AN INEQUALITY OF SCHUR'S TYPE*

L. Milev
(University of Sofia, Bulgaria)

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Abstract

It is proved that the Chebyshev polynomial $\overline{T}_n(x)=T_n(x\cos\frac{\pi}{2n})$, *has the greatest uniform norm on* $[-1,1]$ *of its third derivative among the real polynomials of degree at most* n, *which are bounded by* 1 *in* $[-1,1]$ *and vanish in* -1 *and* 1.

1 Introduction

Let π_n be the set of all real algebraic polynomials of degree not exceeding n. We denote by $\|f\|_{C[-1,1]}:=\max\{|f(x)|:x\in[-1,1]\}$ the uniform norm of f on $[-1,1]$.

According to the inequality of V. A. Markov, the Chebyshev polynomial of first kind $T_n(x):=\cos(n\arccos x)$ has the greatest norm of k-th derivative ($k=1,\cdots,n$) among all polynomials from π_n for which $\|p\|_{C[-1,1]}\leqslant 1$.

This result was extended by Duffin and Schaeffer[4]. They showed that

$$|f^{(k)}(x+iy)|\leqslant|T_n^{(k)}(1+iy)|,\qquad k=1,\cdots,n$$

for every $x\in[-1,1]$, $y\in(-\infty,\infty)$ and every polynomial from π_n provided

$$|f(\eta_j^{(n)})|\leqslant 1,\qquad j=0,\cdots,n,$$

where $\eta_j^{(n)}=\cos\frac{j\pi}{n}$ are the extremal points of $T_n(x)$ in $[-1,1]$.

Extremal problems of Markov's type, under additional restrictions on the polynomials at the endpoints of the interval, have been investigated by Schur[8].

Later problems of Schur's type have been considered by many authors, including Bojanov[1], Bojanov and Rahman[2], Frappier[5].

Denote by $\{\xi_k^{(n)}\}_1^n$ the zeros of $T_n(x)$. Precisely, $\xi_k^{(n)}=\cos\frac{(2k-1)\pi}{2n}$, $k=1,\cdots,n$. Let $a(x):[-1,1]\to[-\xi_1^{(n)},\xi_1^{(n)}]$ be the linear transformation on $[-1,1]$ to $[-\xi_1^{(n)},\xi_1^{(n)}]$, $a(x)=\xi_1^{(n)}x$. Set $\overline{T}_n(x):=T_n(a(x))$.

Suchur[8] proved that if $f\in\pi_n$, $f(-1)=f(1)=0$ then

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$$\| f' \|_{C[-1,1]} \leqslant n\cot \frac{\pi}{2n} \| f \|_{C[-1,1]}.$$

The equality is attained if and only if $f(x)=cT_n(x)$.

Bojanov[1] considered the set P_n of all algebraic polynomials of degree n, which have n zeros in $[-1,1]$. He proved that if $f\in P_n$ and $f(-1)=f(1)=0$, then

$$\| f^{(k)} \|_{L_q[-1,1]} \leqslant \| T_n^{(k)} \|_{L_q[-1,1]} \| f \|_{C[-1,1]}$$

for each $k=1,\cdots,n, 1\leqslant q\leqslant\infty$. He also proved that these inequalities hold for $k=1$ in π_n, and stated the question: Do the inequalities hold in π_n for $k>1$?

We showed in [6] that if $f\in\pi_n, f(-1)=f(1)=0$ then

$$\| f'' \|_{C[-1,1]} \leqslant \| \overline{T}_n'' \|_{C[-1,1]} \| f \|_{C[-1,1]} = n\cot^3 \frac{\pi}{2n} \| f \|_{C[-1,1]}.$$

In this paper we prove the following sharp inequality for the third derivative

$$\| f''' \|_{C[-1,1]} \leqslant \| T_n''' \|_{C[-1,1]} \| f \|_{C[-1,1]}$$

$$= n \frac{\cos^3 \frac{\pi}{2n}}{\sin^5 \frac{\pi}{2n}} (3 - (n^2 + 2)\sin^2 \frac{\pi}{2n}) \| f \|_{C[-1,1]},$$

provided $f\in \pi_n, f(-1)=f(1)=0$. The equality is attained if and only if $f(x)=cT_n(x)$.

2 Some inequalities for the Chebyshev polynomials

Let $\eta_j^{(n)} := \cos \frac{j\pi}{n}, j=0,\cdots,n$ be the extremal points of $T_n(x)$ in $[-1,1]$.

