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# BOUNDEDNESS OF HIGHER ORDER COMMUTATORS OF GENERALIZED FRACTIONAL INTEGRAL OPERATORS ON HARDY SPACES*

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## Abstract

*Let $T_{\mu,b,m}$ be the higher order commutator generated by a generalized fractional integral operator $T_\mu$ and a BMO function b. In this paper, we will study the boundedness of $T_{\mu,b,m}$ on classical Hardy spaces and Herz-type Hardy spaces.*

**Key words** *$(\theta - N)$-type fractional integral operator, commutator, BMO space, Hardy space, Herz-type Hardy space, atom*

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## 1 Introduction and Main Results

Fractional integral operator is one of the important operators in harmonic analysis with background of partial differential equations. In fact, the solution of Laplace equation $\Delta u = f$ for good functions on $R^n$ can be represented by using fractional integral operators acting on $f$. In [1], the authors introduced a new kind of fractional integral operators, namely, the so called $(\theta - N)$-type fractional integral operators, and discussed their boundedness on Hardy spaces, weak Hardy spaces and Herz-type Hardy spaces.

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Ding Yong and Lu Shanzhen[2] have given the boundedness properties of higher order commutators generated by homogenous fractional integral operators and BMO functions on Hardy spaces, weak Hardy spaces and Herz-type spaces.

*Definition* 1.1.[1] Let $\theta$ be a nonnegative nondecreasing function on $(0,\infty)$, $N \in \mathbb{N}\cup\{0\}$, and $0<\mu<n$. $T_\mu$ is called a fractional integral operator, if there exists a measurable function $K(x,y)$ on $\mathbb{R}^n\times\mathbb{R}^n\setminus\{(x,x):x\in\mathbb{R}^n\}$ satisfying

(1) if $0\leqslant|\alpha|\leqslant N$, $|\partial_y^\alpha K(x,y)|\leqslant c|x-y|^{-n+\mu-|\alpha|}$;

(2) if $|y-y'|<|x-y|/2$,

$$\Big|K(x,y)-\sum_{|\alpha|\le N}\frac{1}{\alpha!}\partial_y^\alpha K(x,y')(y-y')^\alpha\Big|\leqslant c\,\theta\Big(\frac{|y-y'|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\mu}}$$

such that

$$T_\mu f(x)=\int_{\mathbb{R}^n}K(x,y)f(y)\mathrm{d}y$$

for all $f\in S(\mathbb{R}^n)$, where $S(\mathbb{R}^n)$ is the space of all the Schwartz functions on $\mathbb{R}^n$.

The higher order commutator generated by $(\theta,N)$-type fractional integral operator $T_\mu$ and a BMO function $b$ is defined by

$$T_{\mu,b,m}=\int K(x,y)(b(x)-b(y))^m f(y)\mathrm{d}y\,,\qquad m\in\mathbb{N}.$$

Recently, Zhang Liqin [3],[4] gave the boundedness of $T_{\mu,b,1}$ on Hardy spaces. In this paper, we will study the boundedness properties of $T_{\mu,b,m}$ on classical Hardy spaces and Herz-type Hardy spaces.We remark that in this paper we are very much motivated by the work of professor Ding Yong in [2], [5].

Now, let us give some definitions and show our results.

*Definition* 1.2.[2] Let b$\in$ BMO($\mathbb{R}^n$), $m\in\mathbb{N}$ and $0<p\le1$. A function $a(x)$ is said to be a $(p,\infty;b^m)$-atom, if

(1) supp$a\subset B(x_0,r)=\{x\in\mathbb{R}^n:|x-x_0|\le r\}$ for some $r>0$;

(2) $\|a\|_\infty\le|B(x_0,r)|^{-1/p}$;

(3) $\int a(x)\mathrm{d}x=\int a(x)b(x)\mathrm{d}x=\cdots=\int a(x)b^m(x)\mathrm{d}x=0.$

A temperate distribution f belongs to the atomic Hardy space $H^p_{b^m}(\mathbb{R}^n)$, if it can be written as $f=\sum_{j=1}^\infty\lambda_j a_j$ in the $S'$-sense, where each $a_j$ is $(p,\infty;b^m)$ -atom, $\lambda_j\in C$ and $\sum_{j=1}^\infty|\lambda_j|^p<\infty$. Moreover, we define the quasi-norm on $H^p_{b^m}(\mathbb{R}^n)$ by

$$\|f\|_{H^p_{b^m}(\mathbb{R}^n)}=\inf\left(\sum_{j=1}^\infty|\lambda_j|^p\right)^{1/p}$$

where the infimum is taken over all decompositions of $f$ as above.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}, C_k = B_k - B_{k-1}, \chi_k = \chi_{C_k}, k \in Z$.

