FUNCTIONAL INEQUALITIES AND BEST APPROXIMATION

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Abstract

We investigate the relationship between best approximations by elements of closed convex cones and the estimation of functionals on an inner product space $(X, \langle \cdot, \cdot \rangle)$ in terms of the inner product on X.

1 Introduction

In some variational problems with inequality constraints one often encounters functionals defined on closed convex cones in an incomplete inner product space $(X, \langle \cdot, \cdot \rangle)$. It then becomes of paramount importance to be able to estimate such functionals in terms of the inner product on X. The first author[3] developed somewhat weaker estimation for such functionals. In this paper we exploit the techniques of Approximation Theory to estimate these functionals in terms of the inner product on X.

This section establishes the notation and terminology that is used throughout. Section 2 investigates the relationship between best approximations by elements of closed convex cones and the estimation of functionals on an inner product space $(X, \langle \cdot, \cdot \rangle)$. Unless otherwise stated, X will always be a real linear space.

Defintion 1. 1. A nonempty subset K of X is said to be a convex cone in X if the following conditions hold:

(K1) $x+y \in K$ for all $x, y \in K$;

(K2) $\alpha x \in K$ for all $x \in X$ and all $\alpha \in \mathbf{R}, \alpha \geqslant 0$.

Defintion 1. 2[4] Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over the real number field

R. An element $x\in X$ is called suborthogonal over an element $y\in X$ if $\langle y,x\rangle\leqslant 0$. We use the notation xSy to mean that x is suborthogonal over y.

If A is a nonempty subset of X, we shall denote by A^S the set of all elements of X that are suborthogonal over all elements of A. That is,

$$A^S=\{y\in X\mid\langle y,x\rangle\leqslant 0 \text{ for all } x\in A\}.$$

Clearly, for every nonempty subset A of X, A^S is a closed convex cone in X.

Defintion **1.3** Let K be a closed convex subset of an inner product space $(X,\langle\cdot,\cdot\rangle)$. For a given $x\in X$, the best approximation to x from K is an element $P_K(x)\in K$ such that

$$\|x-P_K(x)\|=\inf\{\|x-y\|\mid y\in K\}.$$

(It is well-known that if $(X,\langle\cdot,\cdot\rangle)$ is actually a Hilbert space then each $x\in X\backslash K$ has a unique best approximation in K).

Defintion **1.4**

(1) A function $f:X\to\mathbf{R}$ on a linear space X is called sublinear if

$$f(\alpha x+\beta y)\leqslant\alpha f(x)+\beta f(y) \text{ for all } x,y\in X \text{ and all } \alpha\geqslant 0,\beta\geqslant 0.$$

(2) An extended real-valued function F on a real linear space X is called convex if

$$F(\alpha x+\beta y)\leqslant\alpha F(x)+\beta F(y)$$

for all $x,y\in X$ and all $\alpha\geqslant 0,\beta\geqslant 0$ such that $\alpha+\beta=1$.

2 Best Approximations and Estimation of Functionals

In this section we investigate the relationship between best approximations by elements of closed convex cones and the estimation of functionals on an inner product space $(X,\langle\cdot,\cdot\rangle)$ in terms of the inner product on X. We need the following powerful result.

Theorem 2.1 *Let $(X,\langle\cdot,\cdot\rangle)$ be an inner product space over the real number field* **R** *and C a closed convex subset of X with $X\neq C$. If $x_0\in X\backslash C$ and $g_0\in C$, then the following statements are equivalent.*

(1) $g_0=P_C(x_0)$;

(2) $x_0-g_0\in(C-g_0)^S$;

(3) *We have that*

$$\inf_{g\in C}\langle g-x_0,g_0-x_0\rangle=\|g_0-x_0\|^2. \tag{2.1.1}$$

Proof (1)⇒(2). This implication is a classical result (See, for example, [2,Corollary 3.1]). We include its proof for the sake of completeness. Assume that g_0 is the best approximation to x_0 in C. Then, for any $g\in C$ and $0<\lambda<1$, $\lambda g+(1-\lambda)g_0\in C$ since C is convex. Thus,

