SOME CONDITIONS ON BERGMAN SPACE ZERO SETS

By

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0. Introduction

We denote by A^p, $0 < p < \infty$ the Bergman space of functions analytic in the unit disc U and satisfying

$$\|f\|_p^p = \frac{1}{\pi} \int_U \int |f(z)|^p dxdy < \infty.$$

In this work, as in our earlier paper [4], we are concerned with the classification of the zero sets of functions in A^p for various p. It should be noted that in [6] Korenblum obtained a complete characterization of the zeros for the union of all the Bergman spaces. His ideas were fundamental, and many of them have been borrowed or adapted in the present work.

In the last few years there has been a revival of interest in the subject of Bergman space zero sets centering around the remarkable discoveries of Korenblum [7] and Hedenmalm [3]. (See also [2] for an important extension of Hedenmalm's work.)

Most of this paper is devoted to the derivation of sufficient conditions for zero sets for individual A^p spaces. In the first section we develop our basic methods and apply them to the special case of "thin" zero sets, in which the calculations are somewhat simpler and the results somewhat sharper than in the general case. In section 2 we illustrate by examples the sharpness of the results of section 1. Section 3 is devoted to a generalization of our results to arbitrary zero sets. In section 4 we apply our methods to "random zero sets" as treated in [1] and [8], and in section 5 we show how to "move" zeros while preserving sufficient conditions, in the spirit of [7]. In the last section we give some necessary conditions which are stronger than those in [4] but which, unfortunately, still seem weaker than our sufficient conditions.

1. Thin zero sets

A sequence $\{z_k\}$ of not necessarily distinct points in U will be called *thin* if there exists a positive integer M such that there are at most M points of the $\{z_k\}$ in any dyadic rectangle

$$R^{m,n} = \left\{ z \in U : 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}; \frac{m}{2^n} 2\pi \leq \arg z < \frac{m+1}{2^n} 2\pi \right\}.$$

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The smallest such M will be called *the density of the sequence* $\{z_k\}$.

It follows from the work of Korenblum [6] that every thin sequence is a zero set for some A^p. Our concern here is to classify those thin sequences which belong to a given A^p. Our sufficient conditions will be formulated in terms of the following subregions of the disc:

For each θ, $0 \le \theta < 2\pi$, and for each $n \in \mathbb{N}$, define

$$G_{\theta,n} = \{z \in U : |\arg z - \theta| \le n\pi(1 - |z|)\} \cup \{0\}, \tag{1.1}$$

where we choose the value of $\arg z$ which is closest to θ. Thus $G_{\theta,n}$ is roughly a Stolz angle at $e^{i\theta}$. Now for $0 < r < 1$ define

$$G_{r,\theta,n} = \{z \in U : z \in G_{\theta,n} \quad \text{and} \quad |z| \le r\}.$$

With respect to a given set $\{z_k\}$ define

$$\varphi(r,\theta,n) = \frac{1}{n} \times \text{ the number of elements of } \{z_k\} \text{ in the region } G_{r,\theta,n}. \tag{1.2}$$

(1.3) Theorem *Let* $\{z_k\}$ *be a thin sequence in* U *and* n *a natural number. Let* $\varphi(r,\theta,n)$ *be the associated counting function. Then* $\{z_k\}$ *is an* A^p *zero set, with each* z_k *repeated according to the multiplicity of the zero there, if*

$$\int_0^{2\pi} \left[\log \int_0^1 e^{\,p\varphi(r,\theta,n)} r\,dr\right] d\theta < \infty.$$

For the proof we shall assume without loss of generality that no $z_k = 0$, and we shall construct explicitly a function in A^p which vanishes on the given set. To that end we first associate to each z_k the canonical factor

$$B_k(z) = \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z} \tag{1.4}$$

and a "covergence factor"

$$S_k^{(n)}(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_{k,n}(\theta)\right\}, \tag{1.5}$$

where $d\mu_{k,n}$ is a measure on the unit circle T which equals $\frac{1}{n}d\theta$ on the interval $\{e^{i\theta} : |\theta - \arg z_k| \le \min[n\pi(1 - |z_k|), \pi]\}$ and zero elsewhere. To say that $S_k^{(n)}(z)$ is a convergence factor is to say that $B_k(z)S_k^{(n)}(z) \approx 1$. We quantify that statement in the following lemma, which is a simple adaptation of Lemma 3, section 3.3 in [6].

