A MAXIMAL FUNCTION CHARACTERIZATION OF HARDY SPACES ON SPACES OF HOMOGENEOUS TYPE*

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Abstract

A new maximal function is introduced in the dual spaces of test function spaces on spaces of homogeneous type. Using this maximal function, we get new characterization of atomic H^p spaces.

1 Introduction

The purpose of this paper is to give a new maximal function characterization for H^p spaces defined on spaces of homogeneous type. With this aim, we first define a Hardy-type spaces $\tilde{H}_p$ in the dual space of the test function spaces $\mathscr{M}(\beta,\gamma)$. Then we prove that every element in $\tilde{H}_p$ have a decomposition in series of p-atoms, conversely, each distribution on $\mathscr{M}(\beta,\gamma)$ which can be denoted by a series of p-atoms with coefficients satisfy some conditions belongs to $\tilde{H}_p$. Finally, we show that the atom H^p spaces, as defined in[1], can be identified with $\tilde{H}_p$. The results in this paper are generalization of theory of Macia and Segovia[4].

We begin by recalling spaces of homogeneous type. Let X be a set. A quasi-metric d on X is a function $d(x,y):X\times X\to[0,\infty]$ satisfying:

(1.1.i) $d(x,y)=0$ if and only if $x=y$,

(1.1.ii) $d(x,y)=d(y,x)$ for all $x,y\in X$,

(1.1.iii) There exists a constant $A<\infty$ such that for all x,y and z in X,

$$d(x,y)\leqslant A[d(x,z)+d(z,y)]. \quad (1.2)$$

Any quasi-metric defines a topology, for which the balls $B(x,r)=\{y\in X:d(y,x)<r\}$ form a base. However, the balls themselves need not to be open when $A>1$.

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Definition 1.3[1] A space of homogeneous type (X,d,μ) is a set X together with a quasi-metric d and a nonnegative measur μ on X such that $\mu(B(x,r))<\infty$ for all $x\in X$, and all $r>0$, and there exists a constant $A'<\infty$ such that for all $x\in X$ and all $r>0$

$$\mu(B(x,2r))\leqslant A'\mu(B(x,r)). \tag{1.4}$$

In [3] Macia and Segovia have shown that one can replace d by another quasi-metric ρ such that there exist $c>0$ and some $\theta, 0<\theta<1$ satisfying

$$\rho(x,y)\sim\inf\{\mu(B): B \text{ is a ball containing } x \text{ and } y\} \tag{1.5}$$

$$|\rho(x,y)-\rho(x',y)|\leqslant C_\rho(x,x')^\theta[\rho(x,y)+\rho(x',y)]^{1-\theta}, \quad \text{for all } x,x' \text{ and } y\in X. \tag{1.6}$$

In this paper we assume that $\mu(\{x\})=0$ for all $x\in X$ and $\mu(X)=+\infty$. For the case $\mu(X)<\infty$, see the Remark (5.11) below.

Now we introduce a class of test functions on X.

Definition 1.7 Fix two exponents $0<\beta<\theta$, see [2], and $\gamma>0$. A function ψ defined on X is said to be a test function of type (x_0,d,β,γ), $x_0\in X$ and $d>0$, if ψ satisfies the following conditions:

(i) $|\psi(x)|\leqslant C\dfrac{d^\gamma}{(d+\rho(x,x_0))^{1+\gamma}}$;

(ii) $|\psi(x)-\psi(y)|\leqslant C\left(\dfrac{\rho(x,y)}{d+\rho(x,x_0)}\right)^\beta\dfrac{d^\gamma}{(d+\rho(x,x_0))^{1+\gamma}}$,

for $\rho(x,y)\leqslant\dfrac{1}{2A}[d+\rho(x,x_0)]$.

If ψ is a test function of type (x_0,d,β,γ) we write $\psi\in\mathcal{M}(x_0,d,\beta,\gamma)$ and the norm of ψ in $\mathcal{M}(x_0,d,\beta,\gamma)$ is defined by

$$\|\psi\|_{\mathcal{M}(x_0,d,\beta,\gamma)}=\inf\{C: \text{(i) and (ii) hold}\}.$$

For fixed $x_0\in X$, we denote $\mathcal{M}(\beta,\gamma)=\mathcal{M}(x_0,1,\beta,\gamma)$. It is easy to check that $\mathcal{M}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{M}(\beta,\gamma)$. The dual sapce $(\mathcal{M}(\beta,\gamma))'$ consists of all linear function l from $\mathcal{M}(\beta,\gamma)$ to $\mathscr{C}$ with the property that there exists a finite constant C such that for all $\psi\in\mathcal{M}(\beta,\gamma)$, $|l(\psi)|\leqslant C\|\psi\|_{\mathcal{M}(\beta,\gamma)}$. We denote the natural pairing of elements $f\in(\mathcal{M}(\beta,\gamma))'$ and $\psi\in\mathcal{M}(\beta,\gamma)$ by $\langle f,\psi\rangle$. It is also easy to see that $\mathcal{M}(x_1,d,\beta,\gamma)=\mathcal{M}(\beta,\gamma)$ with equivalent norms for $x_1\in X$ and $d>0$. Thus, $\langle f,\psi\rangle$ is well defined for all $f\in(\mathcal{M}(\beta,\gamma))'$ and $\psi\in\mathcal{M}(x,d,\beta,\gamma)$ with $x\in X$ and $d>0$.

For the convenience, sometime, we call a linear functional on $\mathcal{M}(\beta,\gamma)$ to be a distribution. Given a function $f(x)$ in $L^q(X,d\mu)$, $1\leqslant q\leqslant\infty$, clearly,

$$\langle f,\psi\rangle=\int f(x)\psi(x)d\mu(x)$$

defines a linear functional on $\mathcal{M}(\beta,\gamma)$, we shall say that f is a distribution induced by the function $f(x)$.

Denote $\mathcal{M}_0(x,d,\beta,\gamma)=\{\psi\in\mathcal{M}(x,d,\beta,\gamma): \int\psi(z)d\mu(z)=0\}$.

For $f\in(\mathcal{M}(\beta,\gamma))'$, $0<\beta<\theta$, $\gamma>0$, we define the maximal function $f^*(x)$ of f as

$$f^*(x) = \sup\{|\langle f,\psi\rangle| : \text{for some } d>0,\ \psi\in\mathscr{M}(x,d,\beta,\gamma) \text{ and } \|\psi\|_{\mathscr{M}(x,d,\beta,\gamma)}\leqslant 1\}.$$

For $(1+\beta)^{-1}<p<\infty$, we define the maximal function spaces $\widetilde{H}_p$ as

$$\widetilde{H}_p = \{f\in\mathscr{M}(\beta,\gamma))' : f^*\in L^p(X)\}.$$

If $f\in\widetilde{H}_p$, we define $\|f\|_{\widetilde{H}_p}=\|f^*\|_p$.

2 The completeness of $\widetilde{H}_p$

Obviously, $(\widetilde{H}_p, \|\cdot\|_{\widetilde{H}_p})$ is a quasi-metric space for $(1+\beta)^{-1}<p<1$, and metric space for $1\leqslant p<\infty$. In the following, we will show that $\widetilde{H}_p$ with the metric $\|\cdot\|_{\widetilde{H}_p}$ is complete, so $(\widetilde{H}_p, \|\cdot\|_{\widetilde{H}_p})$ is a Banach space when $1\leqslant p<\infty$.