$$\| x_0 - g_0 \|^2 \leqslant \| x_0 - [\lambda g + (1-\lambda) g_0] \|^2 = \| (x_0 - g_0) - \lambda (g - g_0) \|^2$$
$$= \langle (x_0 - g_0) - \lambda (g - g_0), (x_0 - g_0) - \lambda (g - g_0) \rangle$$
$$= \| x_0 - g_0 \|^2 - 2\lambda \langle x_0 - g_0, g - g_0 \rangle + \lambda^2 \| g - g_0 \|^2$$
$$\Rightarrow 2\lambda \langle x_0 - g_0, g - g_0 \rangle \leqslant \lambda^2 \| g - g_0 \|^2$$
$$\Rightarrow \langle x_0 - g_0, g - g_0 \rangle \leqslant \frac{\lambda}{2} \| g - g_0 \|^2.$$

As $\lambda \to 0$, $\langle x_0 - g_0, g - g_0 \rangle \leqslant 0$.

That is $x_0 - g_0 \in (C - g_0)^S$.

(2)⇒(3). By (2) we have that for all $g \in C$,

$$0 \leqslant \langle x_0 - g_0, g_0 - g \rangle = \langle x_0 - g_0, g_0 - x_0 + x_0 - g \rangle$$
$$= \langle x_0 - g_0, x_0 - g \rangle - \langle x_0 - g_0, x_0 - g_0 \rangle$$
$$= \langle x_0 - g_0, x_0 - g \rangle - \| x_0 - g_0 \|^2,$$

whence

$$\| x_0 - g_0 \|^2 \leqslant \langle x_0 - g_0, x_0 - g \rangle \text{ for all } g \in C.$$

Thus,

$$\| x_0 - g_0 \|^2 \leqslant \inf_{g \in C} \langle x_0 - g_0, x_0 - g \rangle$$
$$\leqslant \langle x_0 - g_0, x_0 - g_0 \rangle = \| x_0 - g_0 \|^2.$$

Therefore

$$\| x_0 - g_0 \|^2 = \inf_{g \in C} \langle x_0 - g_0, x_0 - g \rangle.$$

(3)⇒(1). If (3) holds, then for all $g \in C$

$$\| x_0 - g_0 \|^2 \leqslant \langle x_0 - g_0, x_0 - g \rangle \leqslant \| x_0 - g_0 \| \, \| x_0 - g \|,$$

where the second inequality is the Cauchy-Bunyakowsky-Schwarz Inequality. Thus,

$$\| x_0 - g_0 \| \leqslant \| x_0 - g \| \text{ for all } g \in C.$$

That is $g_0 = P_C(x_0)$.

In the case of closed convex cones the above theorem assumes the following sharper form.

Theorem 2.2 *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over the real number field* **R**, *K a closed convex cone in X with $X \neq K$. If $x_0 \in X \backslash K$ and $g_0 \in K$, then the following statements are equivalent.*

(1) $g_0 = P_K(x_0)$;

(2) $x_0 - g_0 \in K^S \cap g_0^{\perp}$, *where* $g_0^{\perp} = \{x \in X \mid \langle x, g_0 \rangle = 0\}$;

(3) $x_0 - g_0 \in K^S$ *and* $\langle x_0, g_0 \rangle = \| g_0 \|^2$;

Proof The equivalence of (2) and (3) is clear.

To prove the equivalence of (1) and (2) it suffices, by Theorem 2.1, to show that $(K - g_0)^S = K^S \cap g_0^{\perp}$. To that end, let $h \in (K - g_0)^S$. Then

$$\langle h, g - g_0 \rangle \leqslant 0 \text{ for all } g \in K. \tag{2.2.1}$$

Thus, for any $k \in K$,

$$0 \geqslant \langle h, (k + g_0) - g_0 \rangle = \langle h, k \rangle.$$

That is, $h \in K^S$. Taking $g = 0$ in (2.2.1) we get that $\langle h, -g_0 \rangle \leqslant 0$ and, with $g = 2g_0$ we have that $\langle h, g_0 \rangle \leqslant 0$. Thus, $\langle h, g_0 \rangle = 0$, i.e., $h \in g_0^{\perp}$, and so $h \in K^S \cap g_0^{\perp}$, which shows that $(K - g_0)^S \subset K^S \cap g_0^{\perp}$.