(1.6) Lemma *There exists an absolute constant c such that if $0 \neq \beta \in U$, if $\xi = \beta/|\beta|$, and if $B(z)$ and $S^{(n)}(z)$ are as in (1.4) and (1.5), with β in place of z_k, then for all $z \in U$ which satisfy*

$$\frac{1-|\beta|}{1-|z|} \leq \min\left(\frac{1}{n}, \frac{1}{4}\right), \tag{1.7}$$

we have

$$|\log B(z) + \log S^{(n)}(z)| \leq cn\left(\frac{1-|\beta|}{|\xi - z|}\right)^2.$$

(Here $\log B(z)$ is defined for $|z| < |\beta|$ by continuation from the value $\log|\beta|$ at 0.)

Proof Korenblum [6] obtained that if

$$S(z) = \exp\left\{\log\frac{1}{|\beta|}\left[\frac{\xi + z}{\xi - z}\right]\right\},$$

then whenever $1 - |\beta| \leq \frac{1}{4}|\xi - z|$, and in particular if (1.7) holds, we have

$$|\log B(z) + \log S(z)| \leq c\left(\frac{1-|\beta|}{|\xi - z|}\right)^2.$$

We observe that if

$$S_1(z) = \exp\left\{(1-|\beta|)\left[\frac{\xi + z}{\xi - z}\right]\right\}$$

then for $1 - |\beta| \leq \frac{1}{4}(1 - |z|)$

$$|\log S(z) - \log S_1(z)| \leq c\left(\frac{1-|\beta|}{|\xi - z|}\right)^2.$$

Thus we need only estimate $|\log S^{(n)}(z) - \log S_1(z)|$ when (1.7) holds. However

$$\log S^{(n)}(z) - \log S_1(z) = \frac{1}{2\pi n}\int_{\alpha_1}^{\alpha_2}\left[\frac{e^{i\theta} + z}{e^{i\theta} - z} - \frac{\xi + z}{\xi - z}\right]d\theta,$$

where $\alpha_i = \arg\beta \pm n\pi(1 - |\beta|)$ $(i = 1, 2)$. Simplifying we have

$$\log S^{(n)}(z) - \log S_1(z) = \frac{1}{2\pi n}\int_{\alpha_1}^{\alpha_2}\frac{2z(\xi - e^{i\theta})}{(e^{i\theta} - z)(\xi - z)}d\theta.$$

It follows from (1.7) that on the interval of integration $|e^{i\theta} - z| \geq c|\xi - z|$ for an absolute constant $c > 0$. Thus

$$|\log S^{(n)}(z) - \log S_1(z)| \leq cn\left(\frac{1-|\beta|}{|\xi - z|}\right)^2,$$

which gives us the desired estimate.

The following elementary lemma will also be used in the proof of the Theorem.

(1.8) Lemma *Let μ be a Borel probability measure on the topological space X, and let f be a nonnegative μ-measurable function on X, such that* $\log f$ *is μ-integrable. Let*

$$S = \{g : X \to \mathbb{R} : g \textit{ is integrable } d\mu \textit{ and } \int g d\mu = 0\}.$$

Then

$$\inf_{g\in S} \int_X f e^g d\mu = \exp \int_X \log f d\mu.$$

Proof If g is in S, then Jensen's inequality gives that

$$\exp \int_X \log f d\mu = \exp \int_X (\log f + g) d\mu \leq \int_X f e^{g} d\mu.$$

On the other hand, if we define

$$g(x) = \int_X \log f d\mu - \log f(x),$$

then $g \in S$ and we have

$$\int_X f e^{g} d\mu = \int_X \exp\left\{\int_X \log f d\mu\right\} d\mu = \exp \int_X \log f d\mu.$$

