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A Property of *m*-Tuplings Morse Sequence

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Abstract: The m-tuplings Morse sequence, as a fixed point of a constant length substitution, was discussed. An equivalent definition was given by the property of the m-adic development of the integers. Using the combinatorial properties of the m-tuplings Morse sequence, we mainly obtain this sequence is an admissible sequence and the other two sequences generated by it are also admissible sequences, which extends the conclusion that the Thue-Morse sequence is an admissible sequences.

Key words: m-tuplings Morse sequence; admissible sequence; Thue-Morse sequence

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0 Introduction

Let us first introduce the m-tuplings Morse sequence. Let $S=\{0, 1\}$ be a two letters alphabet and S^* be the free monoid generated by S. Consider the following homomorphism on S^*, $\sigma: 0\mapsto 01^{m-1}, 1\mapsto 10^{m-1}$, where 0^{m-1} (respectively 1^{m-1}) represents $m-1$ consecutive 0 (respectively 1) digits. Thus $\sigma^n(0)$ as $n\to\infty$, we define an infinite sequence $u=u_0u_1\cdots u_n\cdots\in\{0,1\}^{\mathbf{N}}$, which is called the m-tuplings Morse sequence. Obviously, the case $m=2$ yields the Thue-Morse sequence and this sequence has been studied intensively by many authors[1-6].

It is easy to check that u satisfies the following relations:

① $u_0=0$, $u_{nm}=u_n$, $u_{nm+k}=1-u_k$, where $1\leqslant k\leqslant m-1$;

② at position $k\cdot m^n$ of the sequence occurs $\sigma^n(0)$, if $u_k=0$ and $\sigma^n(1)$, if $u_k=1$;

③ $u_{c\cdot m^n}=1$ for any $n\geqslant 0$ and $c\in\{1,\cdots,m-1\}$.

Given two sequences $\{\eta_i\}_{i=1}^{\infty}$ and $\{\varepsilon_i\}_{i=1}^{\infty}$, we write $\eta_1\eta_2\cdots<\varepsilon_1\varepsilon_2\cdots$ if there exists an integer $n\geqslant 1$ such that $\eta_i=\varepsilon_i$ for all $1\leqslant i<n$ but $\eta_n<\varepsilon_n$. This is a complete ordering. We write for brevity $\overline{\varepsilon_i}$ instead of $l-\varepsilon_i$ and also $\bar{\varepsilon}$ instead of $\overline{\varepsilon_1\varepsilon_2\cdots}$, where $\varepsilon=\varepsilon_1\varepsilon_2\cdots$ is a finite or infinite sequence of integers in $\{0, 1,\cdots, l\}$.

Let us first give the concept of a sequence is admissible.

Definition 1 A sequence $\varepsilon_1, \varepsilon_2,\cdots$ of integers in $\{0, 1, \cdots, l\}$ is called admissible if the following two conditions are fulfilled:

$$\varepsilon_{n+1}\varepsilon_{n+2}\cdots<\varepsilon_1\varepsilon_2\cdots \text{ whenever } \varepsilon_n<l \tag{1}$$

and

$$\overline{\varepsilon_{n+1}\varepsilon_{n+2}\cdots}<\varepsilon_1\varepsilon_2\cdots \text{ whenever } \varepsilon_n>0 \tag{2}$$

It is easy to construct an admissible sequence. For example, if ε begins with $N\geqslant 2$ consecutive l digits and if there are neither N consecutive l digits nor N consecutive 0 digits later, the sequence ε is admissible.

Recently, Komornik[7] *et al* showed that the Thue-Morse sequence is an admissible sequence and in Ref. [8] even testified that Thue-Morse sequence is the smallest admissible sequence. m-tuplings Morse sequence is a more generalized sequence contains Thue-Morse sequence, in this paper we obtain that the m-tuplings Morse sequence is an admissible sequence and some other two sequences generated by it are also admissible sequences. The following is the main results.