Theorem 2.1 *$(\widetilde{H}_p, \|\cdot\|_{\widetilde{H}_p})$ is complete, i.e., for any Cauchy sequence $\{f_n\}$ is $\widetilde{H}_p$ there exists f in $\widetilde{H}_p$ such that*

(i) *$\{f_n\}$ converges to f in $(\mathscr{M}(\beta,\gamma))'$,*

(ii) *$\|f_n-f\|_{\widetilde{H}_p}\to 0$ when $n\to\infty$.*

In ordere to prove the Theorem (2.1), we need the following Lemma.

Lemma 2.2 *There exists $d_0>0, M\geqslant 1$, such that for any $\psi\in\mathscr{M}(x_0,1,\beta,\gamma)$, $\|\psi\|_{\mathscr{M}(x_0,1,\beta,\gamma)}\leqslant 1$, and any $y_0\in B(x_0,d_0)$, we have $\frac{\psi}{M}\in\mathscr{M}(y_0,1,\beta,\gamma)$.*

Proof Take $d_0=(2C)^{-\frac{1}{\theta}}$, where C and θ are constants in (1.6). Let $\mathscr{A}=\{x\in X: \rho(x_0,x)\leqslant 1, \rho(y_0,x)\leqslant 1\}$. When $x\in\mathscr{A}$, we have that

$$|\rho(x_0,x)-\rho(y_0,x)|\leqslant C_\rho(x_0,y_0)^\theta[\rho(x_0,x)+\rho(y_0,x)]^{1-\theta}\leqslant 2^{-\theta},$$

so,

$$\rho(y_0,x)\leqslant\rho(x_0,x)+2^{-\theta}$$

and

$$\frac{1}{(1+\rho(x_0,x))^{1+\gamma}}\leqslant\frac{1}{((1-2^{-\theta})+\rho(y_0,x))^{1+\gamma}}$$
$$\leqslant\frac{1}{(1-2^{-\theta})^{1+\gamma}}\frac{1}{(1+\rho(x_0,x))^{1+\gamma}}.$$

When $x\notin\mathscr{A}$,

$$|\rho(x_0,x)-\rho(y_0,x)|\leqslant\frac{1}{2}[\rho(x_0,x)+\rho(y_0,x)]^{1-\theta}$$
$$\leqslant\frac{1}{2}[\rho(x_0,x)+\rho(y_0,x)], \frac{1}{2}\rho(y_0,x)\leqslant\frac{3}{2}\rho(x_0,x),$$

so,

$$\frac{1}{(1+\rho(x_0,x))^{1+\gamma}}\leqslant\frac{9}{(1+\rho(y_0,x))^{1+\gamma}}.$$

We take $M_1=\max\{\frac{1}{(1-2^{-\theta})^{1+\gamma}},9\}$, then for any $y_0\in B(x_0,d_0)$ and any $\psi\in\mathscr{M}(x_0,1,\beta,\gamma)$, $\|\psi\|_{\mathscr{M}(x_0,1,\beta,\gamma)}\leqslant 1$, we have

$$|\psi(x)| \leqslant \frac{M_1}{(1+\rho(y_0,x))^{1+\gamma}}.$$

For any x any y in X, $\rho(x,y)\leqslant\frac{1}{2A}[1+\rho(y_0,x)]$. If $\rho(x,y)\leqslant\frac{1}{2A}[1+\rho(x_0,x)]$, then

$$|\psi(x)-\psi(y)|\leqslant\left(\frac{\rho(x,y)}{1+\rho(x_0,x)}\right)^{\beta}\frac{1}{(1+\rho(x_0,x))^{1+\gamma}}$$

$$\leqslant M_1^{2+\gamma}\left(\frac{\rho(x,y)}{1+\rho(y_0,x)}\right)^{\beta}\frac{1}{(1+\rho(y_0,x))^{1+\gamma}}.$$

If $\rho(x,y)\geqslant\frac{1}{2A}[1+\rho(x_0,x)]$, then $\rho(x_0,x)\leqslant\rho(y_0,x)$, and $\rho(y_0,x)\leqslant A[\rho(y_0,y)+\rho(y,x)]\leqslant A_\rho(y_0,y)+\frac{1}{2}[1+\rho(y_0,x)]$, so $\frac{1}{2}[1+\rho(y_0,x)]\leqslant A[\rho(y_0,y)+1]$. Thus

$$|\psi(x)-\psi(y)|\leqslant|\psi(x)|+|\psi(y)|$$

$$\leqslant M_1\left(\frac{1}{(1+\rho(y_0,x))^{1+\gamma}}+\frac{1}{(1+\rho(y_0,y))^{1+\gamma}}\right)$$

$$\leqslant 2AM_1(M_1+(2A)^{1+\gamma}M_1)\left(\frac{\rho(x,y)}{1+\rho(y_0,x)}\right)^{\beta}\frac{1}{(1+\rho(y_0,x))^{1+\gamma}}.$$

Let $M=2AM_1(M_1+(2A)^{1+\gamma}M_1)$, then the result of Lemma is true. We complete the proof of Lemma (2.2).

Now, we prove Theorem 2.1. For any $\psi\in\mathcal{M}(\beta,\gamma)$, $\|\psi\|_{\mathcal{M}(\beta,\gamma)}\leqslant 1$, and any $x\in B(x_0,d_0)$, by Lemma (2.2),

$$|\langle f_n-f_m,\psi\rangle|\leqslant M|\langle f_n-f_m,\frac{\psi}{M}\rangle|\leqslant M(f_n-f_m)^*(x).$$

Then

$$\|f_n-f_m\|_{(\mathcal{M}(\beta,\gamma))'}=\sup_{\|\psi\|_{\mathcal{M}(\beta,\gamma)}\leqslant 1}|\langle f_n-f_m,\psi\rangle|\leqslant M(f_n-f_m)^*(x),$$

for $x\in B(x_0,d_0)$. Taking the p-power and integrating on $B(x_0,d_0)$, we obtain

$$\|f_n-f_m\|_{(\mathcal{M}(\beta,\gamma))'}\leqslant M\left\{(\mu(B(x_0,d_0)))^{-1}\int_{B(x_0,d_0)}(f_n-f_m)^*(x)^p d\mu(x)\right\}^{\frac{1}{p}}$$

$$\leqslant M\mu(B(x_0,d_0))^{-\frac{1}{p}}\|f_n-f_m\|_{H_p}.$$

This shows that $\{f_m\}$ is a Cauchy sequence in $(\mathcal{M}(\beta,\gamma))'$, therefore, there exists $f\in(\mathcal{M}(\beta,\gamma))'$ such that f is the limit of the sequence $\{f_n\}_{n=1}^{\infty}$. This proves (i). The proof of (ii) is the same as in [4] and omited here. We complete the proof of Theorem 2.2.