After establishing these preliminaries, we turn to the proof of Theorem (1.3) for a fixed but arbitrary n. First we note that from Lemma (1.6) it follows that whenever $\sum(1 - |z_k|)^2 < \infty$, which is certainly the case for any thin sequence $\{z_k\}$, the product

$$\prod_{k=1}^{\infty} B_k(z) S_k^{(n)}(z)$$

converges to an analytic function on U which vanishes precisely on $\{z_k\}$. The function we build will be of the form

$$f(z) = \prod_{k=1}^{\infty} B_k(z) S_k^{(n)}(z) \exp \int_0^{2\pi} h(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} \frac{d\theta}{2\pi}, \tag{1.9}$$

where $h(\theta)$ is a real-valued integrable function satisfying $\int_0^{2\pi} h(\theta)d\theta = 0$, so that the last term has value 1 at the origin. $h(\theta)$ will be chosen later.

We now turn to the estimation of $f(z)$ on the circle $|z| = r < 1$. We first show that with $a = \min(1/4, 1/n)$, there is a constant c independent of r such that

$$(1.10) \qquad \left| \prod_{1-|z_k|\le a(1-r)} B_k(z) S_k^{(n)}(z) \right| \le c \quad \text{whenever } |z| = r.$$

However by Lemma (1.6), when $|z| = r$ the above product is bounded by

$$\exp \sum_{1-|z_k|\le a(1-r)} cn \left(\frac{1-|z_k|}{\left|\frac{z_k}{|z_k|} - z\right|} \right)^2 .$$

Since n is a fixed integer, and since $\{z_k\}$ is a thin sequence, we need only bound

$$\sum_k \left(\frac{1-|a_k|}{\left|\frac{a_k}{|a_k|} - z\right|} \right)^2$$

where $\{a_k\}$ is a sequence consisting of precisely one point in each dyadic subrectangle of the annulus $B = \{\xi : 1 - |\xi| \le 2a(1 - |z|)\}$. Now let $\{R_k\}$ denote these dyadic rectangles, and let $A(R_k)$ denote their areas. Let w_k denote the closest point to z in the projection of $\overline{R_k}$ on T. Then there are absolute constants c_i such that

$$\sum \left(\frac{1-|a_k|}{\left|\frac{a_k}{|a_k|} - z\right|} \right)^2 \le c_1 \sum \frac{1}{|w_k - z|^2} A(R_k) \le c_2 \int_B \frac{1}{\left|\frac{\xi}{|\xi|} - z\right|^2} dA(\xi)$$

$$\text{(where } dA = \text{ Lebesgue area measure)}$$

$$\le 4\pi a(1-|z|)c_2 \int_0^{2\pi} \frac{1}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi} = 4\pi a \frac{1-|z|}{1-|z|^2} c_2 \le c_3.$$

Thus we have obtained the desired bound.

Next we note that if $\{z_k\}$ is thin of density M, then there is an absolute constant c such that for all $z \in U$

$$(1.11) \qquad \left| \prod_{a(1-|z|)\le 1-|z_k|<1-|z|} B_\mu(z) S_k^{(n)}(z) \right| \le e^{ca^{-1}M},$$

for the measures μ_k defined in (1.5) belonging to these $S_k^{(n)}$ cannot have total density greater than $ca^{-1}M$ at any point of T.

