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COMMUTATORS OF MULTIPLIERS ON HARDY SPACES*

Pu Zhang
(*Zhejiang Sci-Tech University, China*)

Jiukun Hua
(*Zhejiang Gongshang University, China*)

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Abstract

Let T be the multiplier operator associated to a multiplier m, and $[b,T]$ be the commutator generated by T and a BMO function b. In this paper, the authors have proved that $[b,T]$ is bounded from the Hardy space $H^1(\mathbb{R}^n)$ into the weak $L^1(\mathbb{R}^n)$ space and from certain atomic Hardy space $H^1_b(\mathbb{R}^n)$ into the Lebesgue space $L^1(\mathbb{R}^n)$, when the multiplier m satisfies the conditions of Hörmander type.

Key words *multiplier, commutator, Hardy space, Hörmander condition*

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1 Introduction and Statement of Results

A bounded measurable function m defined on $\mathbb{R}^n$ is called a multiplier. The multiplier operator T associated to m is defined by

$$\big(Tf\big)^{\wedge}(x) = m(x)\hat{f}(x), \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

where $\hat{f}$ denotes the Fourier transform of f and $\mathcal{S}(\mathbb{R}^n)$ the Schwartz test function class.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index of nonnegative integers $\alpha_j (j = 1, 2, \cdots, n)$ with

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$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Denote by D^α the partial differential operators of order α as follows

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Definition 1.1[6]. Let ℓ be a positive integer and $1 \le s \le 2$. We say that the multiplier m satisfies the condition $M(s,\ell)$, and write $m \in M(s,\ell)$, if

$$\sup_{R>0} \left(R^{s|\alpha|-n} \int_{R<|\xi|<2R} |D^\alpha m(\xi)|^s \mathrm{d}\xi \right)^{1/s} < +\infty, \quad \text{for all } |\alpha| \le \ell. \tag{1.2}$$

Condition (1.2) has been known sometimes to be related to multiplier theorems and was profoundly studied by Kurtz-Wheeden in [6], see also Hörmander[5] or Stein[10] for $s = 2$. It is easy to see that $m \in M(s,\ell)$ implies $m \in M(s_1,\ell_1)$ when $s \ge s_1$ and $\ell \ge \ell_1$. The following theorem is the unweighted case of Kurtz-Wheeden's results in [6].

Theorem A[6]. *Let ℓ be a positive integer. If $1 < s \le 2$, $n/s < \ell \le n$ and $m \in M(s,\ell)$, then there exists a positive number $C > 0$ independent of f such that*

(i) $\|Tf\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$, *for* $1 < p < \infty$;

(ii) $|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \le C\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}$, *for all* $\lambda > 0$.

Let $b \in BMO(\mathbb{R}^n)$, we denote by $[b,T]f = bTf - T(bf)$ the commutator generated by T and the function b. There are a few papers on commutators of multipliers, such as [2],[3],[13], and [14]. In 1988, You[13] established boundedness properties of $[b,T]$ on $L^p(\mathbb{R}^n)$ space and got the following theorem.

Theorem B[13]. *Let $b \in$ BMO$(\mathbb{R}^n)$ and ℓ be a positive integer. If $m \in M(s,\ell)$, $1 < s \le 2$ and $n/s < \ell \le n$. Then there is a positive constant $C > 0$ such that*

$$\|[b,T]f\|_{L^p(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{L^p(\mathbb{R}^n)}, \quad \text{for } 1 < p < \infty,$$

where $\|b\|_*$ is the BMO *norm of the function b.*

On the other hand, denote by T_{CZ} the Calderón-Zygmund singular integral with homogeneous kernel and $[b, T_{CZ}]$ the commutator generated by T_{CZ} and the function b. It is well known that $[b, T_{CZ}]$ is neither of weak type $(1,1)$ nor bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, see [8] and [9] for instance, where $H^1(\mathbb{R}^n)$ stands for the Hardy space on $\mathbb{R}^n$. Instead of the weak type $(1,1)$ and the (H^1, L^1) estimates, Alvarez[1] and Peréz[9] studied boundedness properties of $[b, T_{CZ}]$ on a kind of subspaces of Hardy spaces. Recently, Chen and Hu[4] proved that $[b, T_{CZ}]$ is bounded from $H^1(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$.