*Definition* 1.3.[2] Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^q_{Loc}(\mathbb{R}^n \setminus \{0\}) : \| f \|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\| f \|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \| f\chi_k \|^p_{L^q(\mathbb{R}^n)} \right\}^{1/p}$$

with usual modifications made when $p = \infty$.

*Definition* 1.4.[2] Let $1 < q < \infty$, $\alpha \ge n(1 - 1/q)$ and $b \in$ BMO. A function $a$ on $\mathbb{R}^n$ is called a central $(\alpha, q; b^m)$-atom , if

(1) supp $a \subset B(0,r) = \{x \in \mathbb{R}^n : | x | \le r\}$ for some $r > 0$ ;

(2) $\| a \|_q \le | B(0,r) |^{-\frac{\alpha}{n}}$ ;

(3) $\displaystyle\int a(x)\mathrm{d}x = \int a(x)b(x)\mathrm{d}x = \cdots = \int a(x)b^m(x)\mathrm{d}x = 0.$

*Definition* 1.5.[2] Let $b \in$ BMO. $1 < q < \infty$, $\alpha \ge n(1 - 1/q)$ and $0 < p < \infty$. A temperate distribution $f$ belongs to the Herz-type Hardy space $H\dot{K}^{\alpha,p}_{q,b^m}(\mathbb{R}^n)$, if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the $S'$-sense, where each $a_j$ is a central $(\alpha, q; b^m)$ -atom supported on $B_j = B(0, 2^j)$, $\lambda_j \in C$ and $\sum_{j=-\infty}^{\infty} | \lambda_j |^p < \infty$. Moreover, we define the quasinorm on $H\dot{K}^{\alpha,p}_{q,b^m}(\mathbb{R}^n)$ by

$$\|f\|_{H\dot{K}^{\alpha,p}_{q,b^m}(\mathbb{R}^n)} = \inf \Big( \sum_{j=-\infty}^{\infty} | \lambda_j |^p \Big)^{1/p}$$

with the infimum taken over all decompositions of $f$ as above.

In this paper, we obtain following results.

**Theorem 1.1.** *Let* $0 < \mu < n$, $1 < p < n/\mu$, $1/q = 1/p - \mu/n$, $b \in$ BMO. *Then there is a constant* $c > 0$, *independent of* $f$, *such that*

$$\|T_{\mu,b,m} f\|_q \le c\|f\|_p .$$

**Theorem 1.2.** *Let* $0 < \mu < n$, $0 < p \le 1$, $1/q = 1/p - \mu/n$, $N \ge [n(1/p - 1)]$, $b \in$ BMO *and*

$$\int_0^1 \frac{\theta^q(t)}{t^{(\mu-n)q+n+1}} \Big(\log \frac{1}{t} + 1\Big)^{mq} \mathrm{d}t < \infty,$$

*then* $T_{\mu,b,m}$ *is bounded from* $H^p_{b^m}(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$.

**Theorem 1.3.** *Let* $0 < \mu < n$, $1 < q_1 < n/\mu$, $1/q_2 = 1/q_1 - \mu/n$, $0 < p_1 \le p_2 < \infty$, $n(1-1/q_1) \le \alpha < \infty$, $N \ge [\alpha + n(1/q_1 - 1)]$, $b \in \mathrm{BMO}$, *and*

$$\int_0^1 \frac{(\theta(t))^{p_1 \wedge 1}}{t^{p_1 \wedge 1(\alpha - n + n/q_1)+1}} \Big(\log \frac{1}{t} + 1\Big)^{m(p_1\wedge 1)} \mathrm{d}t < \infty,$$

*where* $p_1 \wedge 1 = \min\{p_1, 1\}$, *then* $T_{\mu,b,m}$ *is bounded from* $H\dot{K}^{\alpha,p_1}_{q_1,b^m}(\mathbb{R}^n)$ *into* $\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$.