3 Calderon-Zygmund type lemma

Lemma 3.1 (**covering lemma**[4]) *Let Ω be an open set of finite measure strictly contained in X and $d(x)=\inf\{\rho(x,y):y\notin\Omega\}$. Given $C\geqslant 1$, let $r(x)=(2AC)^{-1}d(x)$. Then there exists a natural number M, which depends on C, and a sequence $\{x_n\}$ such that, denoting $r(x_n)$ by r_n, we have*

(3.2.i) *the balls* $B(x_n,(4A)^{-1}r_n)$ *are pairwise disjoint*,

(3.2.ii) $\bigcup_n B(x_n,r_n)=\Omega$,

(3.2.iii) *for every* n, $B(x_n,Cr_n)\subset\Omega$,

(3.2.iv) *for every* n, $x\in B(x_n,Cr_n)$ *implies that* $Cr_n\leqslant d(x)\leqslant 3A^2Cr_n$,

(3.2.v) *for every* n, *there exists* $y_n\notin\Omega$ *such that* $\rho(x_n,y_n)<3ACr_n$,

(3.2.vi) *for every* n, *the number of balls* $B(x_k,Cr_k)$ *whose intersections with* $B(x_n,Cr_n)$ *are non-empty is at most* M.

Lemma 3.3 (partition of the unity) *Let* Ω *be an open set of finite measure strictly contained in* X. *Consider the sequence* $\{x_n\}$ *and* $\{r_n\}$ *given by Lemma* 3.1 *for* $C=5A$. *Then, there exists a sequence* $\{\varphi_n(x)\}$ *of non-negative functions satisfying*

(3.4.i) supp $\varphi_n\subset B(x_n,2r_n)$,

(3.4.ii) $\varphi_n(x)\geqslant\frac{1}{M}$, *for* $x\in B(x_n,r_n)$,

(3.4.iii) *there exists* C *such that for every* n, $\varphi_n\in\mathscr{M}(x_n,r_n,\beta,\gamma)$ *and* $\|\varphi_n\|\leqslant Cr_n$,

(3.4.iv) $\sum_n\varphi_n(x)=\chi_\Omega(x)$.

Proof Let $\eta(s)$ be an infinitely differentiable function on $[0,\infty)$ such that $0\leqslant\eta(s)\leqslant 1$, $\eta(s)=1$ if $0\leqslant s\leqslant 1$ and $\eta(s)=0$ for $s\geqslant 2$. For every n, we define

$$\psi_n(x)=\eta(\frac{\rho(x,x_n)}{r_n}).$$

These functions ψ_n are non-negative, with supp $\psi_n\subset B(x_n,2r_n)$ and by (3.2.ii) and (3.2.vi), satisfy

$$1\leqslant\sum_n\psi_n(x)\leqslant M,\ \text{for every } x\in\Omega.$$

It is easy to prove that $\psi_n\in\mathscr{M}(x_n,r_n,\beta,\gamma)$ and $\|\psi_n\|_{\mathscr{M}(x_n,r_n,\beta,\gamma)}\leqslant Cr_n$ where C is independent of n.

We define $\varphi_n(x)$ by $\varphi_n(x)=0$ if $x\notin\Omega$ and $\varphi_n(x)=\psi_n(x)/\sum_n\psi_n(x)$ if $x\in\Omega$. Then $\{\varphi_n(x)\}$ satisfies Lemma 3.3.

Lemma 3.5 *Let* $\{\varphi_n(x)\}$ *be the partition of unity in Lemma* 3.3 *associated to some open set* Ω, *then for every* n, *the linear mapping*

$$S_n(\psi)(x)=\varphi_n(x)[\int\varphi_n(z)d\mu(z)]^{-1}\int(\psi(x)-\psi(z))\varphi_n(z)d\mu(z)$$

is continuous from $\mathscr{M}(\beta,\gamma)$ *to* $\mathscr{M}_0(\beta,\gamma)$.

Proof Considering that $\mathscr{M}(\beta,\gamma)=\mathscr{M}(x,d,\beta,\gamma)$ with the equivalent norms for all $x\in X$ and $d>0$, we can easily prove the Lemma for $\mathscr{M}(x_n,r_n,\beta,\gamma)$. The details are omited.

Lemma 3.6[4] *Let* $0<\beta$, $1<q(1+\beta)$ *and* M *a positive integer. There exists a constant* $C_{\beta,q,M}$ *such that given any sequence of points* $\{x_n\}$ *and any sequence of positive number* $\{r_n\}$, *satisfying the condition that no point in* X *belongs to more than* M *balls* $B(x_n,r_n)$, *then*

$$\int\Big[\sum_n\Big(\frac{r_n}{r_n+\rho(x_n,z)}\Big)^{1+\beta}\Big]^q d\mu(z)\leqslant C_{\beta,q,M}\mu(U_nB(x_n,r_n)).$$

Lemma 3.7 (Calderon-Zygmund type Lemma) *Suppose that* $f \in \tilde{H}_p$, $(1+\beta)^{-1} < p < \infty$. *Let* $t > 0$ *and* $\Omega = \{x \in X : f^*(x) > t\}$. *This set is open set and* $\mu(\Omega) < \infty$. *Let* $\{\varphi_n(x)\}$ *be the partition of the unity in Lemma* 3.3 *associated to* Ω *and let* $\{S_n\}$ *be the linear transformations defined in Lemma* 3.5. *If we define the distribution* b_n *by*

$$\langle b_n, \psi \rangle = \langle f, S_n(\psi) \rangle, \text{ for } \psi \in \mathscr{M}(\beta, \gamma), \tag{3.8}$$

then

$$b_n^*(x) \leqslant Ct \left(\frac{r_n}{r_n + \rho(x_n, x)} \right)^{1+\beta} \chi_{B^c(x_n, 4Ar_n)}(x) + Cf^*(x) \chi_{B(x_n, 4Ar_n)}(x), \tag{3.9}$$

and

$$\int b_n^*(x)^p d\mu(x) \leqslant C \int_{B(x_n, 4Ar_n)} f^*(x)^p d\mu(x). \tag{3.10}$$

Moreover, the series $\sum_n b_n$ *converges in* $(\mathscr{M}(\beta, \gamma))'$ *to a distribution* b *satisfying*

$$b^*(x) \leqslant Ct \sum_n \left(\frac{r_n}{r_n + \rho(x_n, x)} \right)^{1+\beta} + cf^*(x) \chi_\Omega(x), \tag{3.11}$$

$$\int b^*(x)^p d\mu(x) \leqslant C \int_\Omega f^*(x)^p d\mu(x). \tag{3.12}$$

The distribution $g = f - b$ *satisfies*

$$g^*(x) \leqslant Ct \sum_n \left(\frac{r_n}{r_n + \rho(x_n, x)} \right)^{1+\beta} + Cf^*(x) \chi_{\Omega^c}(x). \tag{3.13}$$

Proof First, we prove (3.9). Let $x \notin B(x_n, 4Ar_n)$. We shall show that there exists a constant C independent of n such that for any $\psi \in \mathscr{M}(x, d, \beta, \gamma)$, $\|\psi\|_{\mathscr{M}(x,d,\beta,\gamma)} \leqslant 1$, we have $S_n(\psi) \in \mathscr{M}(y_n, r_n, \beta, \gamma)$ and

$$\|S_n(\psi)\|_{\mathscr{M}(y_n, r_n, \beta, \gamma)} \leqslant C \left(\frac{r_n}{r_n + \rho(x_n, x)} \right)^{1+\beta}, \tag{3.14}$$

where y_n is the point in Ω^c given by (3.2.v).