Motivated by [4] and [9], we consider the same problems for commutators of multiplier operators in this paper. More precisely, we shall prove that $[b,T]$ is bounded from the Hardy space $H^1(\mathbb{R}^n)$ into the weak $L^1(\mathbb{R}^n)$ space and from a kind of atomic Hardy type space $H_b^1(\mathbb{R}^n)$

into $L^1(\mathbb{R}^n)$. We would like to remark that, in [4] and [9], under some conditions imposed on the kernels of T_{CZ}, the Calderón-Zygmund singular integral operator T_{CZ} can be realized by a multiplier. So, we believe that the commutator $[b,T]$ of the multiplier operator T is also neither of weak type $(1,1)$ nor of type (H^1,L^1). To state our results, we first recall the definition of the space $H^1_b(\mathbb{R}^n)$.

Definition 1.2[9]. A function a is called a b-atom if there is a ball $B=B(x_B,r_B)$ centered at x_B with radius r_B such that $\operatorname{supp}(a)\subset B$ and

$$\text{(i) } \|a\|_{L^\infty(\mathbb{R}^n)}\le |B|^{-1};\qquad \text{(ii) } \int a(y)\mathrm{d}y=\int a(y)b(y)\mathrm{d}y=0.$$

The space $H^1_b(\mathbb{R}^n)$ consists of all functions $f\in L^1(\mathbb{R}^n)$ which can be written as $f=\sum\limits_j \lambda_j a_j$, where a_j's are b-atoms and λ_j's are complex numbers with $\sum\limits_j|\lambda_j|<\infty$. Furthermore, we define the quasi-norm on $H^1_b(\mathbb{R}^n)$ by $\|f\|_{H^1_b(\mathbb{R}^n)}=\inf\Big\{\sum\limits_j|\lambda_j|\Big\}$, where the infimum is taken over all the above decompositions of f.

Theorem 1.1. *Let $b\in \mathrm{BMO}(\mathbb{R}^n)$, T be defined as in* (1.1) *and ℓ a positive integer. If $m\in M(s,\ell)$, $1<s\le 2$ and $n/s<\ell\le n$, then there is a positive constant $C>0$, independent of b such that for any $\lambda>0$ and any $f\in H^1(\mathbb{R}^n)$,*

$$\Big|\big\{x\in\mathbb{R}^n:|[b,T]f(x)|>\lambda\big\}\Big|\le C\lambda^{-1}\|b\|_*\|f\|_{H^1(\mathbb{R}^n)}.$$

Theorem 1.2. *Let $b\in \mathrm{BMO}(\mathbb{R}^n)$, T be defined as in* (1.1) *and ℓ a positive integer. If $m\in M(s,\ell)$, $1<s\le 2$ and $n/s<\ell\le n$, then $[b,T]$ is bounded from $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, there is a positive constant $C>0$, independent of f and b such that*

$$\|[b,T]f\|_{L^1(\mathbb{R}^n)}\le C\|b\|_*\|f\|_{H^1_b(\mathbb{R}^n)}.$$

Our paper is arranged as follows. In §2, we give some preliminary materials and prove Theorem 1.1. In the last section, we prove Theorem 1.2.

2 Proof of Theorem 1.1

As in [6] and [12], we denote by $K(x)$ the kernel that corresponds to the inverse Fourier transform of m in the sense of distribution, this says, $K(x)=m^\vee(x)$ in the sense of distribution. Then for $f\in\mathcal{S}(\mathbb{R}^n)$ we have $Tf(x)=(K*f)(x)$, where $g^\vee$ denotes the inverse Fourier transform of the function g. To prove Theorem 1.1, it is natural to consider how the behavior of K reflects the fact that $m\in M(s,\ell)$. We have the following estimates for the kernel K from Strömberg and Torkinsky[12].

Denote by $|x| \sim t$ the fact that the value of x lies in the annulus $\{x \in \mathbb{R}^n : at < |x| < bt\}$, where $0 < a \le 1 < b < \infty$ are constants specified in each instance.

