THE GIBBS DERIVATIVES OF GENERALIZED BRIDGE FUNCTIONS

Rao Xuefang Zhang Qishan
(*Beijing University of Aeronautics and Astronatuics*)

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Abstract

The logical derivatives of p-adic copy-shift and shift-copy generalized Bridge functions are given in this paper. When p equals 2, they are exactly the logical derivatives of natural ordering copy-shift and copy-shift Bridge functions.

1 Introduction

It is well known that exponential functions e^{ikx}, $k=0,\pm1,\pm2,\cdots$ are the characteristic functions of linear differential equations of first order constant coefficients $y'=\lambda y$. Sine ordinary differential operates on the generalized Bridge functions, this results in that the ordinary derivative of the generalized Bridge functions are identically zero. So the generalized Bridge functions do not satisfy above linear differential equation. However, the generaalized Bridge functions system that is a kind of non-sinusoidal orthogonal functions system is similar to Chrestenson Chrestenson fuction system in nature. In the literature[1], a so-called logical derivative operator upon Chrestenson functions was derived so that Chrestenson functions are the characteristic functions of first order logical differential equation. Now we introduce logical differential operator to generalized Bridge functions and obtain resemble conclusion to Chrestenson functions.

Because the construction of generalized Bridge functions is the combination of shift method of block pulse functions with the copy method of Chrestension functions, they can be represented as the products of block pulse functions and Chrestenson functions. We have divided generalized Bridge functions into two types. One is copy-shift, the other is shift-copy. p-adic natural ordering copy-shift generalized Bridge function is defined as

$$gb'_p(\omega,j,m,t) = ch_p(\omega \bmod L),\ j,p^{m-j}t) \times \mathrm{blo}([\omega/L],m-j,t) \qquad (1.1)$$

where $0\leqslant t<1$, $L=p^j$.

p-adic natural ordering shift-copy generalized Bridge funciton is defined as

$$\widetilde{gb}_p(\omega,j,m,t) = \text{blo}(\omega(\text{mod}L),j,p^{m-j}t) \times ch_p([\omega/L],m-j,t) \quad (1.2)$$

Where $0 \leqslant t < 1, L = p^j$.

We shall introduce the concept of logical derivative to these two types of generalized Bridge functions. First we shall give the defintion of logical derivative.

The Definition of Logical Derivative

For any fixed $t \in G = [0,1)$, we can represent t by the p-adic form of $t = \sum_{s=0}^{\infty} t^{(s)} p_{-s-1}, t^{(s)} \in \{0,1,\cdots,p-1\}$. Now we define the logical derivative of $f(t)$ on $G=[0,1)$. If the following sum

$$\sum_{k=0}^{N} p^k \{ \sum_{l=0}^{p-1} A_l f(t \oplus l p^{-k-1}) \} \quad (2.1)$$

converges to a limit while $N \to \infty$, then this limit is called p-adic logical derivative of $f(t)$ at the point t and denoted by $f^{(1)}(t)$, where complex numbers $A_l = A_l(p)$, $l=0,1,\cdots,p-1$ satisfy the following equations (2.2)

$$\sum_{l=0}^{p-1} A_l \gamma^{kl} = k, \qquad k = 0,1,\cdots,p-1 \quad (2.2)$$

$\gamma = \exp\{2\pi i/p\}$. It is readily verified that complex numbers A_l satisfied equations (2.2) can be given by [1]

$$A_0(p) = (p-1)/2,\ A_l(p) = \gamma^l/(1-\gamma^l), \qquad l = 1,2,\cdots,p-1. \quad (2.3)$$

If t admitted of change on $[0,1)$, we can define logical derivative functions as usual.

The Logical Derinative of Generalized Bridge Functions

Set $\omega = \sum_{s=0}^{m-1} \omega^{(s)} p^{m-1-s}$, $\omega^{(s)} \in \{0,1,\cdots,p-1\}, t = \sum_{s=0}^{\infty} t^{(s)} p^{-s-1}, t^{(s)} \in \{0,1,\cdots,p-1\}$, specify the pseudo-product operator $\circ$ of non-negative integer number ω and fractional number $t (0 \leqslant t < 1)$ as $\omega \circ t = \sum_{s=0}^{m-1} \omega^{(m-1-s)} t^{(s)}$, thus the p-adic natural ordering Chrestenson function of the order number ω can be expressed as follows:

$$ch_p(\omega,m,t) = \exp\left\{\frac{2\pi i}{p} \sum_{s=0}^{m-1} \omega^{(m-1-s)} t^{(s)}\right\} = \exp\{\frac{2\pi i}{p} \omega \circ t\} \quad (3.1)$$

We can conclude that p-adic natural ordering copy-shift generalized Bridge functions $gb'_p(\omega,j,m,t), \omega = 0,1,2,\cdots$, are the characteristic functions of first order logical differential equation $\varphi^{(1)} = \lambda\varphi$. Here we shall verify the fact in the following theorem 3.1.