From supp $S_n(\psi) \subset B(x_n, 2r_n)$, we can assume $z \in B(x_n, 2r_n)$. Then it follows that $\rho(z, x_n) \leqslant 2r_n \leqslant \frac{1}{2A}\rho(x_n, x) \leqslant \frac{1}{2A}[d + \rho(x_n, x)]$, we have

$$\begin{aligned} |S_n(\psi)(z)| &\leqslant \varphi_n(z) |\psi(z) - \psi(x_n)| \\ &\quad + \varphi_n(z) \left(\int \varphi_n(z) d\mu(z) \right)^{-1} \int |\psi(x_n) - \psi(z)| \varphi_n(z) d\mu(z) \\ &\leqslant \left(\frac{\rho(z, x_n)}{d + \rho(x_n, x)} \right)^\beta \frac{d^\gamma}{(d + \rho(x_n, x))^{1+\gamma}} + C \frac{r_n^\beta}{(d + \rho(x_n, x))^{1+\gamma}} \\ &\leqslant C \frac{r_n^\beta}{\rho(x_n, x)^{1+\beta}}. \end{aligned}$$

For $z, z' \in X$, $\rho(z, z') \leqslant \frac{1}{2A}[r_n + \rho(y_n, z)]$. Without loss of generality, we assume that $z \in B(x_n, 2r_n)$.

$$|S_n(\psi)(z) - S_n(\psi)(z')| \leqslant \varphi_n(z') |\psi(z) - \psi(z')|$$

$$+ |\varphi_n(z) - \varphi_n(z')| \left(\int \varphi_n(y) d\mu(y)\right)^{-1} \int |\psi(z) - \psi(y)| \varphi_n(y) d\mu(y)$$

$$= I + II.$$

Notice that $r_n + \rho(x_n, x) \leqslant r_n + A[\rho(x_n, z) + \rho(x, z)] \leqslant C\rho(x, z)$, we have

$$I \leqslant \left(\frac{\rho(z,z')}{d + \rho(x,z)}\right)^{\beta} \frac{d^{\gamma}}{(d + \rho(x,z))^{1+\gamma}} \leqslant C \frac{\rho(z,z')^{\beta}}{(r_n + \rho(x_n,x))^{1+\beta}}.$$

If $\rho(z,z') \leqslant \frac{1}{2A}[r_n + \rho(x_n, z)]$, then

$$II \leqslant Cr_n \left(\frac{\rho(z,z')}{r_n + \rho(x_n,z)}\right)^{\beta} \frac{r_n^{\gamma}}{(r_n + \rho(x_n,z))^{1+\gamma}} \frac{r_n^{\beta}}{(r_n + \rho(x_n,x))^{1+\beta}}$$

$$\leqslant C \frac{\rho(z,z')^{\beta}}{(r_n + \rho(x_n,x))^{1+\beta}}.$$

If $\rho(z,z') > \frac{1}{2A}[r_n + \rho(x_n, z)] \geqslant \frac{r_n}{2A}$, then

$$II \leqslant Cr_n \left(\frac{r_n^{\gamma}}{(r_n + \rho(x_n,z))^{1+\gamma}} + \frac{r_n^{\gamma}}{(r_n + \rho(x_n,z'))^{1+\gamma}}\right) \frac{r_n^{\beta}}{(r_n + \rho(x_n,x))^{1+\beta}}$$

$$\leqslant C \frac{\rho(z,z')^{\beta}}{(r_n + \rho(x_n,x))^{1+\beta}}.$$

For every n, by (3. 2. v), $y_n \in B(x_n, 15A^2 r_n)$, then $\rho(y_n, z) \leqslant A[\rho(x_n, y_n) + \rho(x_n, z)] \leqslant 17A^3 r_n$. We have

$$|S_n(\psi)(z)| \leqslant C \left(\frac{r_n}{r_n + \rho(x_n,x)}\right)^{1+\beta} \frac{r_n^{\gamma}}{(r_n + \rho(y_n,z))^{1+\gamma}},$$

and when $\rho(z,z') \leqslant \frac{1}{2A}[r_n + \rho(y_n, z)]$,

$$|S_n(\psi)(z) - S_n(\psi)(z')|$$

$$\leqslant c \left(\frac{r_n}{r_n + \rho(x_n,x)}\right)^{1+\beta} \left(\frac{\rho(z,z')}{r_n + \rho(y_n,z)}\right)^{\beta} \frac{r_n^{\gamma}}{(r_n + \rho(y_n,z))^{1+\gamma}}.$$

This proves (3. 14). By (3. 8), we get that

$$b_n^*(x) \leqslant Cf^*(y_n) \left(\frac{r_n}{r_n + \rho(x_n,x)}\right)^{1+\beta} \leqslant Ct(r_n / r_n + \rho(x_n,x))^{1+\beta}.$$

Let $x \in B(x_n, 4Ar_n)$ and $\psi \in \mathcal{M}(x, d, \beta, \gamma)$, $\|\psi\|_{\mathcal{M}(x,d,\beta,\gamma)} \leqslant 1$. Assume that $d \geqslant r_n$, by the same way as above, we can prove that $S_n(\psi) \in \mathcal{M}(x, r_n, \beta, \gamma)$ and $\|S_n(\psi)\|_{\mathcal{M}(x,r_n,\beta,\gamma)} \leqslant C$. We assume that $d < r_n$, then

$$S_n(\psi)(z) = \varphi_n(z)\psi(z) - \varphi_n(z)\left[\int \varphi_n(y) d\mu(y)\right]^{-1} \int \psi(y)\varphi_n(y) d\mu(y) = h_1(z) - h_2(z).$$

Using the same way above we can prove that $h_1 \in \mathcal{M}(x, d, \beta, \gamma)$, $h_2 \in \mathcal{M}(x, r_n, \beta, \gamma)$ and $\|h_1\|_{\mathcal{M}(x,d,\beta,\gamma)} \leqslant C$, $\|h_2\|_{\mathcal{M}(x,r_n,\beta,\gamma)} \leqslant C$, where C is independent of n and ψ. Then we have

$$|\langle b_n, \psi\rangle| \leqslant |\langle f, S_n(\psi)\rangle| \leqslant |\langle f, h_1\rangle| + |\langle f, h_2\rangle| \leqslant Cf^*(x).$$

We complete the proof of (3. 9).

Taking the p-th power of (3. 9) and integrating on X, by Lemma 3. 6, we get

$$\int b_n^*(x)^p d\mu(x) \leqslant Ct^p \int_{B^c(x_n,4Ar_n)} \left(\frac{r_n}{r_n+\rho(x_n,x)}\right)^{(1+\beta)p} d\mu(x)$$

$$+ C\int_{B(x_n,4Ar_n)} f^*(x)^p d\mu(x)$$

$$\leqslant Ct^p \mu(B(x_n,4Ar_n)) + C\int_{B(x_n,4Ar_n)} f^*(x)^p d\mu(x).$$

Taking into account that $B(x_n,4Ar_n) \subset \Omega$, we get

$$\int b_n^*(x)^p d\mu(x) \leqslant C\int_{B(x_n,4Ar_n)} f^*(x)^p d\mu(x).$$

This proves (3. 10).

Next, let us study the convergence of the series of Σb_n. From Lemma (3. 1), (3. 10) and the fact that $f^*(x)$ is in $L^p(X,d\mu)$, we get that the partial sum of Σb_n is a Cauchy sequence, by Theorem (2. 1), Σb_n converges in $(\mathscr{M}(\beta,\gamma))'$ to a distribution b. Estimates (3. 11) and (3. 12) for $b^*(x)$ are obtained by adding up the estimates (3. 9), (3. 10) and Lemma 3. 6.