Lemma 2.1([12],p.152). *Suppose $m \in M(s,\ell)$, $1 \le s \le 2$. Given $1 \le \tilde{s} < \infty$, let $r \ge 1$ be such that $1/r = \max\{1/s, 1-1/\tilde{s}\}$ and $\tilde{\ell} = \ell - n/r$. Then $K \in \tilde{M}(\tilde{s}, \tilde{\ell})$, this says that K satisfies*

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}} K(x)|^{\tilde{s}} \mathrm{d}x\right)^{1/\tilde{s}} \le C R^{n/\tilde{s}-n-|\tilde{\alpha}|}, \tag{2.1}$$

for all $R > 0$ and all multi-indices $|\tilde{\alpha}| < \tilde{\ell}$. And in addition, if μ denotes the largest integer strictly less than $\tilde{\ell}$ and $\tilde{\ell} = \mu + \nu$,

$$\left(\int_{|x|\sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x-z)|^{\tilde{s}} \mathrm{d}x\right)^{1/\tilde{s}} \le \begin{cases} C\left(\frac{|z|}{R}\right)^{\nu} R^{n/\tilde{s}-n-\mu}, & \text{if } 0 < \nu < 1, \\ C\left(\frac{|z|}{R}\right)\left(\log \frac{R}{|z|}\right) R^{n/\tilde{s}-n-\mu}, & \text{if } \nu = 1, \end{cases} \tag{2.2}$$

for all $|z| < R/2$, $R > 0$, and all multi-indices $\tilde{\alpha}$ with $|\tilde{\alpha}| = \mu$.

To prove our result, we also need the atomic decomposition theorem of the Hardy space, which can be found in [7] and [11].

Definition 2.1. Let $0 < p \le 1$. A function a defined in $\mathbb{R}^n$ is called a $(p,2)$-atom, if

1) supp $a \subset B(x_0, r) \equiv \{x \in \mathbb{R}^n : |x - x_0| < r\}$, for some $r > 0$;

2) $\|a\|_{L^2(\mathbb{R}^n)} \le |B(x_0,r)|^{1/2-1/p}$;

3) $\int_{\mathbb{R}^n} a(x) x^{\gamma} \mathrm{d}x = 0$, for all multi-indices γ with $0 \le |\gamma| \le [n(1/p-1)]$.

Lemma 2.2. *Let $0 < p \le 1$. A distribution f belongs to $H^p(\mathbb{R}^n)$ if and only if $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the sense of distribution, where a_j's are $(p,2)$-atoms and $\lambda_j \in \mathbb{C}$ with $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Furthermore,*

$$\|f\|_{H^p(\mathbb{R}^n)} \sim \inf\left\{\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\Big)^{1/p}\right\},$$

where the infimum is taken over all the above atomic decompositions of f.

In the sequel, for a ball $B \subset \mathbb{R}^n$ and a locally integrable function b, we denote by $b_B = \frac{1}{|B|}\int_B b(x)dx$. Now, we are at the position to prove Theorem 1.1.

Proof of Theorem 1.1. For any given $f \in H^1(\mathbb{R}^n)$, by Lemma 2.2, $f = \sum_j \lambda_j a_j$, where a_j's are $(1,2)$-atoms with $\mathrm{supp}(a_j) \subset B_j \equiv B(x_j, r_j)$ and furthermore, $\|f\|_{H^1(\mathbb{R}^n)} \sim \inf\left\{\sum_j |\lambda_j|\right\}$. Denote by $b_j = b_{B_j}$, then

$$\begin{aligned}[b,T]f(x) &= \sum_{j=-\infty}^{\infty}\lambda_j(b(x)-b_j)Ta_j(x)\chi_{2B_j}(x)\\ &\quad+\sum_{j=-\infty}^{\infty}\lambda_j(b(x)-b_j)Ta_j(x)\chi_{(2B_j)^C}(x)\\ &\quad-T\Big(\sum_{j=-\infty}^{\infty}\lambda_j(b-b_j)a_j\Big)(x)\\ &:=I_1(x)+I_2(x)+I_3(x).\end{aligned}$$

To prove our theorem, it suffices to establish the following estimates,

$$\big|\{x\in\mathbb{R}^n:|I_i(x)|>\lambda/3\}\big|\le C\lambda^{-1}\|b\|_*\|f\|_{H^1(\mathbb{R}^n)},\quad i=1,2,3. \tag{2.3}$$