If $h \in K^S \cap g_0^{\perp}$, then $h \in K^S$ and $h \in g_0^{\perp}$. That is, $\langle h, g \rangle \leqslant 0$ for all $g \in K$ and $\langle h, g_0 \rangle = 0$. This implies that

$$\langle h, g - g_0 \rangle \leqslant 0 \text{ for all } g \in K,$$

i.e., $h \in (K - g_0)^S$, and so $K^S \cap g_0^{\perp} \subset (K - g_0)^S$.

Remark **2.3** Note that the above theorem improves Lemma 2.1 [3] in which additional conditions were imposed.

Remark **2.4** If K is a linear subspace then $K^S = K^{\perp}$ and so condition (2) of the above theorem reduces to the classical condition that g_0 is the best approximation to x_0 in K if and only if $x_0 - g_0 \perp K$.

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over the real number field $\mathbf{R}$ and let $F: X \to \mathbf{R}$ be a continuous convex mapping on X. Denote by

$$F^{\leqslant}(r) := \{x \in X \mid F(x) \leqslant r\}, \quad r \in \mathbf{R}.$$

Assume that r is such that $F^{\leqslant} \neq \Phi$.

The following theorem characterises best approximations by elements of the "level set" $F^{\leqslant}(r)$. This result can also be viewed as an estimation theorem for a continuous convex mapping defined on an inner product space in terms of the inner product which generates the norm $\|\cdot\|$.

Theorem 2.5 *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over the real number field $\mathbf{R}$ and let $F: X \to \mathbf{R}$ be a continuous convex mapping on X. Let $x_0 \in X \backslash F^{\leqslant}(r)$ and $g_0 \in F^{\leqslant}(r)$. Then the following statements are equivalent.*

(1) $g_0 = P_{F^{\leqslant}(r)}(x_0)$;

(2) *One has the estimation*

$$F(x) \geqslant r + \frac{F(x_0) - r}{\|x_0 - g_0\|^2} \langle x - g_0, x_0 - g_0 \rangle \quad \text{for all } x \in F^{\leqslant}(r), \tag{2.5.1}$$

or equivalently,

$$F(x) \geqslant F(g_0) + \frac{F(x_0) - r}{\|x_0 - g_0\|^2} \langle x - g_0, x_0 - g_0 \rangle \text{ for all } x \in F^{\leqslant}(r). \tag{2.5.1$'$}$$

Proof (1)$\Rightarrow$(2): Assume that $g_0 = P_{F^{\leqslant}(r)}(x_0)$. Observe that $F(x_0) > r$ since $x_0 \in X\backslash$

$F^{\leqslant}(r)$. Let $x\in F^{\leqslant}(r)$. Then $F(x)\leqslant r$. Set $\alpha=F(x_0)-r$ and $\beta=r-F(x)$. Then $\alpha>0,\beta\geqslant 0$ and $0<\alpha+\beta=F(x_0)-F(x)$. Consider the element $u=\frac{\alpha x+\beta x_0}{\alpha+\beta}$. By convexity of F we have that

$$F(u)=F\left(\frac{\alpha x+\beta x_0}{\alpha+\beta}\right)\leqslant\frac{\alpha F(x)+\beta F(x_0)}{\alpha+\beta}$$
$$=\frac{(F(x_0)-r)F(x)+(r-F(x))F(x_0)}{F(x_0)-F(x)}=r.$$

That is, $u\in F^{\leqslant}(r)$. As $g_0=P_{F^{\leqslant}(r)}(x_0)$, we have by Theorem 2. 1 that $x_0-g_0\in(F^{\leqslant}(r)-g_0)^S$, i. e., $\langle g-g_0,x_0-g_0\rangle\leqslant 0$ for all $g\in F^{\leqslant}(r)$. In particular, $\langle u-g_0,x_0-g_0\rangle\leqslant 0$. That is,

$$0\geqslant\langle u-g_0,x_0-g_0\rangle=\langle\frac{\alpha x+\beta x_0}{\alpha+\beta}-g_0,x_0-g_0\rangle$$
$$=\frac{1}{\alpha+\beta}\langle\alpha x+\beta x_0-(\alpha+\beta)g_0,x_0-g_0\rangle$$
$$=\frac{1}{\alpha+\beta}\langle\alpha(x-g_0)+\beta(x_0-g_0),x_0-g_0\rangle$$
$$=\frac{\alpha}{\alpha+\beta}\langle x-g_0,x_0-g_0\rangle+\frac{\beta}{\alpha+\beta}\langle x_0-g_0,x_0-g_0\rangle$$
$$=\frac{(F(x_0)-r)}{F(x_0)-F(x)}\langle x-g_0,x_0-g_0\rangle+\frac{(r-F(x))}{F(x_0)-F(x)}\|x_0-g_0\|^2.$$

