}_3, \zeta^{16}_6, \zeta^{16}_8, \zeta^{16}_{10}, \zeta^{16}_{12}, \zeta^{16}_{13}, \zeta^{16}_{15} \\ \zeta^{16}_2, \zeta^{16}_4, \zeta^{16}_5, \zeta^{16}_7, \zeta^{16}_9, \zeta^{16}_{11}, \zeta^{16}_{14}, \zeta^{16}_{16} \end{array}\right. \begin{array}{l} [M^{j,0}_\alpha]^t \\ [M^{j,1}_\alpha]^t, \end{array} \\ R^j_\alpha &= \end{aligned}$$

where ζ^{16}_i, $i = 1, \cdots, 16$ are 16-dimension unit column vectors whose i-th element is 1 and the others are 0. Let

$$M^j_\alpha = [L^j_\alpha, R^j_\alpha]^t.$$

Then, we rearrange the rows in matrix M^j_α following the order 1, 9, 2, 10, 4, 5, 8, 12, 14, 15, 3, 6, 7, 10, 11, 13, 16. Denote by G^j_α the new matrix . We define the function vector $\widetilde{\Psi}^j_\alpha(x)$ by the following scaling equation

$$\left[\begin{array}{c} \Phi^j_\alpha(x) \\ \Psi^j_\alpha(x) \end{array}\right] = G^{j+1}_\alpha R \Phi^{j+1}_\alpha(x). \tag{3.12}$$

Write $\Psi^j_\alpha(x) = [\psi^{j,0}_\alpha(x), \cdots, \psi^{j,11}_\alpha(x)]^t$, then we have

Theorem 3.3. *For $\alpha \in Z^2$, suppose that the functions $\psi^{j,0}_\alpha(x), \cdots, \psi^{j,11}_\alpha(x)$ are defined as the above formula* (3.12). *Let $U^{j,0} = \{ \psi^{j,k}_\alpha(x),\ k = 0,1,2,\ \alpha \in I^0_j, \alpha_i \neq N_j, i = 1,2\}$; $U^{j,1} = \{\psi^{j,k}_\alpha(x),\ k = 3,4,5,\ \alpha \in I^1_j\}$; $U^{j,2} = \{\psi^{j,k}_\alpha(x),\ k = 6,7,8,\ \alpha \in I^2_j\}$; $U^{j,3} = \{\psi^{j,k}_\alpha(x),\ k = 9,10,11,\ \alpha \in I^3_j\}$; $U^{j,4} = \{\psi^{j,r}_{(0,N_j)},\ r = 1,3,4;\ \psi^{j,s}_{(N_j,0)},\ s = 2,4,5;\ \psi^{j,t}_{(N_j,N_j)},\ t = 0,3,5\}$ then the function system $U^j = \cup^4_{i=0} U^{j,i}$ is an orthogonal basis for W_j. We call these functions periodic wavelets.*

Proof. It is easy to verify that $[G^{j+1}_\alpha][G^{j+1}_\alpha]^t$ is a unit matrix. Namely, any pair of rows in G^j_α is orthogonal. Furthermore, the number of the functions in $S^j \cup U^j$ is equal to $\dim V_{j+1}$. This implies the conclusion. The proof is completed.

4 Decomposition and Reconstruction Algorithms

In this section, we establish the decomposition and reconstruction algorithms corresponding to the wavelets constructed in section 3. To this end, for $j \geq 0$, define projection operators $\mathcal{P}_j$ and $\mathcal{Q}_j$: $L^2_*(TT) \to V_j$ and $L^2_*(TT) \to W_j$ respectively as follows. For any $f(x) \in L^2_*(TT)$,

$$\mathcal{P}_j f = \sum_{\alpha \in I^0_j} \sum_{i=0}^{3} \langle f, \varphi^{j,i}_\alpha \rangle \varphi^{j,i}_\alpha, \qquad \mathcal{Q}_j f = \sum_{\alpha \in I^0_j} \sum_{i=0}^{11} \langle f, \psi^{j,i}_\alpha \rangle \psi^{j,i}_\alpha$$

Here, if the functions $\varphi^{j,i}_\alpha$ and $\psi^{j,i}_\alpha$ are not appeared in S^j or U^j, the corresponding terms in the above summations are assumed to be zero. Similarly conventions are for the other functions..