Theorem 3.1

$$gb'^{(1)}_p(\omega,j,m,t)=\begin{cases}(p^{m-j}\omega(\mathrm{mod}L)+\frac{p^{m-j}-1}{2})\cdot gb'_p(\omega,j,m,t),\\ \quad t\in[\frac{[\omega/L]}{p^{m-j}},\frac{[\omega/L]+1}{p^{m-j}}),\\ 0,\quad t\notin[\frac{[\omega/L]}{p^{m-j}},\frac{[\omega/L]}{p^{m-j}}).\end{cases}$$

Proof By virtue of (1.1),

$$gb'_p(\omega,j,m,t)=ch_p(\omega\mathrm{mod}L),j,p^{m-j}t)\times \mathrm{blo}([\omega/L],m-j,t).$$

Since while $t\notin[\frac{[\omega/L]}{p^{m-j}},\frac{[\omega/L]+1}{p^{m-j}})$, $\mathrm{blo}([\omega/L],m-j,t)=0$, hence $gb'_p(\omega,j,m,t)=0$. Thus $gb'^{(1)}_p(\omega,j,m,t)=0$. And while $t\in[\frac{[\omega/L]}{p^{m-j}},\frac{[\omega/L]+1}{p^{m-j}})$, for $k<m-j,l\neq 0$

$$\mathrm{blo}([\omega/L],m-j,t)\times \mathrm{blo}([\omega/L],m-j,t\oplus lp^{-k-1})=0,$$

and $k\geqslant m-j$,

$$\mathrm{blo}([\omega/L],m-j,t)\times \mathrm{blo}([\omega/L],m-j,t\oplus lp^{-k-1})=\mathrm{blo}([\omega/L],m-j,t).$$

Hence

$$\begin{aligned}S_N(t)=&\sum_{k=0}^{N}p^kA_0ch_p(\omega(\mathrm{mod}L),j,p^{m-j}t)\times \mathrm{blo}([\omega/L],m-j,t)\\&+\sum_{k=m-j}^{N}p^k\{\sum_{l=1}^{p-1}A_lch_p(\omega(\mathrm{mod}L),j,p^{m-j}(t\ominus lp^{-k-1})\times \mathrm{blo}([\omega/L],m-j,t)\}\\&=ch_p(\omega(\mathrm{mod}L),j,p^{m-j}t)\times \mathrm{blo}([\omega/L],m-j,t)\\&\left\{\sum_{k=m-j}^{N}p^k\{\sum_{l=0}^{p-1}A_lch_p(\omega(\mathrm{modo}L),j,lp^{m-j-k-1})\}+A_0\frac{p^{m-j}-1}{p-1}\right\},\end{aligned}$$

where $A_0=\frac{p-1}{2}$. Thus the above expression becomes

$$\begin{aligned}&=gb'_p(\omega,j,m,t)\{p^{m-j}\sum_{s=0}^{N-m+j}p^s(\sum_{l=0}^{p-1}A_l\exp\{\frac{2\pi i}{p}(\omega(\mathrm{mod}L)\circ lp^{-s-1})\})\}+\frac{p^{m-j}-1}{2}\}\\&=gb'_p(\omega,j,m,t)\{p^{m-j}\sum_{s=0}^{N-m+j}p^s(\omega(\mathrm{mod}L))^{(m-1-s)}+\frac{p^{m-j}-1}{2}\}.\end{aligned}$$

While $N\to\infty, S_N(t)\to(p^{m-j}\omega(\mathrm{mod}L)+\frac{p^{m-j}-1}{2})\cdot gb'_p(\omega,j,m,t)$, we botain.

$$gb'^{(1)}_p(\omega,j,m,t)=(p^{m-j}\omega(\mathrm{mod}L)+\frac{p^{m-j}-1}{2})gb'_p(\omega,j,m,t).$$

When j equals m, that is, they are only made to copy not to shift generalized Bridge funcitons degenerate into Chrestenson functions. For this situation the conclusion from the theorem 3.1 is consistent with the conclusion of literature[1].

For p-adic natural ordering shift-copy generalized Bridge functions we also conclude that they are the characteristic functions of first order logical differential equation. By virtue of the formula (1.2), we know that the mathematical expression of p-adic natural ordering shift-copy generalized Bridge function $\widetilde{gb}_p(\omega,j,m,t)$ is

$$\widetilde{gb}_p(\omega,j,m,t) = \text{blo}(\omega(\text{mod}L),j,p^{m-j}t) \times ch_p([\omega/L],m-j,t).$$

Since while

$$t \notin [\frac{\omega(\text{mod}L)}{p^m} + \frac{s}{p^{m-j}}, \frac{\omega(\text{mod}L)+1}{p^m} + \frac{s}{p^{m-j}}),\ s = 0,1,\cdots,p^{m-j}-1,$$

$$\text{blo}(\omega(\text{mod}L),j,p^{m-j}t) = 0,$$

hence $\widetilde{gb}_p(\omega,j,m,t) = 0$. Thus $\widetilde{gb}_p^{(1)}(\omega,j,m,t) = 0$. While $t \in [\frac{\omega(\text{mod}L)}{p^m} + \frac{s}{p^{m-j}}, \frac{\omega(\text{mod}L)+1}{p^m} + \frac{s}{p^{m-j}})$, $s=0,1,\cdots,p^{m-j}-1$