Thus

$$F(x)\geqslant\frac{(F(x_0)-r)}{\|x_0-g_0\|^2}\langle x-g_0,x_0-g_0\rangle+r\qquad\text{for all } x\in F^{\leqslant}(r),$$

which verifies (2. 5. 1). Since (2. 5. 1) holds for all $x\in F^{\leqslant}(r)$, we have that $F(g_0)\geqslant r$. But $F(g_0)\leqslant r$ since $g_0\in F^{\leqslant}(r)$. Thus $F(g_0)=r$ and so

$$F(x)\geqslant\frac{(F(x_0)-r)}{\|x_0-g_0\|^2}\langle x-g_0,x_0-g_0\rangle+F(g_0)\qquad\text{for all } x\in F^{\leqslant}(r),$$

which verifies (2. 5. 1)$'$.

(2)$\Rightarrow$(1): Assume that (2. 5. 1) holds. Then for all $x\in F^{\leqslant}(r)$,

$$0\geqslant F(x)-r\geqslant\frac{(F(x_0)-r)}{\|x_0-g_0\|^2}\langle x-g_0,x_0-g_0\rangle.$$

Since $F(x_0)-r>0$, we have that

$$\langle x-g_0,x_0-g_0\rangle\leqslant 0\text{ for all } x\in F^{\leqslant}(r).$$

That is, $x_0-g_0\in(F^{\leqslant}(r)-g_0)^S$, whence $g_0=P_{F^{\leqslant}(r)}(x_0)$.

Corollary 2. 6 *Let $p:X\to\mathbf{R}$ be a continuous sublinear functional on the inner product space $(X,\langle\cdot,\cdot\rangle)$. Put $K(p):=\{x\in X\mid p(x)\leqslant 0\}$. Let $x_0\in X\backslash K(p)$ and $g_0\in K(p)$.*

Then the following statements are equivalent.

(1) $g_0=P_{K(p)}(x_0)$;

(2) *One has the estimation*

$$p(x)\geqslant\frac{p(x_0)}{\|x_0-g_0\|^2}\langle x-g_0,x_0-g_0\rangle \text{ for all } x\in K(p), \tag{2.6.1}$$

or equivalently,

$$p(x)\geqslant p(g_0)+\frac{p(x_0)}{\|x_0-g_0\|^2}\langle x-g_0,x_0-g_0\rangle \text{ for all } x\in K(p). \tag{2.6.1$'$}$$

The proof follows from the above theorem by taking F=p *and* r=0.

Remark 2.7 The above corollary improves Theorem 2.4 [3].

Theorem 2.8 *Let* $f:X\to\mathbf{R}$ *be a continuous linear functional on the inner product space* $(X,\langle\cdot,\cdot\rangle)$, $K=\ker(f):=\{x\in X\mid f(x)=0\}$. *Let* $x_0\in X\backslash K$ *and* $g_0\in K$. *Then the following statements are equivalent.*

(1) $g_0=P_K(x_0)$;

(2) *One has the representation*

$$f(x)=\frac{f(x_0)}{\|x_0-g_0\|^2}\langle x,x_0-g_0\rangle \textit{ for all } x\in X. \tag{2.8.1}$$

Moreover

$$\|f\|=\frac{|f(x_0)|}{\|x_0-g_0\|}.$$

Proof (1)$\Rightarrow$(2): Assume that $g_0=P_K(x_0)$. Then $x_0-g_0\perp K$ since K is a closed subspace of X. Now, for each $x\in X$, the element $f(x)x_0-f(x_0)x$ belongs to K, and so

$$\langle f(x)x_0-f(x_0)x,x_0-g_0\rangle=0 \text{ for all } x\in X.$$

Thus,

$$f(x)\langle x_0,x_0-g_0\rangle=f(x_0)\langle x,x_0-g_0\rangle.$$

Since $\langle g_0,x_0-g_0\rangle=0$, we have that

$$f(x)\langle x_0-g_0,x_0-g_0\rangle=f(x_0)\langle x,x_0-g_0\rangle,$$

whence

$$f(x)=\frac{f(x_0)}{\|x_0-g_0\|^2}\langle x,x_0-g_0\rangle \text{ for all } x\in X,$$

which verifies (2.8.1).