Lemma 1 *The Chebyshev polynomials satisfy the inequality*

$$|T_n''(x)| \leqslant T_n''(\eta_1^{(n)})$$

for each $x\in [-\eta_1^{(n)},\eta_1^{(n)}]$. *For* $n>2$ *the equality is attained if and only if* $x=-\eta_1^{(n)}$ *or* $x=\eta_1^{(n)}$.

Proof For $n=2,3$ the assertion is verified directly on the polynomials $T_2(x)=2x^2-1$ and $T_3(x)=4x^3-3x$. Suppose now that $n\geqslant 4$.

After a change of the variable $x=\cos\theta$, the wanted inequality is equivalent to

$$|f(\theta)| \leqslant g(\theta)$$

for each $\theta\in [\frac{\pi}{n},\pi-\frac{\pi}{n}]$, where

$$f(\theta) = \sin n\theta\cos\theta - n\cos n\theta\sin\theta,$$

$$g(\theta) = c_n\sin^3\theta,\ c_n = \frac{n}{\sin^2 \frac{\pi}{n}}.$$

The function $|f(\theta)|$ is even with respect to $\frac{\pi}{2}$. Since $f'(\theta)=(n^2-1)\sin\theta\sin n\theta$, it is clear that $f(\theta)$ decreases in $[\frac{\pi}{n},\frac{2\pi}{n}]$, increases in $[\frac{2\pi}{n},\frac{3\pi}{n}]$, etc. The local maxima of $|f(\theta)|$ are attained in the points $\frac{k\pi}{n}$ and the corresponding extremal values are

$$|f(\frac{k\pi}{n})| = n\sin\frac{k\pi}{n}, \qquad k = 1,\cdots,n-1.$$

The sequence $\{|f(\frac{k\pi}{n})|\}_{k=1}^{[n/2]}$ is increasing and the sequence $\{|f(\frac{k\pi}{n})|\}_{k=[n/2]+1}^{n-1}$ is decreasing. The function $g(\theta)$ increases in $[\frac{\pi}{n},\frac{\pi}{2}]$ and decreases in $[\frac{\pi}{2},\pi-\frac{\pi}{n}]$, $g(\theta)$ is even with respect to $\frac{\pi}{2}$. It is seen that $f(\frac{\pi}{n})=g(\frac{\pi}{n})$.

We shall consider two cases.

Case 1: $\theta\in[\frac{\pi}{n},\frac{2\pi}{n}]\cup[\frac{(n-2)\pi}{n},\frac{(n-1)\pi}{n}]$.

Without loss of generality we may assume that $\theta\in[\frac{\pi}{n},\frac{2\pi}{n}]$. Let us denote by θ_1 the unique zero of $f(\theta)$ in this interval. For $n\geqslant 4$ we have

$$f(\frac{11\pi}{8n}) = \sin\frac{11\pi}{8}\cos\frac{11\pi}{8n} - n\cos\frac{11\pi}{8}\sin\frac{11\pi}{8n}$$

$$> \sin\frac{11\pi}{8} - \frac{11}{4}\cos\frac{11\pi}{8} > 0,$$

hence

$$\theta_1 > \frac{11\pi}{8n}.$$

Taking into account the attitude of $|f(\theta)|$ and $g(\theta)$ in $[\frac{\pi}{n},\frac{2\pi}{n}]$, it is sufficient to prove that

$$|f(\frac{2\pi}{n})| < g(\frac{11\pi}{8n}),$$

which is eqiuvalent to

$$\sin\frac{2\pi}{n}\sin^2\frac{\pi}{n} < \sin^3\frac{11\pi}{8n}.$$

The last inequality is verified directly for $n=4,5,6$ and for $n\geqslant 7$ it follows from the inequalities

$$\sin\frac{2\pi}{n}\sin^2\frac{\pi}{n} < \frac{2\pi^3}{n^3}$$

and

$$\sin\frac{11\pi}{8n} > \frac{55}{8n}\sin\frac{\pi}{5}.$$

Case 2: $\theta\in[\frac{2\pi}{n},\frac{(n-2)\pi}{n}]$.

We may assume that $n\geqslant 6$. It is sufficient to prove that

$$|f(\frac{(k+1)\pi}{n})| < g(\frac{k\pi}{n}), \qquad k = 2,\cdots,[\frac{n}{2}]-1,$$

or, equivalently,

$$\sin\frac{(k+1)\pi}{n}\sin^2\frac{\pi}{n} < \sin^3\frac{k\pi}{n}, \qquad k = 2,\cdots,[\frac{n}{2}]-1.$$

The following inequalities are true for $n\geqslant 6$: $\cos\frac{\pi}{n}\geqslant\frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{n}>\frac{2}{n}$. Hence

$$\sin\frac{3\pi}{n}<\frac{3\pi}{n}<\frac{6\sqrt{3}}{n}<8\sin\frac{\pi}{n}\cos^3\frac{\pi}{n}$$

and the inequality for $k=2$ is established.