Theorem 1 The sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ is admissible.

Theorem 2 If $l=2p+1$ is an odd integer, then the sequence $\{p+\varepsilon_i\}_{i=1}^{\infty}$ is admissible.

Theorem 3 If $l=2p$ is an even integer, then the sequence $\{p+\varepsilon_i-\varepsilon_{i-1}\}_{i=1}^{\infty}$ is admissible.

1 The Property of the *m*-Tuplings Morse Sequence

The m-tuplings Morse sequence can be defined by a property of the m-adic development of the integers. Moreover this property is simple enough to be recognizable by a finite automaton. Let us denote by $\mathbf{N}_0$ (respectively $\mathbf{N}_1$) the set of integers n such that $(n+1)$-th letter of the m-tuplings Morse sequence is 0 (respectively 1).

For any integer n, we denote by $B_m(n)=\#\{i, n_i\neq 0\}$, if $n=\sum_{i\geqslant 0} n_i m^i$, $n_i\in\{0,1,\cdots,m-1\}$.

Proposition 1 With the notations above, we have

$$\mathbf{N}_0=\{n\in\mathbf{N}, B_m(n)\text{ is even}\},$$
$$\mathbf{N}_1=\{n\in\mathbf{N}, B_m(n)\text{ is odd}\}$$

Proof Obviously $\mathbf{N}_0$ and $\mathbf{N}_1$ form a partition of $\mathbf{N}$. It is thus enough to show by induction over r the following property:

$$(\mathrm{P}_r)\qquad \begin{cases} n\in\mathbf{N}_0 \text{ and } n<m^r \Rightarrow B_m(n) \text{ is even} \\ n\in\mathbf{N}_1 \text{ and } n<m^r \Rightarrow B_m(n) \text{ is odd.}\end{cases}$$

Note that P_0 is obvious. Suppose that P_r is true and consider $n\in\mathbf{N}_0$ (respectively $\mathbf{N}_1$) such that $m^r\leqslant n<m^{r+1}$. Then $n=k\cdot m^r+n'$ with $n'\in\mathbf{N}_1$ (respectively $\mathbf{N}_0$), where $n'<m^r$ and $1\leqslant k\leqslant m-1$. As $B_m(n)=B_m(k\cdot m^r+n')=1+B_m(n')$, P_{r+1} follows.

In order to prove Theorem 1, consider the m-adic expansions

$$i=\delta_k m^k+\cdots+\delta_0,$$

where $\quad\delta_0,\cdots,\delta_k\in\{0, 1,\cdots, m-1\}$.

Let i' be the smallest j such that $\delta_j\neq 0$ and $\tilde{i}$ be the number of j such that $\delta_j\neq 0$.

Proof of Theorem 1

ⓐ Let $\varepsilon_i=0$ for some i. Then $\tilde{i}\geqslant 2$ and we can write $i=c_1 m^{n_1}+c_2 m^{n_2}+j$ with $n_1>n_2$, $0\leqslant j<m^{n_2}$ and c_1, $c_2\in\{1,\cdots, m-1\}$, thus we have $\varepsilon_j=\varepsilon_i=0$, if $j\neq 0$, we will verify that

$$\varepsilon_{i+1}\varepsilon_{i+2}\cdots<\varepsilon_{j+1}\varepsilon_{j+2}\cdots \tag{3}$$

We discuss it in two cases:

① If $n_1\geqslant n_2+2$, then we have

$$\varepsilon_{i+k}=\varepsilon_{j+k}\text{ for }1\leqslant k\leqslant m^{n_2}-j-1,$$

and

$$\varepsilon_{i+m^{n_2}-j}=\varepsilon_{c_1 m^{n_1}+(c_2+1)m^{n_2}}=0<1=\varepsilon_{m^{n_2}}=\varepsilon_{j+m^{n_2}-j}.$$

Thus (3) is satisfied.