Thus it remains only to bound

$$\prod_{|z_k|\le r=|z|} B_k(z)S_k(z)\exp\int_0^{2\pi} h(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\frac{d\theta}{2\pi}. \tag{1.12}$$

Estimating each $|B_k(z)|$ by 1, we find that in absolute value this product is bounded by

$$\exp\int_0^{2\pi} P_z(\theta)\left[h(\theta)d\theta+\sum_{|z_k|\le r} d\mu_{k,n}(\theta)\right],$$

where $P_z(\theta)$ is the Poisson kernel for the point z, namely

$$P_z(\theta)=\frac{1}{2\pi}\operatorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z}.$$

However, our construction gives that for every $(r,\theta)\in U$

$$\sum_{|z_k|\le r} d\mu_{k,n}(\theta)=\varphi(r,\theta,n)d\theta. \tag{1.13}$$

Putting together all the pieces we have for f as defined in (1.9) and for $|z|=r$

$$|f(z)|\le c\exp\int_0^{2\pi} P_z(\theta)[h(\theta)+\varphi(r,\theta,n)]d\theta.$$

If $0<p<\infty$ we have

$$\begin{aligned}|f(z)|^p&\le c\exp\int_0^{2\pi} P_z(\theta)[ph(\theta)+p\varphi(r,\theta,n)]d\theta\\ &\le c\int_0^{2\pi}\exp[ph(\theta)+p\varphi(r,\theta,n)]P_z(\theta)d\theta\end{aligned}$$

since for each fixed z, $P_z(\theta)d\theta$ is a probability measure on $[0,2\pi]$.

Now we write $z=re^{i\alpha}$, and integrate the above inequality to obtain

$$\int_0^{2\pi}|f(re^{i\alpha})|^p d\alpha\le c\int_0^{2\pi}\exp[ph(\theta)+p\varphi(r,\theta,n)]d\theta.$$

Thus

$$\|f\|_p^p\le\frac{c}{\pi}\int_0^{2\pi} e^{ph(\theta)}\left[\int_0^1 e^{p\varphi(r,\theta,n)}rdr\right]d\theta,$$

where we are still free to choose h as we wish, just so

$$\int_0^{2\pi} h(\theta)\frac{d\theta}{2\pi} = 0.$$

If we now choose $h(\theta)$ as in Lemma (1.8), we obtain that

$$\|f\|_p^p \leq c\exp\left\{\frac{1}{\pi}\int_0^{2\pi}\left[\log\int_0^1 e^{p\varphi(r,\theta,n)}rdr\right]d\theta\right\}.$$

Since the last integral was assumed finite, we conclude that $f \in A^p$, and thus $\{z_k\}$ is an A^p zero set.

2. Some examples

In this section we illustrate the sharpness of the conditions in section 1 by means of some simple examples. First we note a necessary condition for thin zero sets, which is implicit in [1]. Namely, if $f \in A^p$, if $f(0) \neq 0$, and if $\{z_k\}$ are the zeros of f repeated according to multiplicity, then exponentiation of Jensen's formula gives that for $0 < r < 1$

$$|f(0)|^p \prod_{|z_k|\leq r}\left(\frac{r}{|z_k|}\right)^p \leq \frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta = o\left(\frac{1}{1-r}\right).$$

Thus

$$\prod_{|z_k|\leq r}\frac{r}{|z_k|} = o\left(\frac{1}{(1-r)^{1/p}}\right),$$

and since the $\{z_k\}$ are thin, they number no more than $0\left(\frac{1}{1-r}\right)$ in the disc $|z| \leq r$, and therefore we can conclude that

$$\prod_{|z_k|\leq r}\frac{1}{|z_k|} = o\left(\frac{1}{(1-r)^{1/p}}\right). \tag{2.1}$$

In light of this condition and our sufficient conditions from section 1, we shall examine the zero sets of the functions

$$f(z) = \prod_{k=1}^{\infty}\left(1+\mu z^{\beta^k}\right); \quad \beta > 2 \quad \text{integral}, \quad \mu > 1,$$

which were studied in [4]. Let $\{z_\ell\}$ denote the zero set of f, which is clearly a thin set, and for each $k > 1$ define

$$a_k = \left(\frac{1}{\mu}\right)^{\beta^{-k}}. \tag{2.2}$$

Thus

$$\frac{1}{\log(1/a_k)} = \beta^k / \log \mu$$

so

$$\frac{1}{c}\beta^k \le \frac{1}{1-a_k} \le c\beta^k \quad \text{for some } c \text{ independent of } k.$$