It remains to prove the inequality (3. 13). Assume that $x \in \Omega$, then there exists k such that $x \in B(x_k,r_k)$. By (3. 2. vi) we know that the set J of all integers n such that $B(x_n, 4Ar_n) \cap B(x_k,2Ar_k) \neq \varnothing$ has at most M elements. Moreover, by (3. 2. iv), for every $n \in J$, r_n satisfies $(3A^2)^{-1}r_k \leqslant r_n \leqslant 3A^2 r_k$. Let $\psi \in \mathscr{M}(x,d,\beta,\gamma)$ and $\|\psi\|_{\mathscr{M}(x,d,\beta,\gamma)} \leqslant 1$. If $d \leqslant r_k$, then

$$\langle g,\psi\rangle = \langle f,\psi\rangle - \sum_n \langle b_n,\psi\rangle$$

$$= \langle f,\psi\rangle - \sum_{n\in J}\langle f,S_n(\psi)\rangle - \sum_{n\notin J}\langle b_n,\psi\rangle$$

$$= \langle f,\tilde{\psi}\rangle - \sum_{n\in J}\langle f,\tilde{\varphi}_n\rangle - \sum_{n\notin J}\langle b_n,\psi\rangle,$$

where

$$\tilde{\psi} = (1 - \sum_{n\in J}\varphi_n)\psi,$$

$$\tilde{\varphi}_n(z) = \varphi_n(z)\left[\int \varphi_n(y)d\mu(y)\right]^{-1}\int \psi(y)\varphi_n(y)d\mu(y), \qquad \text{for } n \in J.$$

Notice that $\tilde{\psi}(z)=0$ for $z \in B(x_k,2Ar_k)$, it is easy to prove that $\tilde{\psi} \in \mathscr{M}(y_k,d,\beta,\gamma)$ and $\tilde{\varphi}_n \in \mathscr{M}(y_k,r_k,\beta,\gamma)$ for $n \in J$, $\|\tilde{\psi}\|_{\mathscr{M}(y_k,d,\beta,\gamma)} \leqslant C$, $\|\tilde{\varphi}_n\|_{\mathscr{M}(y_k,r_k,\beta,\gamma)} \leqslant C$, where C is independent of n,k and ψ. When $n \notin J$, we have $x \notin B(x_n,4Ar_n)$, using the proof of (3. 9), we have

$$\sum_{n\notin J} |\langle b_n,\psi\rangle| \leqslant Ct\sum_{n\notin J}\left(\frac{r_n}{r_n+\rho(x_n,x)}\right)^{1+\beta}.$$

So

$$|\langle g,\psi\rangle| \leqslant |\langle f,\tilde{\psi}\rangle| + \sum_{n\in J}|\langle f,\tilde{\varphi}_n\rangle| + \sum_{n\notin J}|\langle b_n,\psi\rangle|$$

$$\leqslant Cf^*(y_k) + \sum_{n\notin J} Ct\left(\frac{r_n}{r_n+\rho(x_n,x)}\right)^{1+\beta}$$

$$\leqslant Ct\sum_n\left(\frac{r_n}{r_n+\rho(x_n,x)}\right)^{1+\beta}$$

If $d>r_k$, then

$$|\langle g,\psi\rangle| = |\langle f,\psi\rangle| + \sum_{n\in J}|\langle b_n,\psi\rangle| + \sum_{n\notin J}|\langle b_n,\psi\rangle|.$$

It is easy to prove that $\psi\in\mathscr{M}(y_k,d,\beta,\gamma)$ and $\|\psi\|_{\mathscr{M}(y_k,d,\beta,\gamma)}\leqslant C$, we obtain,

$$|\langle f,\psi\rangle| \leqslant Cf^*(y_k) \leqslant Ct \leqslant Ct\left(\frac{r_n}{r_n+\rho(x_n,x)}\right)^{1+\beta},$$

and

$$\sum_{n\in J}|\langle b_n,\psi\rangle| \leqslant C\sum_{n\in J}b_n^*(y_k)$$
$$\leqslant Ct\sum_{n\in J}\left(\frac{r_n}{r_n+\rho(x_n,y_k)}\right)^{1+\beta}$$
$$\leqslant Ct\sum_{n\in J}\left(\frac{r_n}{r_n+\rho(x_n,x)}\right)^{1+\beta}.$$

On the other hand

$$\sum_{n\notin J}|\langle b_n,\psi\rangle| \leqslant \sum_{n\notin J}b_n^*(x) \leqslant \sum_{n\notin J}Ct\left(\frac{r_n}{r_n+\rho(x_n,x)}\right)^{1+\beta}.$$

Therefore we have shown that if $x\in\Omega$, then (3.13) holds.

If $x\notin\Omega$, then

$$|\langle g,\psi\rangle|\leqslant |\langle f,\psi\rangle| + |\langle b,\psi\rangle|$$
$$\leqslant f^*(x) + b^*(x) \leqslant f^*(x) + Ct\sum_{n}\left(\frac{r_n}{r_n+\rho(x_n,x)}\right)^{1+\beta}.$$

This completes the proof of Lemma 3.7.

Lemma 3.15 *Let $\eta(s)$ be an infinitely differentiable function defined on $[0,\infty)$ such that $0\leqslant\eta(s)\leqslant1$, $\eta(s)=1$ for $0\leqslant s\leqslant1$ and $\eta(s)=0$ for $s\geqslant2$. For $t>0$, $x,y\in X$, we define*

$$S_t(x,y) = \left[\int\eta(\rho(x,z)/t)d\mu(z)\right]^{-1}\eta(\rho(x,y)/t).$$

Then we have

(i) supp $S_t(x,y)\subset\{(x,y):\rho(x,y)\leqslant2t\}$,

(ii) $0\leqslant S_t(x,y)\leqslant C\dfrac{t^\gamma}{(t+\rho(x,y))^{1+\gamma}}$,

(iii) $|S_t(x,y)-S_t(x',y)|+|S_t(y,x)-S_t(y,x')|\leqslant C\left(\dfrac{\rho(x,x')}{t+\rho(x,y)}\right)^\beta\dfrac{t^\gamma}{(t+\rho(x,y))^{1+\gamma}}$,

for all x,x' and $y\in X$, $\rho(x,x')\leqslant\frac{1}{2A}[t+\rho(x,y)]$,

(iv) $\int S_t(x,y)d\mu(y) = 1, \quad \int S_t(x,y)d\mu(x) \leqslant C.$

Lemma 3.16 *Let $\{S_t(x,y)\}_{t>0}$ be the family of functions as in Lemma 3.15. Then for any $0<\beta'<\beta$, and for $\psi\in\mathscr{M}(\beta,\gamma)$,*

$$\psi_t(z) = \int S_t(z,y)\psi(y)d\mu(y)$$

converges to $\psi(z)$ in $\mathscr{M}(\beta',\gamma)$ as t goes to zero.

Corollary 3.17 *Let $k(z)$ belong to the closure of $\mathscr{M}(\beta,\gamma)$ in $L^p(X)$, $1\leqslant p<\infty$. Then, if*

$$k_t(z) = \int S_t(z,y)k(y)d\mu(y),$$

where $\{S_t(z,y)\}_{t>0}$ as in Lemma 3.15, we have

$$\lim_{t\to 0}\|k_t - k\|_p = 0,$$

moreover, $|k(z)|\leqslant Ck^*(z)$ for almost every where on $z\in X$.