(2)$\Rightarrow$(1): Assume that (2.8.1) holds. Then for each $x\in K$,

$$0=f(x)=\frac{f(x_0)}{\|x_0-g_0\|^2}\langle x,x_0-g_0\rangle\Rightarrow\langle x,x_0-g_0\rangle=0.$$

That is $x_0-g_0\perp K$, and so $g_0=P_K(x_0)$.

Rewrite (2.8.1) as

$$f(x) = \langle x, z\rangle \text{ where } z = \frac{f(x_0)(x_0 - g_0)}{\| x_0 - g_0 \|^2}.$$

Then by the Cauchy-Bunyakowsky-Schwarz Inequality $|f(x)| = |\langle x, z\rangle| \leqslant \|x\| \ \|z\|$. Thus $\|f\| \leqslant \|z\|$. Since $f(z) = \langle z, z\rangle = \|z\|^2$, we have that $\|f\| \geqslant \|z\|$. Hence

$$\|f\| = \|z\| = \frac{|f(x_0)|}{\|x_0 - g_0\|}.$$

Theorem 2.9 *Let $(X, \langle \cdot , \cdot \rangle)$ be an inner product space over the real number field* $\mathbf{R}$ *and let $F: X \to \mathbf{R}$ be a continuous convex mapping on X. Let $x_0 \in X \backslash F^{<}(r)$ and $g_0 \in F^{<}(r)$ such that $g_0 = P_{F^{<}(r)}(x_0)$, $-x_0 \in F^{<}(r)$, and $\|x_0\| > \|g_0\|$. Then we have the estimation*

$$F(x) \geqslant \begin{cases} r + \dfrac{F(x_0) - r}{\|x_0 - g_0\|^2}\langle x - g_0, x_0 - g_0\rangle, & \text{for all } x \in F^{<}(r), \\ r + \dfrac{r - F(-x_0)}{\|x_0\|^2 - \|g_0\|^2}\langle x - g_0, x_0 - g_0\rangle, & \text{for all } x \in X\backslash F^{<}(r). \end{cases}$$

Proof The first part is restatement of Theorem 2.5.

If $x \in X\backslash F^{<}(r)$, then $F(x) > r$. Also, $-x_0 \in F^{<}(r)$ implies that $F(-x_0) \leqslant r$. Let $\alpha = F(x) - r$ and $\beta = r - F(-x_0)$. Then $\alpha > 0, \beta \geqslant 0$ and $0 < \alpha + \beta = F(x) - F(-x_0)$. Consider the element $w = \dfrac{\alpha(-x_0) + \beta x}{\alpha + \beta}$. By convexity of F,

$$F(w) = F\left(\frac{\alpha(-x_0) + \beta x}{\alpha + \beta}\right) \leqslant \frac{\alpha F(-x_0) + \beta F(x)}{\alpha + \beta}$$

$$= \frac{(F(x) - r)F(-x_0) + (r - F(-x_0))F(x)}{F(x) - F(-x_0)} = r.$$

That is, $w \in F^{<}(r)$. Since $g_0 = P_{F^{<}(r)}(x_0)$, $\langle w - g_0, x_0 - g_0\rangle \leqslant 0$. That is,

$$0 \geqslant \langle w - g_0, x_0 - g_0\rangle = \left\langle \frac{-\alpha x_0 + \beta x}{\alpha + \beta} - g_0, x_0 - g_0\right\rangle$$

$$= \frac{1}{\alpha + \beta}\langle -\alpha x_0 + \beta x - \alpha g_0 - \beta g_0, x_0 - g_0\rangle$$

$$= \frac{1}{\alpha + \beta}\langle -\alpha(x_0 + g_0) + \beta(x - g_0), x_0 - g_0\rangle$$

$$= \frac{1}{\alpha + \beta}[-\alpha\langle x_0 + g_0, x_0 - g_0\rangle + \beta\langle x - g_0, x_0 - g_0\rangle]$$