However,

$$\prod_{|z_\ell| \le a_k} \frac{1}{|z_\ell|} = \mu^k \ge c\left(\frac{1}{1-a_k}\right)^{\log\mu/\log\beta}.$$

Thus by (2.1) $\{z_k\}$ can be an A^p zero set only if $p < \log\beta/\log\mu$. Now if in fact $p < \log\beta/\log\mu$ we can choose n so large that $p(\log\mu + 2/n) < (\log\beta)(1-1/n)$. For that n we calculate the corresponding $\varphi(r,\theta,n)$ as in (1.2). Note that with a_k as in (2.2), on the circle $|z| = a_k$, f has β^k evenly spaced zeros. Thus for every θ the number of zeros satisfying

$$|z_\ell| = a_k; \quad |\arg z_\ell - \theta| \le n\pi(1-a_k)$$

is bounded by $n\beta^k(1-a_k)+2 \le n\beta^k \log(1/a_k)+2 = n\log\mu + 2$. Since there are k such circles in the disc $|z| \le a_k$, we obtain that $\varphi(a_k,\theta,n) \le k\log\mu + 2k/n$ for all θ. By our choice of n,

$$p\varphi(a_k,\theta,n) \le kp\Big(\log\mu + \frac{2}{n}\Big) \le k\left[(\log\beta)\left(1-\frac{1}{n}\right)\right].$$

Thus

$$\exp\{p\varphi(a_k,\theta,n)\} \le \beta^{k(1-1/n)} \le c\left(\frac{1}{1-a_k}\right)^{1-1/n},$$

by (2.2), and even though this inequality has been verified only for $r = a_k$ it is easily seen that with a larger c we have generally

$$\exp\{p\varphi(r,\theta,n)\} \le c\left(\frac{1}{1-r}\right)^{1-1/n}, \quad 0<r<1, \quad 0 \le \theta < 2\pi.$$

Therefore the sufficient condition of Theorem (1.3) is satisfied, and $\{z_\ell\}$ is an A^p zero set for all $p < \log\beta/\log\mu$. Of course this does not mean that the function f itself belongs to all of these spaces. However, it is easily seen that the associated function,

$$g(z) = \prod_{k=1}^{\infty} \frac{1+\mu z^{\beta^k}}{1+\frac{1}{\mu}z^{\beta^k}} = \prod_{k=1}^{\infty} \mu\frac{\frac{1}{\mu}+z^{\beta^k}}{1+\frac{1}{\mu}z^{\beta^k}},$$

which is essentially a Blaschke product, satisfies

$$|g(z)| \le c\mu^k \quad \text{when } |z|^{\beta^k} = \frac{1}{\mu},$$

and therefore $g \in A^p$ for all $p < \log\beta/\log\mu$.

3. General zero sets

In this section we derive sufficient conditions for A^p zero sets which are not necessarily thin. Our method is essentially as in section 1. However, when the set $\{z_k\}$ is not thin, the estimates (1.10) and (1.11) are no longer valid. Thus we cannot estimate the product $\prod_{k=1}^{\infty} B_k(z)S_k^{(n)}(z)$ on the circle of the radius r by means of those terms alone which correspond to zeros satisfying $|z_k| \leq r$. However, there is a simple if not wholly satisfactory solution to that problem; namely that instead of transferring a particular factor $B_k(z)S_k^{(n)}(z)$ from the main term (1.12) to the error term (1.10) immediately at $|z| = |z_k|$ one must do so when $1 - |z| = p(z_k)\log^{1+\varepsilon} p(z_k)(1 - |z_k|)$, where

(3.1) $\quad p(z_k) =$ the number of $\{z_\ell\}$ in the dyadic rectangle containing z_k.

The precise theorem is as follows.