The proofs of Lemma 3.15, Lemma 3.16 and Corollary 3.17 are simple computations and omited here.

4 Some properties of $\widetilde{H}_p$

In the following, we assume that for all $\eta>0$, $C_0^\eta(X)$ is dense in $L^1(X)$, so $C_0^\eta(X)$ is dense in $L^p(X)$ for $1\leqslant p<\infty$. Furthermore for all $0<\beta<\theta,\gamma>0$, $\mathscr{M}(\beta,\gamma)$ is dense in $L^p(X)$, $1\leqslant p<\infty$, see [2].

Theorem 4.1 *For any $f(z)$ in $L^p(X)$, $1<p<\infty$, the dstribution f induced by $f(z)$ in $\widetilde{H}_p$ and there exists a constant C independent of $f(z)$ such that*

$$\|f\|_{\widetilde{H}_p}\leqslant C\|f\|_p.$$

Proof For $x\in X,\psi\in\mathscr{M}(x,d,\beta,\gamma)$, $\|\psi\|_{\mathscr{M}(x,d,\beta,\gamma)}\leqslant 1$, we have

$$\begin{aligned}|\langle f,\psi\rangle| &= \left|\int f(z)\psi(z)d\mu(z)\right| \\ &\leqslant \int |f(z)|\frac{d^\gamma}{(d+\rho(x,z))^{1+\gamma}}d\mu(z)\leqslant CM(f)(x),\end{aligned}$$

where M is the Hardy-Littlewood maximal function. So $f^*(x)\leqslant CM(f)(x)$, and $\|f\|_{\widetilde{H}_p}$
$\leqslant C\|M(f)\|_p\leqslant C\|f\|_p$, for $1<p<\infty$.

Theorem 4.2 *If a distribution $f\in\widetilde{H}_p$ for $1<p<\infty$, then there exists a function $\tilde{f}(z)$ such that $|\tilde{f}(z)|\leqslant Cf^*(z)$ and*

$$\langle f,\psi\rangle = \int\tilde{f}(z)\psi(z)d\mu(z)$$

for every $\psi\in\mathscr{M}(\beta,\gamma)$.

Proof For $\varepsilon>0$, let $\{\varphi_k^\varepsilon(z)\}$ be a partition of the unity for X such that supp $\varphi_k^\varepsilon(z)\subset B(z_k^\varepsilon,\varepsilon)$ and for any give n $z\in X$, $\varphi_k^\varepsilon(z)\neq 0$ holds for no more than N values of k. If $\{S_t(z,y)\}_{t>0}$ is the family of functions in Lemma 3.15, then when t is small enough, for $\psi\in\mathscr{M}(\beta',\gamma)$, $\beta'>\beta$, we have

$$\psi_t(z) = \int S_t(z,y)\psi(y)d\mu(y) = \lim_{\varepsilon\to 0}\sum\psi(z_k^\varepsilon)\int S_t(z,y)\varphi_y^\varepsilon(y)d\mu(y), \qquad (4.3)$$

where the limit is taken in $\mathscr{M}(\beta',\gamma)$.

For $z \in X$, let $0<\varepsilon<(2A)^{-1}$, we have

$$
\begin{aligned}
&|\psi_t(z)-\sum_k \psi(z_k^t)\int S_t(z,y)\phi_k^t(y)d\mu(y)| \\
&\leqslant \sum \int S_t(z,y)\phi_k^t(y)\,|\psi(y)-\psi(z_k^t)|\,d\mu(y) \\
&\leqslant C\sum\int S_t(z,y)\phi_k^t(y)\|\psi\|\frac{\varepsilon^{\beta}}{(1+\rho(x_0,z_k^t))^{1+\beta+\gamma}}d\mu(y) \\
&\leqslant C\|\psi\|\varepsilon^{\beta}/(1+\rho(x_0,z))^{1+\beta+\gamma}\leqslant C\|\psi\|\varepsilon^{\beta}/(1+\rho(x_0,z))^{1+\gamma}. \qquad (4.4)
\end{aligned}
$$

The last inequality comes from the fact that $1+\rho(x_0,z)\leqslant 1+A[\rho(x_0,z_k^t)+\rho(z_k^t,z)]\leqslant 1+A[\rho(x_0,z_k^t)+Ct]\leqslant C[1+\rho(x_0,z_k^t)]$.

For $z,z'\in X$, and $\rho(z,z')\leqslant\frac{1}{2A}[1+\rho(x_0,z)]$, we have

$$
\begin{aligned}
&|\psi_t(z)-\sum_k\psi(z_k^t)\int S_t(z,y)\phi_k^t(y)d\mu(y)-\psi_t(z') \\
&\quad+\sum_k\psi(z_k^t)\int S_t(z',y)\phi_k^t(y)d\mu(y)| \\
&\leqslant \sum\int|S_t(z,y)-S_t(z',y)|\,|\psi(y)-\psi(z_k^t)|\phi_k^t(y)d\mu(y) \\
&\leqslant C\|\psi\|\frac{\varepsilon^{\beta}\rho(z,z')^{\beta}}{t^{\beta}(1+\rho(x_0,z))^{\beta}}\frac{1}{(1+\rho(x_0,z))^{1+\gamma}}. \qquad (4.5)
\end{aligned}
$$

So (4.4) and (4.5) imply (4.3).

Now, by Lemma 3.16, given $\lambda>0$, for $\psi\in\mathcal{M}(\beta',\gamma)$,

$$|\langle f,\psi\rangle|<|\langle f,\psi_t\rangle|+\lambda \qquad (4.6)$$

holds for t small enough. On the other hand, by (4.3), we have

$$|\langle f,\psi_t\rangle|<\sum|\psi(z_k^t)|\cdot|\langle f,\int S_t(\cdot,y)\phi_k^t(y)d\mu(y)\rangle|+\lambda \qquad (4.7)$$

for ε small enough. We can assume that $\varepsilon<t$. Let $A_k^t(z)=\int S_t(z,y)\phi_k^t(y)d\mu(y)$. It is easy to show that there exists a constant C such that $(C\int\phi_k^t(y)d\mu(y))^{-1}A_k^t(\cdot)\in\mathcal{M}(x,t,\beta,\gamma)$ for all $x\in B(z_k^t,\varepsilon)$. Therefore, we get for every $z\in B(z_k^t,\varepsilon)$,

$$|\langle f,A_k^t\rangle|\leqslant Cf^*(z)\int\phi_k^t(y)d\mu(y)\leqslant C\int f^*(y)\phi_k^t(y)d\mu(y).$$

Going back to (4.7), we get

$$|\langle f,\psi_t\rangle|\leqslant C\sum|\psi(z_k^t)|\int f^*(z)\phi_k^t(z)d\mu(z)+\lambda.$$

On the other hand, since $\psi\in\mathcal{M}(\beta',\gamma)$, for every $z\in B(z_k^t,\varepsilon)$,

$$|\psi(z_k^t)-\psi(z)|\leqslant\|\psi\|\left(\frac{\varepsilon}{1+\rho(z,x_0)}\right)^{\beta}\frac{1}{(1+\rho(z,x_0))^{1+\gamma}}.$$