Since $b\in \mathrm{BMO}(\mathbb{R}^n)$, then for any nonnegative integer k and any ball B, there holds $|b_{2^{k+1}B}-b_B|\le C(k+1)\|b\|_*$ (see [11], p.141), and then

$$\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b(x)-b_B|^p\mathrm{d}x\right)^{1/p}\le C(k+1)\|b\|_*,\quad \text{for all } 1\le p<\infty. \tag{2.4}$$

Since $n/s<\ell\le n$ and $1<s\le 2$, by Theorem A, T is bounded from $L^2(\mathbb{R}^n)$ into itself. Applying the Hölder inequality, (2.4) and the size condition of a_j, we obtain

$$\begin{aligned}\|(b-b_j)T(a_j)\chi_{2B_j}\|_{L^1(\mathbb{R}^n)} &\le C\Big(\int_{2B_j}|b(x)-b_j|^2\mathrm{d}x\Big)^{1/2}\|a_j\|_{L^2(\mathbb{R}^n)}\\ &\le C\Big(\frac{1}{|2B_j|}\int_{2B_j}|b(x)-b_j|^2\mathrm{d}x\Big)^{1/2}\le C\|b\|_*.\end{aligned}$$

Consequently,

$$\begin{aligned}|\{x\in\mathbb{R}^n:|I_1(x)|>\lambda/3\}| &\le 3\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|\|(b-b_j)T(a_j)\chi_{2B_j}\|_{L^1(\mathbb{R}^n)}\\ &\le C\lambda^{-1}\|b\|_*\sum_{j=-\infty}^{\infty}|\lambda_j|\le C\lambda^{-1}\|b\|_*\|f\|_{H^1(\mathbb{R}^n)}.\end{aligned}$$

By Theorem A, T is of weak type (1,1). Applying the Hölder's inequality and the size condition of a_j, we have

$$\begin{aligned}|\{x\in\mathbb{R}^n:|I_3(x)|>\lambda/3\}| &\le C\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|\|(b-b_j)a_j\|_{L^1(\mathbb{R}^n)}\\ &\le C\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|\Big(\int_{2B_j}|b(x)-b_j|^2dx\Big)^{1/2}\|a_j\|_{L^2(\mathbb{R}^n)}\\ &\le C\|b\|_*\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|\le C\lambda^{-1}\|b\|_*\|f\|_{H^1(\mathbb{R}^n)}.\end{aligned}$$

To check (2.3) for $i = 2$, firstly, we consider the case of $0 < \ell - n/s \leq 1$. Set $\tilde{s} = s$ and $\tilde{\ell} = \ell - n \cdot \max\{1/s, 1/\tilde{s}'\}$. Noting that $1 < s \leq 2 \leq s' = \tilde{s}'$, we have $\tilde{\ell} = \ell - n/s$. This fact and Lemma 2.1 imply $K \in \tilde{M}(s, \tilde{\ell})$. Denote by $K_j(x, y) = K(x - y) - K(x - x_j)$ for simplicity.

If $0 < \tilde{\ell} < 1$, by (2.2), then for any $y \in B_j$ and any positive integer k,

$$\left(\int_{2^{k+1}B_j \setminus 2^k B_j} |K_j(x,y)|^s \mathrm{d}x\right)^{1/s} \leq C2^{-(k+1)(\ell - n/s)}(2^{k+1}r_j)^{-n/s'}. \tag{2.5}$$

If $\tilde{\ell} = 1$, we choose a real number ε with $0 < \varepsilon < 1$ such that $t^{1-\varepsilon}\log(1/t) \leq C$ when $0 < t \leq 1/2$. Then by (2.2), for any $y \in B_j$ and any positive integer k, we can obtain

$$\begin{aligned}\left(\int_{2^{k+1}B_j \setminus 2^k B_j} |K_j(x,y)|^s \mathrm{d}x\right)^{1/s} &\leq C\left(\frac{|y - x_j|}{2^{k+1}r_j}\right)\left(\log \frac{2^{k+1}r_j}{|y - x_j|}\right)(2^{k+1}r_j)^{n/s-n} \\ &\leq C\left(\frac{|y - x_j|}{2^{k+1}r_j}\right)^{\varepsilon}(2^{k+1}r_j)^{n/s-n} \\ &\leq C2^{-(k+1)\varepsilon}(2^{k+1}r_j)^{-n/s'}.\end{aligned} \tag{2.6}$$