② If $n_1=n_2+1$, when $c_2<m-1$ we have

$$\varepsilon_{i+k}=\varepsilon_{j+k}\text{ for }1\leqslant k\leqslant m^{n_2}-j-1,$$

and

$$\varepsilon_{i+m^{n_2}-j}=\varepsilon_{c_1 m^{n_1}+(c_2+1)m^{n_2}}=0<1=\varepsilon_{m^{n_2}}=\varepsilon_{j+m^{n_2}-j},$$

which implies (3) again. When $c_2=m-1$ we have

$$\varepsilon_{i+k}=\varepsilon_{j+k}\text{ for }1\leqslant k\leqslant 2m^{n_2}-j-1,$$

and

$$\varepsilon_{i+2m^{n_2}-j}=\varepsilon_{(c_1+1)m^{n_1}+m^{n_2}}=0<1=\varepsilon_{2m^{n_2}}=\varepsilon_{j+2m^{n_2}-j},$$

which implies (3) again.

If $j=0$, in the same way as above we can prove (1), if not iterating (3) eventually we can also obtain (1).

ⓑ Let $\varepsilon_i=1$ for some i, we can write $i=c_3 m^n+j$ with $0\leqslant j<m^n$, thus $\varepsilon_j=0$ if $j\neq 0$. We can get

$$\overline{\varepsilon_{i+k}}=\varepsilon_{j+k}\text{ for }1\leqslant k\leqslant m^n-j-1,$$

and

$$\overline{\varepsilon_{i+m^n-j}}=\overline{\varepsilon_{(c_3+1)m^n}}=0<1=\varepsilon_{m^n}=\varepsilon_{j+m^n-j}.$$

Hence

$$\overline{\varepsilon_{i+1}\varepsilon_{i+2}\cdots}<\varepsilon_{j+1}\varepsilon_{j+2}\cdots \tag{4}$$

If $j=0$, in the same way we can prove (2), if not, we also can get (2) by combining (4) and (3).

2 The Admissible Property of the Sequence $\{p+\varepsilon_i\}_{i=1}^{\infty}$ for $l=2p+1$

Lemma 1 We have

$$\varepsilon_{n+1}\varepsilon_{n+2}\cdots<\varepsilon_1\varepsilon_2\cdots\text{ and }\overline{\varepsilon_{n+1}\varepsilon_{n+2}\cdots}<\varepsilon_1\varepsilon_2\cdots$$

for all $n\geqslant 1$.

Proof We can easily get the above relations for $n=1, 2,\cdots, m$. When $n\geqslant m+1$, considering the m-adic expansion of n,

$$n=\delta_k m^k+\cdots+\delta_0.$$

Let

$$t=\delta_k m^k+\cdots+\delta_1 m.$$

If $\varepsilon_n=1$, then we have $\varepsilon_t=0$ and $\varepsilon_{n+1}\varepsilon_{n+2}\cdots\leqslant\varepsilon_{t+1}\varepsilon_{t+2}\cdots$. By the Theorem 1 we have $\varepsilon_{t+1}\varepsilon_{t+2}\cdots<\varepsilon_1\varepsilon_2\cdots$, hence we have the first relation. The second relation is also satisfied directly by Theorem 1 because $\varepsilon_n=1$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ is an admissible sequence.

If $\varepsilon_n=0$, then we have $\varepsilon_t=1$ and $\varepsilon_{n+1}\varepsilon_{n+2}\cdots\geqslant\varepsilon_{t+1}\varepsilon_{t+2}\cdots$, that is $\overline{\varepsilon_{n+1}\varepsilon_{n+2}\cdots}\leqslant\overline{\varepsilon_{t+1}\varepsilon_{t+2}\cdots}$. From Theorem 1 we know that when $\varepsilon_t=1$, we have $\overline{\varepsilon_{t+1}\varepsilon_{t+2}\cdots}<\varepsilon_1\varepsilon_2\cdots$. Combining with this we obtain $\overline{\varepsilon_{n+1}\varepsilon_{n+2}\cdots}<\varepsilon_1\varepsilon_2\cdots$, hence we have the second relation. The first relation is also satisfied directly by Theorem 1.