Let $k\geqslant 3$. Then

$$\sin\frac{(k+1)\pi}{n}\sin^2\frac{\pi}{n}<\frac{\pi^3(k+1)}{n^3}<\frac{8k^3}{n^3}<\sin^3\frac{k\pi}{n},$$

as desired.

This completes the proof of Lemma 1.

We shall denote by $P_n^{(\alpha,\beta)}(x)$ the Jacoby polynomials. Precisely, $P_n^{(\alpha,\beta)}(x)$ is the polynomial from π_n which is orthogonal in $[-1,1]$ with a weight

$$w(x)=(1-x)^\alpha(1+x)^\beta,$$

to every polynomial of degree $n-1$ and normalized by the condition

$$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}.$$

Using the fact that except for a constant factor $T_n^{(k)}(x)$ is $P_{n-k}^{(\frac{2k-1}{2},\frac{2k-1}{2})}(x)$ and some results for ultrspherical polynomials (see [7] and [9]), we obtain the following properties of the k-th derivative of the Chebyshev polynomial:

(i) The function $y=T_n^{(k)}(x)$ satisfies the differential equation

$$(1-x^2)y''-(2k+1)xy'+(n^2-k^2)y=0.$$

(ii) The largest zero x_1 of $T_n^{(k)}(x)$ satisfies the inequality

$$x_1>\sqrt{\frac{n-k-1}{n+k}},$$

(it follows from the inequality (6.2.13) in [9], p. 119).

(iii) The sequence formed by the relative maxima of $|T_n^{(k)}(x)|$ in the interval $[0,1]$ and by the value of this function at $x=1$ is increasing.

Theorem 1 *The Chebyshev polynomials satisfy the inequality*

$$|T_n'''(x)|\leqslant T_n'''(\xi_1^{(n)})$$

for each $x\in[-\xi_1^{(n)},\xi_1^{(n)}]$. *For* $n>3$ *the equality is attained if and only if* $x=-\xi_1^{(n)}$ *or* $x=\xi_1^{(n)}$.

Proof All interior local maxima of $T'''_n(x)$ are located in the interval $(-\xi_1^{(n)},\xi_1^{(n)})$. By virtue of (iii), the inequality is equivalent to

$$|T_n'''(x_1)|<T_n'''(\xi_1^{(n)}),\tag{1}$$

where x_1 is the largest zero of $T_n^{(4)}(x)$.

For $n=4,\cdots,7$ the assertion is verified directly on the polynomials $T_4(x),\cdots,T_7(x)$. We suppose that $n\geqslant 8$.

Using differential equation (i), we obtain

$$T_n'''(x_1) = \frac{(n^2 - 4)T_n''(x_1)}{5x_1}.$$

Let us note that

$$T_n'(\xi_1^{(n)}) = \frac{n}{\sin\frac{\pi}{2n}}, \quad T_n''(\xi_1^{(n)}) = \frac{n\cos\frac{\pi}{2n}}{\sin^3\frac{\pi}{2n}}.$$

Hence, by (i), we get

$$T_n'''(\xi_1^{(n)}) = \frac{n}{\sin^5\frac{\pi}{2n}}(3 - (n^2 + 2)\sin^2\frac{\pi}{2n}).$$

Therefore the inequality (1) may be written in the form

$$(n^2 - 4)|T_n''(x_1)| < 5x_1 \frac{n}{\sin^5\frac{\pi}{2n}}(3 - (n^2 + 2)\sin^2\frac{\pi}{2n}). \tag{2}$$

Since $x_1 \in [-\eta_1^{(n)}, \eta_1^{(n)}]$, Lemma 1 implies that

$$|T_n''(x_1)| < T_n''(\eta_1^{(n)}) = \frac{n^2}{\sin^2\frac{\pi}{n}}.$$

On the other hand, by (ii),

$$x_1 > \sqrt{\frac{n-5}{n+4}}.$$

Now, it sufficient to prove the stronger inequality

$$n(n^2 - 4)\sin^3\frac{\pi}{2n} < 20\sqrt{\frac{n-5}{n+4}}\cos^2\frac{\pi}{2n}(3 - (n^2 + 2)\sin^2\frac{\pi}{2n}).$$

We have for $n \geqslant 8$:

$$\begin{aligned} 20\sqrt{\frac{n-5}{n+4}}\cos^2\frac{\pi}{2n}(3 - (n^2 + 2)\sin^2\frac{\pi}{2n}) &> \\ > 10\cos^2\frac{\pi}{16}(3 - (n^2 + 2)\frac{\pi^2}{4n^2}) &> \\ > 10\cos^2\frac{\pi}{16}(3 - \frac{\pi^2}{4} - \frac{\pi^2}{128}) &> \\ > 4 > \frac{\pi^3}{8} > n(n^2 - 4)\sin^3\frac{\pi}{2n}&. \end{aligned}$$

The theorem is proved.