Denoted by $c_\alpha^{j,i} = \langle f, \varphi_\alpha^{j,i}\rangle$, $d_\alpha^{j,i} = \langle f, \psi_\alpha^{j,i}\rangle$. Then, for $\alpha \in I_j^0$, $i = 0,1,2,3$, we have

$$\begin{aligned} c_\alpha^{j,i} &= \langle f, \varphi_\alpha^{j,i}\rangle \\ &= \langle f, (\zeta_i^{16})^t G_\alpha^{j+1} R\Phi_\alpha^{j+1}(x)\rangle \\ &= (\zeta_i^{16})^t G_\alpha^{j+1} C_\alpha^{j+1}, \end{aligned} \tag{4.1}$$

with

$$\begin{aligned} C_\alpha^{j+1} &= \langle f, R\Phi_\alpha^{j+1}(x)\rangle \\ &= (\langle f, \varphi_\alpha^{j+1,0}\rangle, \cdots, \langle f, \varphi_{\alpha_{j+1}^3}^{j+1,3}\rangle)^t \\ &= (c_\alpha^{j+1,0}, \cdots, c_{\alpha_{j+1}^3}^{j+1,3})^t. \end{aligned}$$

Similarly, we have

$$d_\alpha^{j,i} = (\zeta_{i+4}^{16})^t G_\alpha^{j+1} C_\alpha^{j+1}. \tag{4.2}$$

On the contrary, for $\alpha \in I_{j+1}^0$, $i = 0,1,2,3$, we see that one of $\{\alpha_{j+1}^i,\ i = 0,1,2,3\}$ must be in I_j^0, say, $\alpha_{j+1}^k \in I_j^0$, with some $k \in \{0,1,2,3\}$. Then we have

$$\begin{aligned} c_\alpha^{j+1,i} &= \langle f, \varphi_\alpha^{j+1,i}\rangle \\ &= \langle \mathcal{P}_j f + \mathcal{Q}_j f, \varphi_\alpha^{j+1,i}\rangle \\ &= \langle \sum_{\beta\in I_j^0}\sum_{l=0}^{3} c_\beta^{j,l}\varphi_\beta^{j,l}, \varphi_\alpha^{j+1,i}\rangle + \langle \sum_{\beta\in I_j^0}\sum_{l=0}^{11} d_\beta^{j,l}\psi_\beta^{j,l}, \varphi_\alpha^{j+1,i}\rangle \\ &= \langle \sum_{\beta\in I_j^0}\sum_{l=0}^{3} c_\beta^{j,l}(\zeta_l^{16})^t G_\beta^{j+1} R\Phi_\beta^{j+1}(x), \varphi_\alpha^{j+1,i}\rangle \\ &\quad + \langle \sum_{\beta\in I_j^0}\sum_{l=0}^{11} d_\beta^{j,l}(\zeta_{l+4}^{16})^t G_\beta^{j+1} R\Phi_\beta^{j+1}(x), \varphi_\alpha^{j+1,i}\rangle \\ &= \sum_{l=0}^{3}(\zeta_l^{16})^t G_\alpha^{j+1}\zeta_{4k+i}^{16} c_{\alpha_{j+1}^k}^{j,l} + \sum_{l=0}^{11}(\zeta_{l+4}^{16})^t G_\alpha^{j+1}\zeta_{4k+i}^{16} d_{\alpha_{j+1}^k}^{j,l}. \end{aligned} \tag{4.3}$$

We call (4.1) and (4.2) decomposition formulas and (4.3) the reconstruction formula. For there are only 8 nonzero elements in every row or every column of the dilation matrix G_α^{j+1}, we can see that the decomposition and reconstruction formulas involve only 8 terms which is very simple in practical computation. Furthermore, when the underlying function $\omega(x)$ is symmetric, *i.e.* $\omega(x) = \omega(-x)$, there are only 4 terms in the scaling relations and the same in the decomposition and reconstruction formulas as well. In fact, if $\omega(x) = \omega(-x)$, the sequence $\{c_k\}$ in (2.1) satisfies $c_k = c_{-k}$, for all $k \in Z^2$. Hence, we have the following proposition.

Proposition 4.1. *Suppose that the original function $\omega(x)$ satisfies $\omega(x) = \omega(-x)$, and $\sigma\delta_k^{j;\alpha}$, $\delta\sigma_k^{j;\alpha}$ are defined by (3.5), then $\sigma\delta_k^{j;\alpha} = \delta\sigma_k^{j;\alpha} = 0$, for $\alpha \in Z^2$.*

5 Periodic Wavelets and Fourier Series

In this section, we will show that special periodic scaling functions constructed in (3.3) convergs to cosine and sine functions which implies that the scaling functions constructed in this paper have some stationary properties.

Suppose $\omega(x)$ is continuous, $\mathrm{supp}\,\omega \subset [-\frac{T}{2}, \frac{T}{2}]^2$ and satisfy the partition of unity, $\sum_{k\in Z^2} \omega(x+kh) = 1$, for $x \in R^2$. Define the operator A^j: $C[0,T]^2 \to [0,T]^2$ by

$$A^j f(x) = \sum_{\mu\in Z(K_j)} f(\mu h_j)\Omega_\mu^j(x),$$

where $\Omega_\mu^j(x)$ is defined in Definition 2.2 and $C[0,T]^2$ is the continuous function space on $[0,T]^2$. Then, we have the following theorem.