Proof of Theorem 2

The case $l=1$ has already been proved in Theorem 1, we can thus assume that $l\geqslant 3$.

In order to prove $\{p+\varepsilon_i\}_{i=1}^{\infty}$ is admissible, we only need to prove

$$p+\varepsilon_{n+i}<(p+\varepsilon_i) \text{ and } \overline{p+\varepsilon_{n+i}}<p+\varepsilon_i,$$

which equivalent to

$$p+\varepsilon_{n+i}<p+\varepsilon_i \text{ and } p+1-\varepsilon_{n+i}<p+\varepsilon_i,$$

which has been proved in Lemma 1.

3 The Admissible Property of the Sequence $\{p+\varepsilon_i-\varepsilon_{i-1}\}_{i=1}^{\infty}$ for $l=2p$

We first consider the case $l=2$, set

$$\lambda_i=1+\varepsilon_i-\varepsilon_{i-1}.$$

Then we have two equivalent definitions of the sequence λ_i.

Lemma 2 We have $\lambda_1=2$, $\lambda_{m^{n+1}}=3-\lambda_{m^n}$ for $n=0,1,\cdots$ and $\lambda_{c\cdot m^n+k}=2-\lambda_k$, where $c\in\{1,\cdots,m-1\}$ and $1\leqslant k<m^n$.

Proof We have $\varepsilon_0=0$ and $\varepsilon_{m^n}=1$ for any non-negative integer n. With this two results we have

$$\lambda_1=1+\varepsilon_1-\varepsilon_0=2.$$

Using the relation

$$\lambda_i=1+\varepsilon_i-\varepsilon_{i-1}=\varepsilon_i+\varepsilon_{mi-1}.$$

We have

$$\begin{aligned}\lambda_{m^{n+1}}+\lambda_{m^n}&=\varepsilon_{m^{n+1}}+\varepsilon_{m^{n+2}-1}+\varepsilon_{m^n}+\varepsilon_{m^{n+1}-1}\\&=1+(1-\varepsilon_{m^{n+1}-1})+1+\varepsilon_{m^{n+1}-1}\\&=3\end{aligned}$$

and

$$\begin{aligned}\lambda_{c\cdot m^n+k}+\lambda_k&=\varepsilon_{c\cdot m^n+k}+\varepsilon_{c\cdot m^{n+1}+mk-1}+\varepsilon_k+\varepsilon_{mk-1}\\&=(1-\varepsilon_k)+(1-\varepsilon_{mk-1})+\varepsilon_k+\varepsilon_{mk-1}\\&=2\end{aligned}$$

In order to get another equivalent definition, consider the m-adic expansions

$i=\delta_k m^k+\cdots+\delta_0$, where $\delta_0,\cdots,\delta_k\in\{0,1,\cdots,m-1\}$. Let i' be the smallest j such that $\delta_j\neq 0$ and $\tilde{i}$ be the number of j such that $\delta_j\neq 0$.

Lemma 3 When $\delta_{i'}=1$ we have

$$\lambda_i=\begin{cases}0, & \text{if } i' \text{ is even and } \tilde{i} \text{ is even}\\ 2, & \text{if } i' \text{ is even and } \tilde{i} \text{ is odd}\\ 1, & \text{if } i' \text{ is odd}\end{cases}$$

When $\delta_{i'}\geqslant 2$ we have

$$\lambda_i=\begin{cases}0, & \text{if } i' \text{ is odd and } \tilde{i} \text{ is even}\\ 2, & \text{if } i' \text{ is odd and } \tilde{i} \text{ is odd}\\ 1, & \text{if } i' \text{ is even}\end{cases}$$

Lemma 4 We have

$$\lambda_{i+1}\lambda_{i+2}\cdots<\lambda_1\lambda_2\cdots \tag{5}$$

for all $i\geqslant 1$.