So,

$$|\psi(z_k^t)|\leqslant|\psi(z)|+\|\psi\|\varepsilon^{\beta}/(1+\rho(z,x_0))^{1+\beta+\gamma},$$

for every $z\in B(z_k^i,\varepsilon)$. Thus

$$|\langle f,\psi_i\rangle|\leqslant C\sum_k\int f^*(z)|\psi(z)|\phi_k^i(z)d\mu(z)$$
$$+C\|\psi\|\varepsilon^{\beta}\sum_k\int f^*(z)\phi_k^i(z)\frac{d\mu(z)}{(1+\rho(z,x_0))^{1+\gamma}}+\lambda$$
$$\leqslant C\int f^*(z)|\psi(z)|d\mu(z)+C\|\psi\|\varepsilon^{\beta}\int f^*(z)/(1+\rho(z,x_0))^{1+\gamma}d\mu(z)+\lambda.$$

Since ε is small, we obtain

$$|\langle f,\psi_i\rangle|\leqslant C\int f^*(z)|\psi(z)|d\mu(z)+\lambda. \tag{4.8}$$

From (4.6) and (4.8), and taking into account that λ is any positive number, we get

$$|\langle f,\psi\rangle|\leqslant C\int f^*(z)|\psi(z)|d\mu(z), \tag{4.9}$$

for any $\psi\in\mathscr{M}(\beta',\gamma)$, $f\in\tilde{H}_p$, $\beta<\beta'$, $1<p<\infty$. Because $\mathscr{M}(\beta',\gamma)$ is dense in $L^{p'}(X)$, $\frac{1}{p'}+\frac{1}{p}=1$, so the distribution f can be extended into a continuous linear functional on $L^{p'}(X)$. Thus there exists a unique function $\tilde{f}(z)$ in $L^p(X)$, such that for any $g(z)$ in $L^{p'}(X)$, $\langle f,g\rangle=\int\tilde{f}(z)g(z)d\mu(z)$. Specially, for $\psi\in\mathscr{M}(\beta,\gamma)$, we have $\langle f,\psi\rangle=\int\tilde{f}(z)\psi(z)d\mu(z)$. By Corollary 3.17, we have $|\tilde{f}(z)|\leqslant C\tilde{f}^*(z)=Cf^*(z)$, this ends the proof of Theorem (4.2).

From Theorem 4.1 and Theorem 4.2, we get that $\tilde{H}_p$ can be identified with L^p for $1<p<\infty$. By the Lemma 3.7 and Theorem 4.2, we can prove the following results as in [4].

Theorem 4.10 *For $1\leqslant q<\infty$ and $(1+\beta)^{-1}<p\leqslant 1$, we have that $L^q\cap\tilde{H}_p$ is dense in $\tilde{H}_p$.*

Lemma 4.11 *If $f(z)\in L^q(X)\cap\tilde{H}_p$, $(1+\beta)^{-1}<p\leqslant 1$, $1\leqslant q<\infty$, then with the same notations used in Lemma 3.7, we have*

(4.11.i) *if* $m_n=[\int\varphi_n(z)d\mu(z)]^{-1}\int f(y)\varphi_n(y)d\mu(y)$, *then* $|m_n|\leqslant Ct$,

(4.11.ii) *if* $b_n(z)=[f(z)-m_n]\varphi_n(z)$, *then the distribution induced by* $b_n(z)$ *coincided with* b_n,

(4.11.iii) *the series* $\Sigma_n b_n(z)$ *converges for every* $z\in X$ *and in* $L^q(X)$, *if* $\Sigma_n b_n(z)=b(z)$, *then the distribution induced by* $b(z)$ *coincided with* b,

(4.11.iv) *let* $g(z)=f(z)-b(z)$, *then*

$$g(z)=f(z)\chi_{\Omega^c}(z)+\sum m_n\varphi_n(z),$$
$$|g(z)|\leqslant Ct,$$

moreover, the distribution induces by $g(z)$ *coincided with* g.

5 Atomic decomposition of $\tilde{H}_p$ and atomic H^p space for $(1+\beta)^{-1}<p\leqslant 1$

Definition 5.1 Let $0<\beta<\theta$ and $(1+\beta)^{-1}<p\leqslant 1$. We say that a function $a(z)$ is a p-

atom, if there exists a ball B such that

(i) supp $a(z)\subset B$,

(ii) $\|a\|_\infty \leqslant \mu(B)^{-\frac{1}{p}}$,

(iii) $\int a(z)d\mu(z) = 0$.

Lemma 5.2 *Let $h(z)\in L^q(X)$, $1\leqslant q\leqslant\infty$, with support in $B=B(x_0,r)$ and $\int h(z)d\mu(z) = 0$. Then $h\in\tilde{H}_p$ for $(1+\beta)^{-1}<p\leqslant 1$, and*

$$\|h\|_{\tilde{H}_p} \leqslant C\mu(B)^{\frac{1}{p}-\frac{1}{q}}\|h\|_q,$$

where C does not depend on $h(z)$.

Proof Assume that $\psi\in\mathscr{M}(x,d,\beta,\gamma)$, $\|\psi\|_{\mathscr{M}(x,d,\beta,\gamma)}\leqslant 1$. Let $x\in B(x_0,2Ar)$, then

$$\left|\int h(z)\psi(z)d\mu(z)\right| \leqslant \int |h(z)| \frac{d^\gamma}{(d+\rho(x,z))^{1+\gamma}}d\mu(z) \leqslant CM(h)(x),$$

so

$$h^*(x) \leqslant CM(h)(x).$$

Now considering the case $x\notin B(x_0,2Ar)$, then $\rho(x,x_0)\geqslant 2Ar$, for any $z\in B(x_0,r)$, we have $\rho(z,x_0)\leqslant r\leqslant\frac{1}{2A}[d+\rho(x_0,x)]$, and

$$\begin{aligned}
|\langle h,\psi\rangle| &\leqslant \int |h(z)|\,|\psi(z)-\psi(x_0)|d\mu(z)\\
&\leqslant \|h\|_q\left(\int_B\left(\frac{r^\beta}{(d+\rho(x,x_0))^\beta}\frac{d^\gamma}{(d+\rho(x,x_0))^{1+\gamma}}\right)^{q'}d\mu(z)\right)^{\frac{1}{q'}}\\
&\leqslant \|h\|_q\mu(B)^{\frac{1}{q'}}\frac{r^\beta}{\rho(x,x_0)^{1+\beta}}.
\end{aligned}$$

(in which $\frac{1}{q'}+\frac{1}{q}=1$). So,

$$h^*(x) \leqslant \|h\|_q\mu(B)^{\frac{1}{q'}}\frac{r^\beta}{\rho(x,x_0)^{1+\beta}}.$$

Thus, we have

$$\begin{aligned}
\|h\|_{\tilde{H}_p}^p &\leqslant C\int_{B(x_0,2Ar)} M(h)(x)^p d\mu(x) + \int_{B(x_0,2Ar)}\left(\|h\|_q\mu(B)^{\frac{1}{q'}}\frac{r^\beta}{\rho(x,x_0)^{1+\beta}}\right)^p d\mu(x)\\
&\leqslant C\mu(B)^{1-\frac{p}{q}}\|h\|_q^p.
\end{aligned}$$

This ends the proof of Lemma 5.2.

By Lemma 5.2, we obtain

Lemma 5.3 *Let $a(z)$ be a p-atom, and $(1+\beta)^{-1}<p\leqslant 1$, then the distribution a on $\mathscr{M}(\beta,\gamma)$ induced by $a(z)$ belongs to $\tilde{H}_p$ and*

$$\int a^*(z)^p d\mu(z) \leqslant C < \infty,$$

where C is independent of the p-atom.

