ON GUNDERSEN'S QUESTION FOR THE UNICITY OF MEROMORPHIC FUNCTIONS*

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Abstract

The uniqueness of meromorphic functions that share four values are investigated, some results are obtained to show that if two nonconstant entire functions share three finite values IM, then the functions necessarily share all three values CM.

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1 Introduction and Main Results

Let $f(z)$ be a meromorphic function in the complex plane, we say that f and g share the value $a(a = \infty$ is allowed) provided that $f(z) = a$ if and only if $g(z) = a$. Usually, we will state whether a shared value is by CM (counting multiplicities) or IM (ignoring multiplicities). Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions that share the value a IM. The following well known theorems on uniqueness of meromorphic functions are due to R.Nevanlinna (see [8]).

Theorem A. *Let f and g be nonconstant meromorphic functions. If they share five distinct values (∞ is allowed)IM, then $f \equiv g$.*

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Theorem B. *If f and g are distinct nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 CM,then f is a Möbius transformation of g;two of the values,say, a_1 and a_2, are Picard values,and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In 1976 L.Rubel asked the following question: whether CM can be replaced by IM in Theorem B with the same conclusion or not? Gundersen found a counterexample giving a negative answer to the question (see ref. [1]). Meanwhile, G.G.Gundersen showed that: Suppose $f(z)$ and $g(z)$ are two nonconstant meromorphic functions that share three values CM, and share the fourth value IM, then they share all four values CM. Shortly, it can be stated as 3CM+1IM=4CM. Furthermore,G.G.Gundersen proved that(See [2], [3]) 2CM+2IM=4CM is also true.

Theorem C. *If two nonconstant meromorphic functions share two values IM, and share two other values CM, then f and g share all four values CM.*

Now it remains the so called "1CM+3IM question" open,which is: If two nonconstant meromorphic functions share three values IM and share a fourth value CM,then do the functions necessarily share all four values CM?

G.G.Gundersen(see [4]) asked a sub-question:If two nonconstant entire functions share three finite values IM, then do the functions necessarily share all three values CM?

Notice that ∞ can be regarded as a value shared by entire functions $f(z)$ and $g(z)$ with the property $\overline{N}(r, f) = 0, \overline{N}(r, g) = 0$. We consider the following extension of Gundersen's question:

Question 1. *Let nonconstant meromorphic functions $f(z)$ and $g(z)$ share four values a_1, a_2, a_3, a_4.If one of the values,say, a_1 satisfies $\overline{N}(r, \frac{1}{f-a_1}) = S(r, f)$, then do the functions share all four values CM?*

To the above question 1, we obtained a partial result in 2003(see ref. [6]).

Theorem D. *Let nonconstant meromorphic functions $f(z)$ and $g(z)$ share the values a_1, a_2, a_3, a_4, and satisfy $\overline{N}(r, \dfrac{1}{f-a_1}) = S(r, f)$. If the equality*

$$\overline{N}_{(1,2)+(2,1)}\left(r, \frac{1}{f-a_k}\right) = S(r, f) \tag{1}$$

holds for some $k(k \in \{2, 3, 4\}$, then $f(z)$ and $g(z)$ share a_1, a_2, a_3, a_4 CM.

Recently, we acquire the complete result to question 1 as follows.

Theorem 1. *Let two meromorphic functions $f(z)$ and $g(z)$ share four values a_1, a_2, a_3, a_4. If $\overline{N}\left(r, \dfrac{1}{f-a_1}\right) = S(r, f)$, then $f(z)$ and $g(z)$ share a_1, a_2, a_3, a_4 CM.*

By Theorem 1 the conjecture of 1CM+3IM=4CM tends to be true. Particularly, it gives an affirmative answer to Gundersen's question.

Corollary 1. *If two nonconstant entire functions share three finite values IM, then the functions necessarily share all three values CM.*

2 Main Lemmas

Lemma 1.(See ref. [6], ,Lemma 2.3). *Let nonconstant meromorphic functions f and g share four values a_1, a_2, a_3, ∞ IM.Suppose $\overline{N}(r,f) = S(r,f)$, and $f \not\equiv g$. Let*

$$\Delta = \left(\frac{f'(f-a_3)}{(f-a_1)(f-a_2)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)}\right)^2 \frac{(f-a_1)(f-a_2)(g-a_1)(g-a_2)}{(f-g)^2}.$$

Then

1) $m(r,\Delta) = S(r,f)$;

2) $N(r,\Delta) = \sum_{j=1}^{2} \overline{N}_{(1,2)+(2,1)}(r, \frac{1}{f-a_j}) + S(r,f)$.