## 2 Proof of Theorems

**Lemma 2.1.**[2] *Let* $b \in BMO(\mathbb{R}^n)$, $l > 1$, *then*

$$\int_{2^{j+1}B} | b(x) - b_B |^{ml} \,\mathrm{d}x \le c \,\|b\|_*^{ml} j^{ml} \,| 2^{j+1} B \,| \,.$$

**Lemma 2.2.**[5] *Suppose that* $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $b \in BMO$. *Then for* $\mu(x), v(x) \in A(p,q)$ *and* $\mu(x)v(x)^{-1} = v^m(x)$, *there is a constant c independent of f such that* $M^m_{1,\alpha,b}$ *satisties*

$$\Big(\int_{\mathbb{R}^n} [M^m_{1,\alpha,b} f(x) v(x)]^q \mathrm{d}x\Big)^{1/q} \le c \Big(\int_{\mathbb{R}^n} | f(x)\mu(x) |^p\Big)^{1/p},$$

*where*

$$M^m_{\Omega,\alpha,b} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} | \Omega(x-y) || b(x) - b(y) |^m | f(y) | \,\mathrm{d}y.$$

*Proof of Theorem* 1.1. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$ with $0 < \mu - \varepsilon < \mu + \varepsilon < n$, we choose a $\delta > 0$ such that

$$\delta^{2\varepsilon} = M^m_{1,\mu+\varepsilon,b} f(x) / M^m_{1,\mu-\varepsilon,b} f(x).$$

Write

$$\begin{aligned} T_{\mu,b,m} f(x) &= \int_{|x-y|<\delta} K(x,y)(b(x)-b(y))^m f(y)\mathrm{d}y \\ &\quad + \int_{|x-y|\ge\delta} K(x,y)(b(x)-b(y))^m f(y)\mathrm{d}y = I_1 + I_2. \end{aligned}$$

We have

$$\begin{aligned} | I_1 | &\le \int_{|x-y|<\delta} \frac{1}{| x-y |^{n-\mu}} | b(x) - b(y) |^m | f(y) | \,\mathrm{d}y \\ &\le \sum_{j=0}^{\infty} \int_{2^{-j-1}\delta \le |x-y| < 2^{-j}\delta} (2^{-j-1}\delta)^{\mu-n} | b(x) - b(y) |^m | f(y) | \,\mathrm{d}y \\ &\le 2^{n-\mu} \sum_{j=0}^{\infty} \frac{(2^{-j}\delta)^{\varepsilon}}{(2^{-j}\delta)^{n-\mu+\varepsilon}} \int_{|x-y|<2^{-j}\delta} | b(x) - b(y) |^m | f(y) | \,dy \le c\, \delta^{\varepsilon} M^m_{1,\mu-\varepsilon,b} f(x). \end{aligned}$$

Similarly,

$$\begin{aligned} |I_2| &\le \int_{|x-y|\ge\delta} \frac{1}{|x-y|^{n-\mu}} |b(x)-b(y)|^m |f(y)| \,dy \\ &\le \sum_{j=1}^{\infty} \int_{2^{j-1}\delta\le|x-y|<2^j\delta} (2^{j-1}\delta)^{\mu-n} |b(x)-b(y)|^m |f(y)| \,dy \\ &\le 2^{n-\mu} \sum_{j=1}^{\infty} \frac{(2^j\delta)^{-\varepsilon}}{(2^j\delta)^{n-\mu-\varepsilon}} \int_{|x-y|<2^j\delta} |b(x)-b(y)|^m |f(y)| \,dy \le c\,\delta^{-\varepsilon} M^m_{1,\mu+\varepsilon,b} f(x). \end{aligned}$$