Theorem 5.4 *Let* $(1+\beta)^{-1}<p\leqslant 1$. *For any sequence* $\{a_i(z)\}$ *of* p*-atoms and a sequence* $\{\lambda_i\}$ *of numbers satisfying* $\sum_i|\lambda_i|^p<\infty$, *then there exists* $f\in\tilde{H}_p$ *such that* $f=\sum_i\lambda_i a_i$ *and*

$$\int f^*(z)^p d\mu(z)\leqslant C\sum_i|\lambda_i|^p.$$

Proof For any large positive integers n and m with $n<m$, by Lemma 5.3 we get

$$\int\Big(\sum_{i=n}^{m}\lambda_i a_i\Big)^*(z)^p d\mu(z)\leqslant C\sum_{i=n}^{m}|\lambda_i|^p.$$

Then, by Theorem 2.1, there exists $f\in(\mathscr{M}(\beta,\gamma))'$ such that $f=\sum_i\lambda_i a_i$ and $f^*(z)\leqslant\sum_i|\lambda_i|a^*(z)$. This implies that Theorem 5.4 is true.

In the following, we shall prove that if $f\in\tilde{H}_p$ then f can be expanded into a series of multiples of p-atoms. In order to do this, we need the following Lemma.

Lemma 5.5 *Let* $h(z)$ *in* $L^2(X)$, $|h(z)|\leqslant 1$. *Assume that for some* $(1+\beta)^{-1}<q<1$, $h\in\tilde{H}_q$. *Then for every* p *with* $q<p\leqslant 1$, *there exist a sequence of* p*-atoms* $\{a_k(z)\}$, *and a numerical sequence* $\{\lambda_k\}$ *such that* $h=\sum\lambda_k a_k$, *and*

$$\Big(\sum|\lambda_k|^p\Big)^{\frac{1}{p}}\leqslant C\|h\|_{\tilde{H}_p}.$$

Theorem 5.6 *For* $0<\beta<\theta$, $(1+\beta)^{-1}<p\leqslant 1$. *If* $f\in\tilde{H}_p$, *then there exist a sequence of* p*-atoms* $\{a_n\}$ *and a numerical sequence* $\{\lambda_n\}$ *such that* $f=\sum\lambda_n a_n$, *and there exist two constants* C' *and* C'' *independent of* f *such that*

$$C'\|f\|_{\tilde{H}_p}\leqslant\Big(\sum_k|\lambda_n|^p\Big)^{\frac{1}{p}}\leqslant C''\|f\|_{\tilde{H}_p}.$$

Theorem 5.7 *For* $0<\beta<\theta,\gamma>0$, $(1+\beta)^{-1}<p\leqslant 1$, $\mathscr{M}_0(\beta,\gamma)$ *is dense in* $\tilde{H}_p$.

The proofs of Lemma 5.5, Theorem 5.6 and Theorem 5.7 are similar to the proofs in [4], we omit the details.

In the following, we recall the basic theory of atomic H^p spaces defined in [1]. Let $0<\beta<\infty$, lip(β) denote the set of all functions $\psi(z)$ defined on X such that there exists a constant C satisfying

$$|\psi(x)-\psi(y)|\leqslant C\rho(x,y)^\beta,$$

for every x and y in X. The least constant C for which this condition holds is denoted by $\|\psi\|_{\mathrm{lip}(\beta)}$. It is easy to prove that lip(β) with this norm $\|\cdot\|_{\mathrm{lip}(\beta)}$ is a Banach space. When $\beta=0$, lip(0) is defined as the Banach space of all function ψ in BMO such that for every ball B and $\varepsilon>0$ there exsts a bounded continuous function φ satisfying

$$\int_B|\psi(z)-\varphi(z)|d\mu(z)<\varepsilon,$$

endowed with the norm $\|\cdot\|_{\mathrm{BMO}}$.

Let $a(z)$ be a p-atom and $\psi(z)$ in lip$(\frac{1}{p}-1)$. Then $\langle a,\psi\rangle=\int a(z)\psi(z)d\mu(z)$ defined a linear functional on lip$(\frac{1}{p}-1)$, and

$$|\langle a,\psi\rangle| \leqslant C\|\psi\|_{\mathrm{lip}(\frac{1}{p}-1)}. \tag{5.8}$$

where C is independent of $\psi(z)$ and $a(z)$. Moreover, for every sequence of p-atoms $\{a_i(z)\}$ and every numerical sequence $\{\lambda_i\}$, we have

$$|\sum_i \lambda_i\langle a_i,\psi\rangle| \leqslant C\sum|\lambda_i|\,\|\psi\|_{\mathrm{lip}(\frac{1}{p}-1)}.$$

This shows that

$$\langle f,\psi\rangle = \sum \lambda_i\langle a_i,\psi\rangle \tag{5.9}$$

is a bounded linear functional on $\mathrm{lip}(\frac{1}{p}-1)$. The norm of f as an element of the dual space of $\mathrm{lip}(\frac{1}{p}-1)$ is bounded by $C(\sum_i|\lambda_i|^p)^{\frac{1}{p}}$.

We define H^p as the linear space of all bounded linear fnctionals f on $\mathrm{lip}(\frac{1}{p}-1)$ which can be represented as (5.9) where $\{a_i\}$ is a sequence of p-atoms and $\{\lambda_i\}$ is a numerical sequence such that $\sum_i|\lambda_i|^p<\infty$. For $f\in H^p$, we define

$$\|f\|_{H^p} = \inf\{\sum_i|\lambda_i|^p\}^{\frac{1}{p}},$$

where the infimum is taken over all possible representation of f of the form (5.9). Using the methed in [4], we can prove

Theorem 5.10 *Let* $0<\beta<\theta$ *and* $(1+\beta)^{-1}<p\leqslant 1$. *For every* f *in* H^p, *we denote by* $\tilde{f}$ *the restriction of* f *to* $\mathscr{M}(\beta,\gamma)$. *Then* $\mathscr{T}(f)=\tilde{f}$ *defines an injective linear transformation from* H^p *onto* $\tilde{H}_p$. *Moreover, there exist two positive numbers* C_1 *and* C_2 *such that*

$$C_1\|f\|_{H^p} \leqslant \|\tilde{f}\|_{\tilde{H}_p} \leqslant C_2\|f\|_{H^p}$$

holds for every f *in* H^p.

Remark 5.11 The theory above is studied for spaces of homogeneous type X with infinite measure. In fact, we can prove that the all results in this paper is true for the case that $\mu(X)<\infty$. We omit the details.

Remark 5.12 From the Theorem 5.10, we can see that for fixed p, $(1+\theta)^{-1}<p\leqslant 1$, $\tilde{H}_p$ do not depend on the choice of β and γ which define the basic test function space $\mathscr{M}(\beta,\gamma)$, as long as β and γ satisfy $0<\beta<\theta,\gamma>0$ and $(1+\beta)^{-1}<p$.

Remark 5.13 From the Theorem 5.10, the space $\tilde{H}_p$ can be identified with the atomic H^p space, so the theory above essentially give a maximal function characterization of atomic H^p on spaces of homogeneous type.

Remark 5.14 For the case $X=R^n$, Hardy space H^p have definition for all $0<p\leqslant 1$. In order to study the similiar characterization of H^p for R^n, we need some new test function spaces. We will discuss these details elsewhere.

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