Proof Let $i=\delta_k m^k+\cdots+\delta_0$, where $\delta_0,\cdots,\delta_k\in\{0,1,\cdots,m-1\}$. We prove by induction on $\tilde{i}=\#\{j,\delta_j\neq 0\}$.

Case 1 $\tilde{i}=1$, that is $i=c\cdot m^n$ for some $n\geqslant 0$ and $c\in\{1,\cdots,m-1\}$. Then $\lambda_{i+1}=1$ if $n=0$ and $\lambda_{i+1}=0$ if $n>0$. Hence $\lambda_{i+1}<2=\lambda_1$ in all cases. Then we obtain (4).

Case 2 $\tilde{i}=2$, that is $i=c_1\cdot m^{n_1}+c_2\cdot m^{n_2}$ for $n_1>n_2\geqslant 0$ and $c_1, c_2\in\{1,\cdots,m-1\}$.

If $n_1\geqslant n_2+2$, from Lemma 3 we have

$$\lambda_{i+k}=\lambda_k \text{ for } k=1,\cdots,m^{n_2}-1.$$

But

$$\lambda_{i+m^{n_2}}=\lambda_{c_1\cdot m^{n_1}+(c_2+1)m^{n_2}}=\begin{cases}0, & \text{if } n_2 \text{ is odd}\\ 1, & \text{if } n_2 \text{ is even}\end{cases}$$

and

$$\lambda_{m^{n_2}}=\begin{cases}1, & \text{if } n_2 \text{ is odd}\\ 2, & \text{if } n_2 \text{ is even}\end{cases}$$

Then we have $\lambda_{i+m^{n_2}}<\lambda_{m^{n_2}}$, which proves (5).

If $n_1=n_2+1$, when $c_2\neq m-1$ we have (5) in the same way as above.

When $c_2=m-1$ we have

$$\lambda_{i+k}=\lambda_k \text{ for } k=1,\cdots,2m^{n_1}-1$$

But

$$\lambda_{i+2m^{n_2}}=\lambda_{(c_1+1)m^{n_1}+m^{n_2}}=\begin{cases}1, & \text{if } n_1 \text{ is odd}\\ 0, & \text{if } n_1 \text{ is even}\end{cases}$$

and

$$\lambda_{2m^{n_2}}=\begin{cases}2, & \text{if } n_1 \text{ is odd}\\ 1, & \text{if } n_1 \text{ is even}\end{cases}$$

Then we obtain $\lambda_{i+2m^{n_2}}<\lambda_{2m^{n_2}}$. That testifies (5).

Case 3 $\tilde{i}>2$, then we can write $i=c_1\cdot m^{n_1}+c_2\cdot m^{n_2}+j$ for $n_1>n_2\geqslant 0$, $1\leqslant j<m^{n_2}$ and $c_1, c_2\in\{1,\cdots,m$

$-1\}$.

If $n_1 \geqslant n_2+2$, we have

$$\lambda_{i+k} = \lambda_{j+k} \text{ for } k = 1,\cdots,m^{n_2}-j-1.$$

But

$$\lambda_{i+m^{n_2}-j} = \lambda_{c_1\cdot m^{n_1}+(c_2+1)m^{n_2}} = \begin{cases} 0, & \text{if } n_2 \text{ is odd} \\ 1, & \text{if } n_2 \text{ is even} \end{cases}$$

and

$$\lambda_{j+m^{n_2}-j} = \lambda_{m^{n_2}} = \begin{cases} 1, & \text{if } n_2 \text{ is odd} \\ 2, & \text{if } n_2 \text{ is even} \end{cases}$$

That is $\lambda_{i+m^{n_2}-j} < \lambda_{j+m^{n_2}-j}$.