3 The inequality

We shall recall some resuts, connected with the extremal problem

$$N_k(\xi) = \max\{|f^{(k)}(\xi)| : f \in \pi_n, \|f\| \leqslant 1\},$$

where $1 \leqslant k \leqslant n, \xi \in [-1, 1]$.

The interval $[-1,1]$ can be partitioned into two groups of subintervals:

$$\{[\alpha_i^{(k)},\beta_i^{(k)}]\}_{i=1}^{n-k+1},\quad \alpha_1^{(k)}=-1,\quad \beta_{n-k+1}^{(k)}=1,$$

called Chebyshev intervals, and their complements $\{(\beta_i^{(k)},\alpha_{i+1}^{(k)})\}_{i=1}^{n-k}$.

It was shown in [3] that

$$N_k(\xi)=|T_n^{(k)}(\xi)|, \tag{3}$$

for $\xi\in[\alpha_i^{(k)},\beta_i^{(k)}], i=1,\cdots,n-k+1$ and

$$N_k(\xi)\leqslant\max\{|T_n^{(k)}(\beta_i^{(k)})|,\ |T_n^{(k)}(\alpha_{i+1}^{(k)})|\}, \tag{4}$$

for $\xi\in(\beta_i^{(k)},\alpha_{i+1}^{(k)}), i=1,\cdots,n-k$.

Lemma 2[1] *Let $p\in\pi_n$ such that $p(-\xi_1^{(n)})=p(\xi_1^{(n)})=0$. If $|p(x)|\leqslant 1$ on $[-\xi_1^{(n)},\xi_1^{(n)}]$ then $|p(x)|\leqslant|T_n(x)|$ for each $|x|\geqslant\xi_1^{(n)}$.*

Lemma 3[4] *The points $-\xi_1^{(n)}$ and $\xi_1^{(n)}$ are located in the two end Chebyshev intervals. Precisely, $-\xi_1^{(n)}\in(-1,\beta_1^{(k)}), \xi_1^{(n)}\in(\alpha_{n-k+1}^{(k)},1)$.*

Theorem 2 *Suppose that $f(x)$ is a real algebraic polynomial of degree at most n, satisfying the conditions $f(-1)=f(1)=0$. Then*

$$|f'''(x)|\leqslant n\frac{\cos^3\frac{\pi}{2n}}{\sin^5\frac{\pi}{2n}}\left(3-(n^2+2)\sin^2\frac{\pi}{2n}\right)\|f\|_{C[-1,1]},$$

for each $x\in[-1,1]$. The equality is attained if and only if $f(x)=c\overline{T}_n(x)$ and $x=\pm1$.

Proof Without loss of generality we may assume that $\|f\|_{C[-1,1]}\leqslant 1$. Let $p(x)=f(\frac{x}{\xi_1^{(n)}})$. Lemma 2 implies that $|p(x)|\leqslant 1$ on $[-1,1]$. Let x be an arbitrary point in $[-\xi_1^{(n)},\xi_1^{(n)}]$. By (3), (4) and Lemma 3, there is $y\in[-\xi_1^{(n)},\xi_1^{(n)}]$ such that $|p'''(x)|\leqslant|T_n'''(y)|$. Applying Theorem 1 we obtain $|T_n'''(y)|\leqslant|T_n'''(\xi_1^{(n)})|$.

We conclude that $|p'''(x)|\leqslant T_n'''(\xi_1^{(n)})$ on $[-\xi_1^{(n)},\xi_1^{(n)}]$. It follows that

$$|f'''(x)|=(\xi_1^{(n)})^3|p'''(\xi_1^{(n)}x)|\leqslant(\xi_1^{(n)})^3T_n'''(\xi_1^{(n)})$$

$$=n\frac{\cos^3\frac{\pi}{2n}}{\sin^5\frac{\pi}{2n}}\left(3-(n^2+2)\sin^2\frac{\pi}{2n}\right)=\overline{T}_n'''(1)$$

for each $x\in[-1,1]$.

It is seen from Theorem 1 that the equality is attained if and only if $f(x)=c\overline{T}_n(x)$ and $x=\pm1$. The theorem is proved.