Thus, by the above selection of $\delta$ we get

$$|T_{\mu,b,m}f(x)| \le c\,(\delta^{\varepsilon} M^m_{1,\mu-\varepsilon,b}f(x) + \delta^{-\varepsilon} M^m_{1,\mu+\varepsilon,b}f(x)) = c\,(M^m_{1,\mu-\varepsilon,b}f(x))^{1/2}(M^m_{1,\mu+\varepsilon,b}f(x))^{1/2}.$$

Noting $1 < p < n/\mu$, there is an $\varepsilon > 0$, such that

$$p < n/(\mu+\varepsilon).$$

Let $1/q_1 = 1/p-(\mu-\varepsilon)/n$, $1/q_2 = 1/p-(\mu+\varepsilon)/n$, $l_1 = 2q_1/q$, $l_2 = 2q_2/q$, then $q_1, q_2 > 0$, $l_2 > 1$ and $1/l_1 + 1/l_2 = 1$. Thus, by Lemma 2.2, one has

$$\begin{aligned} \|T_{\mu,b,m}f\|_q^q &\le c \int_{\mathbb{R}^n} |M^m_{1,\mu-\varepsilon,b}f(x)|^{q/2} |M^m_{1,\mu+\varepsilon,b}f(x)|^{q/2} \,dx \\ &\le c\Big(\int_{\mathbb{R}^n} |M^m_{1,\mu-\varepsilon,b}f(x)|^{ql_1/2} \,dx\Big)^{1/l_1} \Big(\int_{\mathbb{R}^n} |M^m_{1,\mu+\varepsilon,b}f(x)|^{ql_2/2} \,dx\Big)^{1/l_2} \\ &= c\Big(\int_{\mathbb{R}^n} |M^m_{1,\mu-\varepsilon,b}f(x)|^{q_1} \,dx\Big)^{q/2q_1} \Big(\int_{\mathbb{R}^n} |M^m_{1,\mu+\varepsilon,b}f(x)|^{q_2} \,dx\Big)^{q/2q_2} \\ &\le c\Big(\int_{\mathbb{R}^n} |f(x)|^p \,dx\Big)^{q/2p} \Big(\int_{\mathbb{R}^n} |f(x)|^p \,dx\Big)^{q/2p} = c\|f\|_p^q\,. \end{aligned}$$

*Proof of Theorems* 1.2. By a standard argument, it suffices to show that there exists a constant $c > 0$ such that for each $(p, \infty; b^m)$-atom $a$,

$$\|T_{\mu,b,m}(a)\|_q \le c\,.$$

Now let us fix a $(p, \infty; b^m)$-atom $a$ with supp $a \subset B = B(x_0, r)$. Write

$$\|T_{\mu,b,m}(a)\|_q^q = \int_{|x-x_0|<4r} |T_{\mu,b,m}(a)|^q dx + \int_{|x-x_0|\ge 4r} |T_{\mu,b,m}(a)|^q dx = L_1 + L_2.$$

For $L_1$, we choose $p_1, q_1$ such that $1 < p_1 < n/\mu, 1/q_1 = 1/p_1 - \mu/n$. Using Theorem 1.1, we have that $T_{\mu,b,m}$ is bounded from $L^{p_1}(\mathbb{R}^n)$ into $L^{q_1}(\mathbb{R}^n)$.

$$\begin{aligned} L_1 &\le \Big(\int_{|x-x_0|<4r} |T_{\mu,b,m}(a)(x)|^{q_1} dx\Big)^{q/q_1} \Big(\int_{|x-x_0|<4r} dx\Big)^{1-\frac{q}{q_1}} \\ &\le c\|a\|_{p_1}^q (4r)^{n(1-q/q_1)} \le c\|a\|_\infty^q |B|^{q/p_1} r^{n(1-q/q_1)} \le c\,. \end{aligned}$$