Noting that $\int_{B_j} |a_j(y)|\mathrm{d}y \leq |B_j|^{1/2}\|a_j\|_{L^2(\mathbb{R}^n)} \leq 1$. By the cancellation condition of a_j, the Hölder inequality, (2.4), (2.5) and (2.6), we have

$$\begin{aligned}\left\|(b - b_j)(Ta_j)\chi_{(2B_j)^C}\right\|_{L^1(\mathbb{R}^n)} &= \int_{(2B_j)^C} |b(x) - b_j|\left|\int_{B_j} K_j(x,y)a_j(y)\mathrm{d}y\right|\mathrm{d}x \\ &\leq C\int_{B_j} |a_j(y)|\sum_{k=1}^{\infty}\int_{2^{k+1}B_j\setminus 2^k B_j} |b(x) - b_j||K_j(x,y)|\mathrm{d}x\mathrm{d}y \\ &\leq C\int_{B_j} |a_j(y)|\sum_{k=1}^{\infty}\left\{\left(\int_{2^{k+1}B_j} |b(x) - b_j|^{s'}\mathrm{d}x\right)^{1/s'}\right. \\ &\quad \left.\times\left(\int_{2^{k+1}B_j\setminus 2^k B_j} |K_j(x,y)|^s\mathrm{d}x\right)^{1/s}\right\}\mathrm{d}y \\ &\leq C\|b\|_*\int_{B_j} |a_j(y)|\mathrm{d}y\sum_{k=1}^{\infty}(k+1)2^{-\varepsilon(k+1)} \leq C\|b\|_*,\end{aligned}$$

where ε is the same as in (2.6) when $\tilde{\ell} = 1$ and $\varepsilon = \ell - n/s$ when $0 < \tilde{\ell} < 1$ as in (2.5).

Then

$$\begin{aligned}\left|\{x \in \mathbb{R}^n : |I_2(x)| > \lambda/3\}\right| &\leq C\lambda^{-1}\left\|\sum_{j=-\infty}^{\infty} \lambda_j(b - b_j)(Ta_j)\chi_{(2B_j)^C}\right\|_{L^1(\mathbb{R}^n)} \\ &\leq C\lambda^{-1}\|b\|_*\sum_{j=-\infty}^{\infty}|\lambda_j| \leq C\lambda^{-1}\|b\|_*\|f\|_{H^1(\mathbb{R}^n)}.\end{aligned}$$

This shows that (2.3) is true for $i = 2$ and $0 < \ell - n/s \leq 1$.

Next, we consider the case of $\ell - n/s > 1$. Set $\ell_1 = n/s + 1$ if n/s is an integer, then $n/s < \ell_1 \leq \ell$ and $\ell_1 - n/s = 1$. If n/s is not an integer, we choose $\ell_1 = \ell - [\ell - \frac{n}{s}]$, where $[\ell - \dfrac{n}{s}]$

is the integer part of $\ell - \frac{n}{s}$, says, $[\ell - \frac{n}{s}]$ is the greatest integer which is less than or equals to $\ell - \frac{n}{s}$, and it easy to see that $n/s < \ell_1 < \ell$ and $\ell_1 - n/s < 1$. Noting that $m \in (s, \ell)$ implies $m \in M(s, \ell_1)$, instead of considering $m \in M(s, \ell)$, we consider $m \in M(s, \ell_1)$ in the estimates of (2.3) for $i = 2$. So, (2.3) is true when $i = 2$ and $n/s < \ell \leq n$.

Thus, the proof of Theorem 1.1 is completed.

3 Proof of Theorem 1.2

Proof of Theorem 1.2. We need to prove that there is a constant $C > 0$ independent of b, such that for any $f \in H_b^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} |[b, T]f(x)|\mathrm{d}x \leq C\|b\|_*\|f\|_{H_b^1(\mathbb{R}^n)}$.