4 Additional results and comments

We obtain here the following inequality of Duffin-Schaeffer-Schur type:

Theorem 3 *Suppose that $p\in\pi_n$ such that $p(-a)=p(a)=0$ for some $a>0$. Let $|p(x)|$*

$\leqslant 1$ *for* $x = a\dfrac{\eta_j^{(n)}}{\xi_1^{(n)}}$, $j = 1, \cdots, n-1$. *Then*

$$|p^{(k)}(x+iy)| \leqslant (\xi_1^{(n)}/a)^k |T_n^{(k)}(1+iy(\xi_1^{(n)}/a))|$$

for $-a/\xi_1^{(n)} \leqslant x \leqslant a/\xi_1^{(n)}$, $-\infty < y < \infty$ *and* $k = 1, \cdots, n$. *The equality is possible if and only if* $p(z) = \pm T_n((\xi_1^{(n)}/a)z)$.

As particular case of Theorem 3 (for $a=1$) we get an earlier result of Frappier [5, Theorem 2′].

Our approach is essentially different and is based on the

Lemma 4 *Let* $q \in \pi_n$ *such that* $q(-\xi_1^{(n)}) = q(\xi_1^{(n)}) = 0$. *If* $|q(\eta_j^{(n)})| \leqslant 1$ *for* $j = 1, \cdots, n-1$, *then*

$$|q(x)| \leqslant |T_n(x)|$$

for each $|x| \geqslant \xi_1^{(n)}$.

Lemma 4 may be considered as a discrete version of Lemma 2. The proof of Lemma 4 is similar to that of Lemma 2 and we omit it.

Proof of Theorem 3.

Let us consider the polynomial $q(z) := p((a/\xi_1^{(n)})z)$. It follows that $q(-\xi_1^{(n)}) = q(\xi_1^{(n)}) = 0$ and $|q(\eta_j^{(n)})| \leqslant 1$ for $j = 1, \cdots, n-1$. Lemma 4 implies that $|q(\eta_0^{(n)})| \leqslant 1$ and $|q(\eta_n^{(n)})| \leqslant 1$. Then the Theorem of Duffin and Schaeffer [4, Theorem Ⅰ] gives

$$|q^{(k)}(x+iy)| \leqslant |T_n^{(k)}(1+iy)|$$

for each $x \in [-1,1]$, $y \in (-\infty, \infty)$ and $k = 1, \cdots, n$ with equality if and only if $q(z) = \pm T_n(z)$. But $p^{(k)}(z) = (\xi_1^{(n)}/a)^k q^{(k)}((\xi_1^{(n)}/a)z)$, hence

$$\begin{aligned}|p^{(k)}(x+iy)| &= (\xi_1^{(n)}/a)^k |q^{(k)}((\xi_1^{(n)}/a)x + iy(\xi_1^{(n)}/a))| \\ &\leqslant (\xi_1^{(n)}/a)^k |T_n^{(k)}(1+iy(\xi_1^{(n)}/a))|\end{aligned}$$

for each $x \in [-a/\xi_1^{(n)}, a/\xi_1^{(n)}]$, $y \in (-\infty, \infty)$ and $k = 1, \cdots, n$.

The proof is completed.

Remark The estimates for the derivatives of the polynomial, resulting from Theorem 3, are not exact on the smaller interval $[-a, a]$. That is why Theorem 3 does not contain Theorem 2. This may be seen from the following example, too:

Let the polynomial $f \in \pi_n$ satisfies the conditions of Theorem 2, i. e. $f(-1) = f(1) = 0$. Without any restriction we may assume that $\| f \|_{C[-1,1]} \leqslant 1$.

Then we may apply Theorem 3 (for $a=1$) to f. In particular, for $k=3$ and $y=0$ this gives

$$|f'''(x)| \leqslant (\xi_1^{(n)})^3 T_n'''(1)$$

for each $x \in [-1/\xi_1^{(n)}, 1/\xi_1^{(n)}]$, with equality if and only if $f(x) = \pm T_n(x) = \pm T_n(\xi_1^{(n)} x)$ and $x = \pm 1/\xi_1^{(n)}$.

This estimate is not sharp for $x \in [-1,1]$-the interval where the conditions on f are imposed. The sharp inequality

$$|f^{n}(x)| \leqslant T_n^{n}(1) = (\xi_1^{(n)})^3 T_n^{n}(\xi_1^{(n)})$$

for each $x \in [-1,1]$, with equality if and only if $f(x) = \pm T_n(x)$ and $x = \pm 1$, is given in Theorem 2.

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Department of Mathematics
University of Sofia
Blvd. James Boucher 5
1126, Sofia
Bulgaria