For $L_2$, noting $|x-x_0|>4r$, by the vanishing condition of $a$, we obtain

$$\begin{aligned}
|T_{\mu,b,m}(a)(x)| &\le c\int_B\Big|K(x,y)-\sum_{|s|\le N}\frac{1}{s\,!}\partial_y^s K(x,x_0)(y-x_0)^s\Big|\,|b(x)-b(y)|^m\,|a(y)|dy\\
&\le c\int_B\theta\Big(\frac{|y-x_0|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\mu}}|a(y)|(|b(x)-b_B|^m+|b(y)-b_B|^m)dy\\
&\le c\theta\Big(\frac{2r}{|x-x_0|}\Big)\frac{1}{|x-x_0|^{n-\mu}}\Big(\int_B|a(y)||b(x)-b_B|^m\,dy+\int_B|a(y)||b(y)-b_B|^m\,dy\Big)\\
&\le c\theta\Big(\frac{2r}{|x-x_0|}\Big)\frac{1}{|x-x_0|^{n-\mu}}\Big(|b(x)-b_B|^m\|a\|_\infty|B|+\|a\|_\infty\|b\|_*^m|B|\Big)\\
&\le c\theta\Big(\frac{2r}{|x-x_0|}\Big)\frac{1}{|x-x_0|^{n-\mu}}|B|^{1-1/p}\Big(|b(x)-b_B|^m+\|b\|_*^m\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
L_2 &\le c\sum_{i=2}^{\infty}\int_{2^ir<|x-x_0|\le 2^{i+1}r}\Big|\theta\Big(\frac{2r}{|x-x_0|}\Big)\frac{1}{|x-x_0|^{n-\mu}}r^{n(1-1/p)}\Big(|b(x)-b_B|^m+\|b\|_*^m\Big)\Big|^q dx\\
&\le c\sum_{i=2}^{\infty}\Big|\theta(2^{-i+1})(2^ir)^{\mu-n}r^{n(1-1/p)}\Big|^q\Big(\int_{|x-x_0|\le 2^{i+1}r}|b(x)-b_B|^{mq}dx+\int_{|x-x_0|\le 2^{i+1}r}\|b\|_*^{mq}dx\Big).
\end{aligned}$$

If $0<p\le\frac{n}{n+\mu}$, then $0<q\le 1$. Choose $l>\frac{1}{q}$, we have

$$\begin{aligned}
\int_{|x-x_0|\le 2^{i+1}r}|b(x)-b_B|^{mq}dx &\le \Big(\int_{|x-x_0|\le 2^{i+1}r}|b(x)-b_B|^{mql}dx\Big)^{1/l}\Big(\int_{|x-x_0|\le 2^{i+1}r}dx\Big)^{1-1/l}\\
&\le c\,\Big(\|b\|_*^{mql}i^{mql}(2^{i+1}r)^n\Big)^{1/l}(2^{i+1}r)^{n(1-1/l)}\le c\,\|b\|_*^{mq}i^{mq}(2^{i+1}r)^n.
\end{aligned}$$

If $\frac{n}{n+\mu}<p\le 1$, then $q>1$,

$$\int_{|x-x_0|\le 2^{i+1}r}|b(x)-b_B|^{mq}dx\le c\,\|b\|_*^{mq}i^{mq}(2^{i+1}r)^n.$$

Thus, we have

$$\int_{|x-x_0|\le 2^{i+1}r}|b(x)-b_B|^{mq}dx\le c\,\|b\|_*^{mq}i^{mq}(2^{i+1}r)^n$$

for all $0<p\le 1, 1/q=1/p-\mu/n$. Therefore,

$$\begin{aligned}
L_2 &\le c\sum_{i=2}^{\infty}\theta^q(2^{-i+1})(2^ir)^{(\mu-n)q}r^{n(1-1/p)q}\Big(c\,\|b\|_*^{mq}i^{mq}(2^{i+1}r)^n+\|b\|_*^{mq}(2^{i+1}r)^n\Big)\\
&\le c\,\|b\|_*^{mq}\sum_{i=2}^{\infty}\theta^q(2^{-i+1})2^{i((\mu-n)q+n)}i^{mq}\le c\,\|b\|_*^{mq}\int_0^1\frac{\theta^q(t)}{t^{(\mu-n)q+n+1}}\Big(\log\frac{1}{t}+1\Big)^{mq}dt\le c\,.
\end{aligned}$$

Combining the estimates for $L_1$ and $L_2$, we finish the proof.