$$\Rightarrow 0 \geqslant -\alpha\langle x_0 + g_0, x_0 - g_0\rangle + \beta\langle x - g_0, x_0 - g_0\rangle$$

$$= -\alpha(\|x_0\|^2 - \|g_0\|^2) + \beta\langle x - g_0, x_0 - g_0\rangle$$

$$= (r - F(x))(\|x_0\|^2 - \|g_0\|^2)$$
$$+ (r - F(-x_0))\langle x - g_0, x_0 - g_0\rangle$$

$$\Rightarrow \quad F(x)(\|x_0\|^2 - \|g_0\|^2) \geqslant r(\|x_0\|^2 - \|g_0\|^2)$$
$$+ (r - F(-x_0)\langle x - g_0, x_0 - g_0\rangle,$$

whence

$$F(x) \geqslant r + \frac{r - F(-x_0)}{\|x_0\|^2 - \|g_0\|^2}\langle x - g_0, x_0 - g_0\rangle \qquad \text{for all } x \in X\backslash F^{<}(r).$$

Corollary 2.10 *Let $p:X\to\mathbb{R}$ be a continuous sublinear functional on the inner product space $(X,\langle\cdot,\cdot\rangle)$, $K(p):=\{x\in X\,|\,p(x)\leqslant 0\}$. Let $x_0\in X\backslash K(p)$ and $g_0\in K(p)$ such that $g_0=P_{K(p)}(x_0)$, $-x_0\in K(p)$, and $\|x_0\|>\|g_0\|$ then we have the estimation*

$$p(x) \geqslant \begin{cases} \dfrac{p(x_0)}{\|x_0 - g_0\|^2}\langle x - g_0, x_0 - g_0\rangle & \text{for all } x \in K(p), \\[2ex] \dfrac{-p(-x_0)}{\|x_0\|^2 - \|g_0\|^2}\langle x - g_0, x_0 - g_0\rangle, & \text{for all } x \in X\backslash K(p). \end{cases}$$

Corollary 2.11 *Let $p:X\to\mathbb{R}$ be a continuous sublinear functional on the inner product space $(X,\langle\cdot,\cdot\rangle)$, $K(p):=\{x\in X\,|\,p(x)\leqslant 0\}$, $x_0\in X\backslash K(p)$ and $g_0\in K(p)$ such that $g_0=P_{K(p)}(x_0)$ and $g_0-x_0\in K(p)$. Then we have the estimation*

$$p(x) \geqslant \begin{cases} \dfrac{p(x_0)}{\|x_0 - g_0\|^2}\langle x - g_0, x_0 - g_0\rangle, & \text{for all } x \in K(p), \\[2ex] \dfrac{-p(g_0 - x_0),}{\|x_0 - g_0\|^2}\langle x - g_0, x_0 - g_0\rangle & \text{for all } x \in X\backslash K(p). \end{cases}$$

Proof The first part is restatement of Corollary 2.6.

Recall that $g_0=P_{K(p)}(x_0)$ if and only if $x_0-g_0\in K(p)^S\cap g_0^{\perp}$. Let $x\in X\backslash K(p)$. Then $p(x)>0$. Set $w_0=x_0-g_0$ and $v=p(x)(-w_0)-p(-w_0)x$. Note that, since $g_0-x_0\in K(p)$, it follows that $p(-w_0)=p(g_0-x_0)\leqslant 0$. Furthermore, since

$$p(v) = p(p(x)(-w_0) - p(-w_0)x) \leqslant p(x)p(-w_0) - p(-w_0)p(x) = 0,$$

we have that $v\in K(p)$. Thus,

$$\begin{aligned} 0 \geqslant \langle v, x_0 - g_0\rangle &= \langle p(x)(-w_0) - p(-w_0)x, x_0 - g_0\rangle \\ &= -p(x)\langle w_0, x_0 - g_0\rangle - p(-w_0)\langle x, x_0 - g_0\rangle \end{aligned}$$

$$\Rightarrow \quad p(x)\langle x_0 - g_0, x_0 - g_0\rangle \geqslant -p(g_0 - x_0)\langle x, x_0 - g_0\rangle,$$

whence

$$p(x) \geqslant \frac{-p(g_0 - x_0)}{\|x_0 - g_0\|^2}\langle x, x_0 - g_0\rangle.$$

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