(3.2) Theorem *Let $\{z_k\}$ be a sequence in U and let $\varepsilon > 0$ and $n \in \mathbb{N}$ be chosen. For each θ, $0 \leq \theta < 2\pi$, define*

$$G_{\theta,n} = \{z \in U : |\arg z - \theta| < \pi n(1 - |z|)\} \cup \{0\}, \tag{3.3}$$

and for $0 < r < 1$ define (with $p(z_k)$ as above)

$$S_r = \left\{z_k : \frac{1 - |z_k|}{1 - r} < \min\left(\frac{1}{4}, \frac{1}{n}, p^{-1}(z_k)\log^{-1-\varepsilon} p(z_k)\right)\right\}. \tag{3.4}$$

Finally, for every r and θ define

$$\varphi(r,\theta) = \frac{1}{n} \times \text{ the number of elements of } \{z_k\} \text{ in } G_{\theta,n} \cap S_r^c,$$

where S_r^c is the complement of S_r.

Then $\{z_k\}$ is an A^p zero set, $0 < p < \infty$, if

$$\int_0^{2\pi} \left[\log \int_0^1 e^{p\varphi(r,\theta)} r\,dr\right] d\theta < \infty.$$

Proof As in Theorem (1.3) we construct

$$f(z) = \prod_{k=1}^{\infty} B_k(z)S_k^{(n)}(z) \exp \int_0^{2\pi} h(\theta)\frac{e^{i\theta} + z}{e^{i\theta} - z}\frac{d\theta}{2\pi}.$$

A perusal of the proof of Theorem (1.3) will indicate that all of the steps there can be repeated here except for one estimate, namely we must show that the "remainder term"

$$\prod_{z_k \in S_r} B_k(re^{i\theta}) S_k^{(n)}(re^{i\theta}) \tag{3.5}$$

is uniformly bounded for all $re^{i\theta} \in U$. In view of Lemma (1.6), it is sufficient to show that

$$\sum_{z_k \in S_r} \left(\frac{1 - |z_k|}{\left| \frac{z_k}{|z_k|} - z \right|} \right)^2 \qquad (|z| = r)$$

is uniformly bounded. However, the above sum equals

$$\sum_{m=0}^{\infty} \sum_{\substack{z_k \in S_r \\ 2^m \le p(z_k) < 2^{m+1}}} \left(\frac{1 - |z_k|}{\left| \frac{z_k}{|z_k|} - z \right|} \right)^2 . \tag{3.6}$$

We turn to the estimation of the inner sum for a fixed m. Now if $z_k \in S_r$ and if $p(z_k) \ge 2^m$, then by (3.4)

$$1 - |z_k| < c2^{-m} m^{-1-\varepsilon}(1 - r) \quad \text{for an absolute constant } c.$$

It follows that if $\{a_k\}$ is a sequence consisting of one point in each dyadic rectangle intersecting the annulus

$$B = \{\xi \in U : 1 - |\xi| \le c2^{-m} m^{-1-\varepsilon}(1 - r)\}.$$

Then for $|z| = r$

$$\sum_{\substack{z_k \in S_r \\ 2^m \le p(z_k) < 2^{m+1}}} \left(\frac{1 - |z_k|}{\left| \frac{z_k}{|z_k|} - z \right|} \right)^2 \le c' 2^{m+1} \sum_k \left(\frac{1 - |a_k|}{\left| \frac{a_k}{|a_k|} - z \right|} \right)^2 .$$

As in the proof of Theorem (1.3), the last sum is bounded by

$$c'' 2^{m+1} \int_B \frac{1}{\left| \frac{\xi}{|\xi|} - z \right|^2} dA(\xi) \le c''' m^{-1-\varepsilon}.$$

It follows immediately that the sum (3.6) and therefore the product(3.5) is uniformly bounded, completing the proof of the theorem.

We now give a modified version of Theorem (3.2) which will be useful in the next section. In order to formulate that theorem we introduce an additional bit of notation: for $\frac{1}{2} \leq |z| < 1$ and $\varepsilon > 0$, let

$$h_\varepsilon(z) = \left[\log_2\left(\frac{1}{1-|z|}\right)\right]^{1+\varepsilon}; \tag{3.7}$$

i.e. if $1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}$, $h_\varepsilon(z) = n^{1+\varepsilon}$, and for $|z| < \frac{1}{2}$, $h_\varepsilon(z) = 1$.