*Proof of Theorem* 1.3. Since $f\in H\dot{K}_{q_1,b^m}^{\alpha,p_1}$, we may write

$$f=\sum_{j=-\infty}^{\infty}\lambda_j\,a_j,$$

where each $a_j$ is a central $(\alpha, q_1; b^m)$-atom with the support $B_j = B(0, 2^j)$ and $\sum_{j=-\infty}^{+\infty} |\lambda_j|^{p_1} < \infty$. Noting that $p_1 \le p_2$, we have

$$\begin{aligned}\left\|T_{\mu,b,m}(f)\right\|^{p_1}_{\dot K^{\alpha,p_2}_{q_2}} &\le c\sum_{k=-\infty}^{+\infty}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-3}|\lambda_j|\,\|T_{\mu,b,m}(a_j)\chi_k\|_{q_2}\Big)^{p_1}\\ &\quad + c\sum_{k=-\infty}^{+\infty}2^{k\alpha p_1}\Big(\sum_{j=k-2}^{+\infty}|\lambda_j|\,\|T_{\mu,b,m}(a_j)\chi_k\|_{q_2}\Big)^{p_1} = c\,J_1 + c\,J_2.\end{aligned}$$

For $J_2$, noting that $T_{\mu,b,m}$ is bounded from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$, we have

$$J_2 \le c\sum_{k=-\infty}^{+\infty}2^{k\alpha p_1}\Big(\sum_{j=k-2}^{+\infty}|\lambda_j|\,\|(a_j)\|_{q_1}\Big)^{p_1} \le c\sum_{k=-\infty}^{+\infty}\Big(\sum_{j=k-2}^{+\infty}|\lambda_j|\,2^{(k-j)\alpha}\Big)^{p_1}.$$

If $0 < p_1 \le 1$, then

$$J_2 \le c\sum_{k=-\infty}^{+\infty}\sum_{j=k-2}^{+\infty}|\lambda_j|^{p_1}\,2^{(k-j)\alpha p_1} \le c\sum_{j=-\infty}^{+\infty}|\lambda_j|^{p_1}\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p_1} \le c\sum_{j=-\infty}^{+\infty}|\lambda_j|^{p_1}.$$

If $p_1 > 1$, then

$$\begin{aligned}J_2 &\le c\sum_{k=-\infty}^{+\infty}\Big(\sum_{j=k-2}^{+\infty}|\lambda_j|^{p_1}\,2^{(k-j)\alpha}\Big)\Big(\sum_{j=k-2}^{+\infty}2^{(k-j)\alpha}\Big)^{\frac{p_1}{p_1'}}\\ &\le c\sum_{j=-\infty}^{+\infty}|\lambda_j|^{p_1}\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha} \le c\sum_{j=-\infty}^{+\infty}|\lambda_j|^{p_1}.\end{aligned}$$

For $J_1$, noting $j \le k-3$, we have $|x-y| \ge 2|y|$ for $x \in C_k,\ y \in B_j$, then

$$\begin{aligned}&|T_{\mu,b,m}a_j(x)|\\ &\le \int_{B_j}\Big|K(x,y) - \sum_{|s|\le N}\frac{1}{s!}\partial_y^s K(x,0)y^s\Big|\,|a_j(y)|\Big(|b(x)-b_{B_j}|^m + |b(y)-b_{B_j}|^m\Big)dy\\ &\le c\int_{B_j}\theta\Big(\frac{|y|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\mu}}|a_j(y)|\Big(|b(x)-b_{B_j})|^m + |b(y)-b_{B_j}|^m\Big)dy\\ &\le c\,\theta\Big(\frac{2^{j+1}}{|x|}\Big)\frac{1}{|x|^{n-\mu}}\Big(\int_{B_j}|a_j(y)|\,|b(x)-b_{B_j}|^m\,dy + \int_{B_j}|a_j(y)|\,|b(y)-b_{B_j}|^m\,dy\Big)\\ &\le c\,\theta(2^{j-k+2})2^{k(\mu-n)}\bigg(\Big(\int_{B_j}|a_j(y)|^{q_1}dy\Big)^{1/q_1}\Big(\int_{B_j}|b(x)-b_{B_j}|^{mq_1'}dy\Big)^{1/q_1'}\\ &\quad + \Big(\int_{B_j}|a_j(y)|^{q_1}dy\Big)^{1/q_1}\Big(\int_{B_j}|b(y)-b_{B_j})|^{mq_1'}dy\Big)^{1/q_1'}\bigg)\\ &\le c\,\theta(2^{j-k+2})2^{k(\mu-n)}2^{-j\alpha}2^{jn(1-\frac{1}{q_1})}\Big(|b(x)-b_{B_j}|^m + \|b\|_*^m\Big),\end{aligned}$$