$$\text{blo}(\omega(\text{mod}L),j,p^{m-j}t) = 1.$$

Because $\text{blo}(\omega\text{mod}L),j,p^{m-j}t)$ is a periodic function whose period is 1, while

$$t \in [\frac{\omega(\text{mod}L)}{p^m} + \frac{s}{p^{m-j}}, \frac{\omega(\text{mod}L)+1}{p^m} + \frac{s}{p^{m-j}}),\qquad s = 0,1,\cdots,p^{m-j}-1,$$

for $k \geqslant m-j-1$, we have

$$\text{blo}(\omega(\text{mod}L),j,p^{m-j}(t \oplus lp^{-k-1})) = \text{blo}(\omega(\text{mod}L),j,p^{m-j}t),$$

for $m-j \leqslant k < m$, $l \neq 0$, we have

$$\text{blo}(\omega(\text{mod}L),j,p^{m-j}(t \oplus lp^{-k-1})) = 0,$$

for $k \geqslant m, l \neq 0$, we have

$$\text{blo}(\omega(\text{mod}L),j,p^{m-j}(t \oplus lp^{-k-1})) = \text{blo}(\omega(\text{mod}L),j,p^{m-j}t).$$

Then

$$S_n(t) = \sum_{k=0}^{N} p^k \{ \sum_{l=0}^{p-1} A_l \widetilde{gb}_p(\omega,j,m,t \oplus lp^{-k-1}) \}.$$

Hence, upon above condition, the sum becomes

$$\begin{aligned} S_N(t) = {} & \text{blo}(\omega(\text{mod}L),j,p^{m-j}t) \times ch_p([\omega/L],m-j,t) \\ & \{ \sum_{k=0}^{M-J-1} p^k [\sum_{l=0}^{p-1} A_1 ch_p([\omega/L],m-j,lp^{-k-1}) \}] \\ & + \sum_{k=m-j}^{m-1} p^k A_0 + \sum_{k=0}^{N} p^k [\sum_{l=0}^{p-1} A_l ch_p([\omega/L],m-j,lp^{-k-1})]\} \\ = {} & \widetilde{gb}_p(\omega,j,m,t) \Big\{ \sum_{k=0}^{m-j-1} p^k \sum_{l=0}^{p-1} A_1 \gamma^{[\omega/L]\cdot lp^{-k-1}} + \frac{p^m - p^{m-j}}{2} + \sum_{k=m}^{N} p^k \sum_{l=0}^{p-1} A_1 \gamma^{[\omega/L]\cdot lp^{-k-1}} \Big\} \\ = {} & \widetilde{gb}_p(\omega,j,m,t) \Big\{ \sum_{k=0}^{m-j-1} [\omega/L]^{m-j-l-k)} p^k + \frac{p^m - p^{m-j}}{2} + \sum_{k=m}^{N} [\omega/L]^{(m-j-l-k)} p^k \Big\} \\ = {} & \widetilde{gb}_p(\omega,j,m,t) \{ [\omega/L] + \frac{p^m - p^{m-j}}{2} \}. \end{aligned}$$

Therefore above result is summarized in the form of theorem 3.2.

Theorem 3.2

$$\widetilde{gb}_p^{(1)}(\omega,j,m,t)=\begin{cases}\{[\omega/L]+\dfrac{p^m-p^{m-j}}{2}\}\widetilde{gb}_p(\omega,j,m,t) \\ \quad t\in[\dfrac{\omega(\mathrm{mod}L)}{p^m}+\dfrac{s}{p^{m-j}}, \dfrac{\omega(\mathrm{mod}L)+1}{p^m}+\dfrac{s}{p^{m-j}}), \\ \quad s=0,1,\cdots,p^{m-j}-1. \\ 0, \quad t\notin[\dfrac{\omega(\mathrm{mod}L)}{p^m}+\dfrac{s}{p^{m-j}},\dfrac{\omega(\mathrm{mod}L)+1}{p^m}+\dfrac{s}{p^{m-j}}), \\ \quad s=0,1,\cdots,p^{m-j}-1.\end{cases}$$

Conclusion

As stated above, we confine our discussion to the situation of p-adic natural ordering generalized Bridge function. In fact, the logical derivatives of other kinds of ordering generalized Bridge functions can be given by the similar expressions. It is easy to see that while p equals 2 above theorems are still valid, that is, while p equals 2, the logical derivatives of natrual ordering copy-shift and shift-copy Bridge functions can be expressed by formula (3.1) and (3.2) respectively.

Because Bridge function system is a kind of orthogonal function system, it has been found applications on secret communications, control theory and information transmission[2]. In our laboratory, the device of multiplex system based upon Bridge functions has been developed. Generalized Bridge funciton system is the generalization of Bridge function system and is also a kind of orthogonal function system. It contains many kinds of orthogonal function system such as Bridge functions system, Chrestenson function system, Watari function system and etc. We are finding its possible applications in signal processing, image processing, mliti-vlaue logic and other fields.

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Department of Electrical Enineering,
Beijing, University of Aernautics and Astronatuics
P. O. Box 0085—202, Beijing 100083,
PRC.