To this end, by a standard argument, it suffices to prove that there exists a positive constant C such that for each b-atom a, the following inequality holds

$$\int_{\mathbb{R}^n} |[b, T]a(x)|\mathrm{d}x \leq C\|b\|_*. \tag{3.1}$$

Fix a b-atom a, by the homogeneity, we can assume that a is supported in a ball $B \equiv B(0, d)$ centered at the origin with radius d. Then,

$$\int_{\mathbb{R}^n} \big|[b, T]a(x)\big|\mathrm{d}x \leq \left(\int_{2B} + \int_{(2B)^C}\right)\big|[b, T]a(x)\big|\mathrm{d}x := J_1 + J_2. \tag{3.2}$$

By Theorem B, $[b, T]$ is bounded from $L^2(\mathbb{R}^n)$ into itself. Applying the Hölder's inequality and the size condition of a, we get

$$\begin{aligned} J_1 &\leq \left(\int_{2B} \big|[b, T]a(x)\big|^2\mathrm{d}x\right)^{1/2} |2B|^{1/2} \leq C\|b\|_*\|a\|_{L^2(\mathbb{R}^n)}|B|^{1/2} \\ &\leq C\|b\|_*\|a\|_{L^\infty(\mathbb{R}^n)}|B| \leq C\|b\|_*. \end{aligned} \tag{3.3}$$

Now, let us consider the second term J_2. By the cancellation condition of a, we have

$$\begin{aligned} J_2 &\leq \int_{(2B)^C}\int_B |K(x-y) - K(x)||b(x) - b_B||a(y)|\mathrm{d}y\mathrm{d}x \\ &\quad + \int_{(2B)^C}\int_B |K(x-y) - K(x)||b(y) - b_B||a(y)|\mathrm{d}y\mathrm{d}x \\ &:= J_2' + J_2''. \end{aligned} \tag{3.4}$$

Without loss of generality, we assume $0 < \ell - n/s \leq 1$. Set $\tilde{s} = s$ and $\tilde{\ell} = \ell - n/s$. From Lemma 2.1 we have $K \in \tilde{M}(s, \tilde{\ell})$. Similar to (2.5) and (2.6), there is a real number ε with $0 < \varepsilon < 1$, then for any positive integer k and $y \in B$,

$$\left(\int_{2^{k+1}B\backslash 2^k B} |K(x-y) - K(x)|^s\mathrm{d}x\right)^{1/s} \leq C2^{-\varepsilon(k+1)}(2^{k+1}d)^{-n/s'}. \tag{3.5}$$

By the size condition of a and using the Hölder's inequality, (2.4) and (3.5), we have

$$\begin{aligned} J_2' &\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\int_B |K(x-y)-K(x)||b(x)-b_B||a(y)|\mathrm{d}y\mathrm{d}x \\ &\le \int_B |a(y)|\sum_{k=1}^{\infty}\left\{\left(\int_{2^{k+1}B}|b(x)-b_B|^{s'}\mathrm{d}x\right)^{1/s'}\right. \\ &\quad\times\left.\left(\int_{2^{k+1}B\setminus 2^kB}|K(x-y)-K(x)|^s\mathrm{d}x\right)^{1/s}\right\}\mathrm{d}y \\ &\le C\|b\|_*|B|^{-1}\int_B\sum_{k=1}^{\infty}(k+1)2^{-\varepsilon(k+1)}dy \le C\|b\|_*. \end{aligned} \tag{3.6}$$

Similarly, by the size condition of a, (2.4) and (3.5), we can obtain

$$\begin{aligned} J_2'' &\le |B|^{-1}\sum_{k=1}^{\infty}\int_B |b(y)-b_B|\int_{2^{k+1}B\setminus 2^kB}|K(x-y)-K(x)|\mathrm{d}x\mathrm{d}y \\ &\le C|B|^{-1}\sum_{k=1}^{\infty}\int_B |b(y)-b_B||2^{k+1}B|^{1/s'}\left(\int_{2^{k+1}B\setminus 2^kB}|K(x-y)-K(x)|^s\mathrm{d}x\right)^{1/s}\mathrm{d}y \\ &\le C\sum_{k=1}^{\infty}2^{-\varepsilon(k+1)}|B|^{-1}\int_B |b(y)-b_B|dy \le C\|b\|_*. \end{aligned}$$

This together with (3.4) and (3.6) we have $J_2 \le C\|b\|_*$. And then, by (3.2) and (3.3), we obtain (3.1). So the proof of Theorem 1.2 is complete.

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Zhang Pu

Institute of Mathematics

Zhejiang Sci-Tech University

Hangzhou 310018

P. R. China

Email: puzhang@sohu.com

Hua Jiukun

Department of Computational Science

Zhejiang Gongshang University

Hangzhou 310035

P. R. China