therefore

$$\begin{aligned} J_1 &\le c\sum_{k=-\infty}^{+\infty} 2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j| \theta(2^{j-k+2})2^{k(\mu-n)-j\alpha+jn(1-\frac{1}{q_1})} \\ &\quad \times\Big(\int_{C_k} |b(x)-b_{B_j}|^{mq_2}+\|b\|_*^{mq_2}\mathrm{d}x\Big)^{\frac{1}{q_2}}\Big)^{p_1} \\ &\le c\sum_{k=-\infty}^{+\infty}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j| \theta(2^{j-k+2})2^{k(\mu-n)+(k-j)\alpha+jn(1-\frac{1}{q_1})} \\ &\quad \times\Big(\|b\|_*^{mq_2}(k-j-1)^{mq_2}2^{kn}+\|b\|_*^{mq_2}2^{kn}\Big)^{\frac{1}{q_2}}\Big)^{p_1} \\ &\le c\|b\|_*^{mp_1}\sum_{k=-\infty}^{+\infty}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j| \theta(2^{j-k+2})2^{(k-j)(\alpha-n+\frac{n}{q_1})}(k-j-1)^m\Big)^{p_1}. \end{aligned}$$

If $0< p_1\le 1$, then

$$\begin{aligned} J_1 &\le c\sum_{k=-\infty}^{+\infty}\sum_{j=-\infty}^{k-3} |\lambda_j|^{p_1} \theta^{p_1}(2^{j-k+2})2^{(k-j)(\alpha-n(1-1/q_1))p_1}(k-j-1)^{mp_1} \\ &\le c\sum_{j=-\infty}^{+\infty} |\lambda_j|^{p_1}\sum_{i=1}^{+\infty}\theta^{p_1}(2^{-i})2^{i(\alpha-n(1-1/q_1))p_1}(i+1)^{mp_1} \\ &\le c\sum_{j=-\infty}^{+\infty} |\lambda_j|^{p_1}\int_0^1 \frac{(\theta(t))^{p_1}}{t^{p_1(\alpha-n+n/q_1)+1}}\Big(\log\frac{1}{t}+1\Big)^{mp_1}dt\le c\sum_{j=-\infty}^{+\infty}|\lambda_j|^{p_1}. \end{aligned}$$

If $p_1> 1$, then

$$\begin{aligned} J_1 &\le c\sum_{k=-\infty}^{+\infty}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j|^{p_1} \theta(2^{j-k+2})2^{(k-j)(\alpha-n+n/q_1)}(k-j-1)^m\Big) \\ &\quad \times\Big(\textstyle\sum_{j=-\infty}^{k-3}\theta(2^{i-k+2})2^{(k-j)(\alpha-n+n/q_1)}(k-j-1)^m\Big)^{\frac{p_1}{p_1'}} \\ &\le c\sum_{k=-\infty}^{+\infty}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j|^{p_1} \theta(2^{j-k+2})2^{(k-j)(\alpha-n+n/q_1)}(k-j-1)^m\Big) \\ &\quad \times\int_0^1\frac{(\theta(t))}{t^{\alpha-n+n/q_1+1}}\Big(\log\frac{1}{t}+1\Big)^m\mathrm{d}t \\ &\le c\sum_{j=-\infty}^{+\infty} |\lambda_j|^{p_1}\int_0^1\frac{\theta(t)}{t^{\alpha-n+\frac{n}{q_1}+1}}\Big(\log\frac{1}{t}+1\Big)^m\mathrm{d}t\le c\sum_{j=-\infty}^{+\infty}|\lambda_j|^{p_1}. \end{aligned}$$

Combining the estimates for $L_1$ and $L_2$, we complete the proof.

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