Lemma 2. *Let two distinct nonconstant meromorphic functions f and g share four values $0,1,c,\infty (c \neq 0,1,\infty)$ with the property $\overline{N}(r,f) = S(r,f)$. And let*

$$\Delta_0 = (\frac{f'f}{(f-1)(f-c)} - \frac{g'g}{(g-1)(g-c)})^2 \frac{(f-1)(f-c)(g-1)(g-c)}{(f-g)^2}, \tag{2}$$

$$\Delta_1 = (\frac{f'(f-1)}{f(f-c)} - \frac{g'(g-1)}{g(g-c)})^2 \frac{f(f-c)g(g-c)}{(f-g)^2}, \tag{3}$$

$$\Delta_c = (\frac{f'(f-c)}{f(f-1)} - \frac{g'(g-c)}{g(g-1)})^2 \frac{f(f-1)g(g-1)}{(f-g)^2} \tag{4}$$

and

$$\nabla_c = (1-c)^2\Delta_0 + c^2\Delta_1 - \Delta_c. \tag{5}$$

Then

1) $m(r,\nabla_c) = S(r,f)$;

2) $N(r,\nabla_c) = \overline{N}_2(r,c) + S(r,f)$.

Remark 1. With the same conditions and symbols as in Lemma 2, let

$$\nabla_0 = -(1-c)^2\Delta_0 + c^2\Delta_1 + \Delta_c, \tag{6}$$

$$\nabla_1 = (1-c)^2\Delta_0 - c^2\Delta_1 + \Delta_c, \tag{7}$$

then, similar to Lemma 2, we have

$$N(r,\nabla_0) = \overline{N}_2(r,0) + S(r,f) \tag{8}$$

$$N(r,\nabla_1)=\overline{N}_2(r,1)+S(r,f). \tag{9}$$

Lemma 3. ([6] Lemma 5.7-5.8). *Under conditions of Lemma 2, we have*

1) *if* $z_0\in S_{(2,1)}(f=0=g), a_0=\frac{f''(z_0)}{2}$, *then*

$$\Delta_c(z_0)=\frac{1}{2}c^2(1-c)^2\varphi(z_0)-5c(1-c)a_0;$$

2) *if* $z_1\in S_{(2,1)}(f=1=g), a_0=\frac{f''(z_1)}{2}$, *then*

$$\Delta_c(z_1)=\frac{1}{2}c^2(1-c)^2\varphi(z_0)+5c(1-c)a_0.$$

Similar to the proofs of Lemma 5.7∼ 5.8 in ref. [6], we can prove

Lemma 4. *Under the condition of Lemma 2, let* $z_c\in S_{(2,1)}(f=c=g), a_0=\dfrac{f''(z_c)}{2}$, *then*

1) $\nabla_0(z_c)=\dfrac{1}{2}c^2(1-c)^2\varphi(z_c)+5c(1-c)a_0;$

2) $\nabla_1(z_c)=\dfrac{1}{2}c^2(1-c)^2\varphi(z_c)-5c(1-c)a_0.$

Lemma 5. *Under the condition of Lemma 2, we assert that there exists a positive integer* $p\geq 4$ *such that*

1) *the inequality*

$$\overline{N}_D(r,a)=\overline{N}_2(r,a)+\overline{N}_3(r,a)+\overline{N}_p(r,a)+S(r,f) \tag{10}$$

holds for $a=0,1,c$;

2) *if there exist an integer* $p\geq 4$ *and a value* $a\in\{0,1,c\}$ *such that* $\overline{N}_p(r,a)\neq S(r,f)$, *then*

$$\overline{N}_k(r,0)=0,\quad \overline{N}_k(r,1)=0,\quad \overline{N}_k(r,c)=0 \tag{11}$$

hold for any $k\geq 4, k\neq p$.

3 Techniques for Proving the Theorem

Without loss of generality, we suppose

(H1) $f(z)$ and $g(z)$ share $\infty,0,1,c$ IM with the property $\overline{N}(r,f)=S(r,f)$.

According to Theorem D, we only need to show that the following case (H2) could not occur.