If $n_1 = n_2+1$, when $c_2 \neq m-1$ we have

$$\lambda_{i+k} = \lambda_{j+k} \text{ for } k = 1,\cdots,m^{n_2}-j-1.$$

But

$$\lambda_{i+m^{n_2}-j} = \lambda_{c_1 m^{n_1}+(c_2+1)m^{n_2}} = \begin{cases} 0, & \text{if } n_2 \text{ is odd} \\ 1, & \text{if } n_2 \text{ is even} \end{cases}$$

and

$$\lambda_{j+m^{n_2}-j} = \lambda_{m^{n_2}} = \begin{cases} 1, & \text{if } n_2 \text{ is odd} \\ 2, & \text{if } n_2 \text{ is even} \end{cases}$$

That is $\lambda_{i+m^{n_2}-j} < \lambda_{j+m^{n_2}-j}$.

When $c_2 = m-1$ we have

$$\lambda_{i+k} = \lambda_{j+k} \text{ for } k = 1,\cdots,2m^{n_2}-j-1.$$

But

$$\lambda_{i+2m^{n_2}-j} = \lambda_{(c_1+1)m^{n_1}+m^{n_2}} = \begin{cases} 1, & \text{if } n_2 \text{ is odd} \\ 0, & \text{if } n_2 \text{ is even} \end{cases}$$

and

$$\lambda_{j+2m^{n_2}-j} = \lambda_{2m^{n_2}} = \begin{cases} 2, & \text{if } n_2 \text{ is odd} \\ 1, & \text{if } n_2 \text{ is even} \end{cases}$$

That is $\lambda_{i+2m^{n_2}-j} < \lambda_{j+2m^{n_2}-j}$.

Iterating eventually we can change Case 3 into Case 1 or Case 2. Thus we can prove (5).

Lemma 5 We have

$$\overline{\lambda_{i+1}\lambda_{i+2}\cdots} < \lambda_1\lambda_2\cdots \tag{6}$$

for all $i \geqslant 1$.

Proof Let $i = c\cdot m^n + j$ with $0 \leqslant j < m^n$. We only need to prove

$$\overline{\lambda_{i+1}\lambda_{i+2}\cdots} < \lambda_{j+1}\lambda_{j+2}\cdots \tag{7}$$

combining with the preceding Lemma we can get (6).

① If $j=0$, $\overline{\lambda_{c\cdot m^n+k}} = \lambda_k$, for $k=1,\cdots,m^n-1$, but

$$\overline{\lambda_{cm^n+m^n}} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

And

$$\lambda_{m^n} = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even} \end{cases}$$

Thus we have (7).

② If $j \geqslant 1$, we have $\overline{\lambda_{i+k}} = \lambda_{j+k}$, for $k=1,\cdots,(m-c)m^n-j-1$, but

$$\overline{\lambda_{i+(m-c)m^n-j}} = \overline{\lambda_{m^{n+1}}} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

and

$$\lambda_{j+(m-c)m^n-j} = \lambda_{(m-c)m^n} = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even} \end{cases}$$

Thus we have (7).

Combining the preceeding two lemmas we can easily obtain Theorem 3. Set $u_i = p+\varepsilon_i-\varepsilon_{i-1}$.

Proof of Theorem 3

By Lemma 4, we have

$$\lambda_{i+1}\lambda_{i+2}\cdots < \lambda_1\lambda_2\cdots$$

which is equivalent to

$$u_{i+1}u_{i+2}\cdots < u_1u_2\cdots$$

From Lemma 5, we have shown that

$$\overline{\lambda_{i+1}\lambda_{i+2}\cdots} < \lambda_1\lambda_2\cdots$$

thus we have

$$\overline{u_{i+1}u_{i+2}\cdots} < u_1u_2\cdots$$

Theorem 3 is proved.

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