Theorem 5.1. *For all $f(x) \in C(TT)$, we have $\lim_{j\to\infty} \|A^j f - f\|_\infty = 0$.*

Proof. We note first that $\sum_{\mu\in Z(K_j)} \Omega_\mu^j(x) = 1$ for $x \in [0,T]^2$, therefore,

$$\begin{aligned} |A^j f(x) - f(x)| &\le \sum_{\mu\in Z(K_j)} |f(x) - f(\mu h_j)||\Omega_\mu^j(x)| \\ &= \sum_{\substack{|\mu_i - [\frac{x_i}{h_j}]|\le \frac{K}{2}+1 \\ i=1,2}} |f(x) - f(\mu h_j)||\Omega_\mu^j(x)| \\ &\le M \sum_{\substack{|\mu_i - [\frac{x_i}{h_j}]|\le \frac{K}{2}+1 \\ i=1,2}} |f(x) - f(\mu h_j)| \\ &\le M(K+2) \max_{|x-t|\le(\frac{K}{2}+1)h_j} |f(x) - f(t)|, \end{aligned}$$

where $M = \max_{x\in TT} |\Omega_\mu^j(x)|$. This shows that $\lim_{j\to\infty} \|A^j f - f\|_\infty = 0$.

Corollary 5.1. *Suppose that the scaling functions $\tilde{\varphi}_\alpha^{j,i}(x)$, $i = 0,1,2,3$ are defined by (3.4), then*

$$\lim_{j\to\infty} \tilde{\varphi}_\alpha^{j,3}(x) = \cos\frac{2\pi\alpha\cdot x}{T}, \quad \lim_{j\to\infty} \tilde{\varphi}_\alpha^{j,1}(x) = \sin\frac{2\pi\alpha\cdot x}{T}.$$

Corollary 5.1 shows that, for $g(x) \in C(TT)$, let $P_j g$ be the projection of $g(x)$ on V_j, then, $\langle P_j g, \tilde{\varphi}_\alpha^{j,3}(x)\rangle$, $\langle P_j g, \tilde{\varphi}_\alpha^{j,1}(x)\rangle$, are the "step" approximation of the Fourier coefficients of $g(x)$.

6 An Example

In this section, we will give an example of the orthogonal real-valued periodic wavelets with box-spline constructed by the above procedure. We choose the centered 3-directions box-spline of degree [1,1,1], hence the final two-scale relations will be simpler.

Let m_1, m_2 be positive integers, m_3 be a nonnegative integer such that both $m_1 + m_3$ and $m_2 + m_3$ are even. Denoted by $c = (m_1 + m_3, m_2 + m_3)^t$, $n = m_1 + m_2 + m_3$. The centered 3-directions $e_1 = [1,0]^t$, $e_2 = [0,1]^t$, $e_3 = [1,1]^t$ box-spline of corresponding degree $m = [m_1, m_2, m_3]$ are defined as follows:

$$\widehat{B}(\omega) = \prod_{j=1}^{3} \left(\frac{1 - e^{-i\omega \cdot e_j}}{i\omega \cdot e_j} \right)^{m_j} e^{ic\cdot\omega/2}.$$

We let

$$C(\omega) = \frac{\widehat{B}(2\omega)}{\widehat{B}(\omega)} = 2^{-n} \prod_{j=1}^{3} (1 + e^{-i\omega\cdot e_j}) e^{ic\cdot\omega/2} = \sum_{k\in Z^2} c_k e^{-ik\cdot\omega},$$

then $B(x) = 4 \sum\limits_{k\in Z^2} c_k B(2x - k)$, where the sequence $\{c_k\}$ has only finite nonzero terms.

We choose $\omega(x) = B(x)$, $m = [1,1,1]$, and $T = 6$. We only illustrate our conclusions by the following figures.

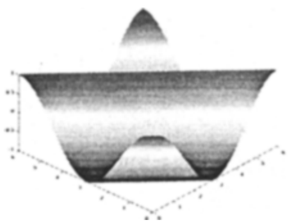

Fig. 1 The figure of $\cos\dfrac{2\pi(x_1 + x_2)}{T}$.

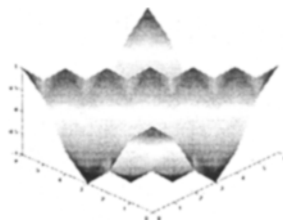

Fig. 2 The figure of $\tilde{\varphi}^{0,3}_{(1,1)}(x)$.

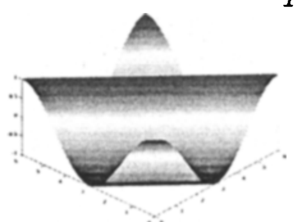

Fig. 3 The figure of $\tilde{\varphi}^{3,3}_{(1,1)}(x)$.

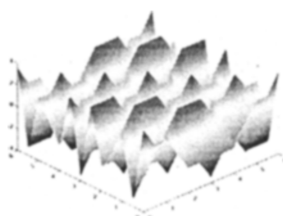

Fig. 4 The figure of $\psi^{0,10}_{(1,1)}(x)$.

From the figures above, we can find that $\tilde{\varphi}^{3,3}_{(1,1)}(x)$ gives a better approximation of $\cos\dfrac{2\pi(x_1 + x_2)}{T}$ than $\tilde{\varphi}^{0,3}_{(1,1)}(x)$. The last figure is one of the corresponding wavelets.

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