(H2) $\overline{N}_2(r,a)\neq S(r,f),\quad a=0,1,c.$

To the constant c, chosen a branch of $\sqrt{z}$ such that $\sqrt{c^2}=c$, we meet two basic cases:

Basic case I: $\sqrt{c^2} = c, \quad \sqrt{(1-c)^2} = (c-1);$

Basic case II: $\sqrt{c^2} = c, \quad \sqrt{(1-c)^2} = -(c-1).$

We merely consider the basic case I, for the basic case II can be managed in the similar way. To the basic case II, there are two sub-cases:

(H3) $\sqrt{c^2} = c, \quad \sqrt{(1-c)^2} = c-1, \quad \sqrt{c^2(1-c)^2} = c(c-1);$

(H4) $\sqrt{c^2} = c, \quad \sqrt{(1-c)^2} = c-1, \quad \sqrt{c^2(1-c)^2} = -c(c-1).$

Let $S = \{z : \varphi(z) = 0\}, D = C \setminus S$. To the chosen branch of $\sqrt{z}$, define

$$\sigma(z) = (f-g)\sqrt{\frac{f'g'}{f(f-1)(f-c)g(g-1)(g-c)}}. \tag{12}$$

Then $\sigma(z)$ is meromorphic on D. From now on, our discussion will be proceeded on D. Since the counting function correspondent to points in the set S is small with respect to f under the assumption (H1), the arguments on counting functions, which will be acquired on D later, also essentially make work on C.

It is obvious that $T(r,\sigma) = S(r,f)$.

Proposition 1. *With conditions and symbols the same as Lemma 2, let*

$$\varphi = \frac{f'g'(f-g)^2}{f(f-1)(f-c)g(g-1)(g-c)}, \tag{13}$$

and

$$H_c = 4\{(1-c)^2\Delta_0 - \frac{1}{2}c^2(1-c)^2\varphi\}\{c^2\Delta_1 - \frac{1}{2}c^2(1-c^2)\varphi\} - \{\nabla_c - \frac{1}{2}c^2(1-c)^2\varphi\}^2. \tag{14}$$

Then $T(r,\varphi) = S(r,f), \quad T(r,H_c) = S(r,f)$.

Furthermore, let

$$H_0 = 4\{c^2\Delta_1 - \frac{1}{2}c^2(1-c)^2\varphi\}\{\Delta_c - \frac{1}{2}c^2(1-c^2)\varphi\} - \{\nabla_0 - \frac{1}{2}c^2(1-c)^2\varphi\}^2, \tag{15}$$

$$H_1 = 4\{(1-c)^2\Delta_0 - \frac{1}{2}c^2(1-c)^2\varphi\}\{\Delta_c - \frac{1}{2}c^2(1-c^2)\varphi\} - \{\nabla_1 - \frac{1}{2}c^2(1-c)^2\varphi\}^2, \tag{16}$$

then

$$H_0 = H_1 = H_c. \tag{17}$$

Since then, H_0, H_1, H_c will be represented by H elsewhere in this paper.

Proposition 2. *We state three facts below.*

(A1) $H(z) \equiv \frac{7}{16}c^4(1-c)^4\varphi^2;$

(A2) *Let k be an integer satisfying $k > 4$, then the equality $\overline{N}_k(r,a) = 0$ holds for $a = 0, 1, c$;*

(A3) *For $a = 0, 1, c$, the equality $\overline{N}_D(r,a) = \overline{N}_2(r,a) + \overline{N}_3(r,a) + \overline{N}_4(r,a) + S(r,f)$ holds, where $H(z) = H_0(z) = H_1(z) = H_c(z)$.*

Since then,the integer p in the definition of $J(z)$ is replaced by 4.

Through the above preparations and arguments, we can exclude the occurring of (H2) by making out the impossibilities of the following cases.

Case 1

$$\overline{N}_3(r,1) + \overline{N}_4(r,1) = S(r,f), \qquad \overline{N}_3(r,c) + \overline{N}_4(r,c) = S(r,f);$$

Case 2

$$\overline{N}_3(r,1) + \overline{N}_4(r,1) = S(r,f), \qquad \overline{N}_3(r,c) + \overline{N}_4(r,c) \neq S(r,f);$$

Case 3

$$\overline{N}_3(r,1) + \overline{N}_4(r,1) \neq S(r,f), \qquad \overline{N}_3(r,c) + \overline{N}_4(r,c) = S(r,f);$$

Case 4

$$\overline{N}_3(r,1) + \overline{N}_4(r,1) \neq S(r,f), \qquad \overline{N}_3(r,c) + \overline{N}_4(r,c) \neq S(r,f).$$

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