(3.8) Theorem *Let $\{z_k\}$ be a sequence in U and let $\varepsilon > 0$ be chosen. Define $h(z) = h_\varepsilon(z)$ and $p(z_k)$ as above. For $0 \leq \theta < 2\pi$ define*

$$G_\theta = \{z \in U : |\arg z - \theta| < \pi(1-|z|)h(z)\} \cup \{0\},$$

and for $0 < r < 1$ let

$$S_r = \left\{z_k : \frac{1-r}{1-|z_k|} \geq \max\left(4, h(z_k), h(z_k)p(z_k)\log^{1+\varepsilon} p(z_k)\right)\right\}, \tag{3.9}$$

and let $\varphi(r,\theta) = \sum_{z_k \in G_\theta \cap S_r^c} h^{-1}(z_k)$. Then $\{z_k\}$ is an A^p zero set if

$$\int_0^{2\pi} \left[\log \int_0^1 e^{p\varphi(r,\theta)} r dr\right] d\theta < \infty.$$

Proof This time the appropriate function is

$$f(z) = \prod_{k=1}^{\infty} B_k(z) S_k^{(h(z_k))}(z) \exp \int_0^{2\pi} h(\theta) \frac{e^{i\theta}+z}{e^{i\theta}-z} \frac{d\theta}{2\pi}.$$

As in the proof of Theorems (1.3) and (3.2) the critical estimate is to show that there is an absolute constant c such that if $|z| = r$, $0 < r < 1$, then

$$\left| \prod_{z_k \in S_r} B_k(z) S_k^{(h(z_k))}(z) \right| \leq c. \tag{3.10}$$

Since $z_k \in S_r$ implies that

$$\frac{1-r}{1-|z_k|} \geq \max(4, h(z_k)),$$

we can use Lemma (1.6) to estimate the log of the above product by

$$c\sum_{z_k\in S_r} h(z_k)\left(\frac{1-|z_k|}{\left|\frac{z_k}{|z_k|}-z\right|}\right)^2 = c\sum_{m=0}^{\infty}\sum_{\substack{z_k\in S_r\\ 2^m\le p(z_k)<2^{m+1}}} h(z_k)\left(\frac{1-|z_k|}{\left|\frac{z_k}{|z_k|}-z\right|}\right)^2.$$

Now if $z_k \in S_r$ and $p(z_k) \geq 2^m$, then

$$\frac{1-r}{1-|z_k|} \geq h(z_k)p(z_k)\log^{1+\varepsilon} p(z_k) \geq h(z_k)2^m m^{1+\varepsilon}.$$

Thus

$$(1-|z_k|)h(z_k) \leq 2^{-m}m^{-1-\varepsilon}(1-r).$$

It follows that if $\{a_k\}$ is a sequence consisting of one point in each dyadic rectangle intersecting the annulus

$$B = \{\xi \in U : (1-|\xi|)h(\xi) \leq 2^{-m}m^{-1-\varepsilon}(1-r)\} \equiv \{z : b_r \leq |z| < 1\},$$

then there is an absolute constant c, such that if $|z| = r$, $0 < r < 1$,

$$\sum_{\substack{z_k\in S_r\\ 2^m\le p(z_k)<2^{m+1}}} h(z_k)\left(\frac{1-|z_k|}{\left|\frac{z_k}{|z_k|}-z\right|}\right)^2 \leq c2^{m+1}\sum_k h(a_k)\left(\frac{1-|a_k|}{\left|\frac{a_k}{|a_k|}-z\right|}\right)^2.$$

As in the proof of Theorem (1.3), the last sum is bounded by

$$\begin{aligned} c_1 2^{m+1}\int_B \frac{\log^{1+\varepsilon}\left(\frac{1}{1-|\xi|}\right)}{\left|\frac{\xi}{|\xi|}-z\right|^2}\,dA(\xi) &= \frac{c_1 2^{m+1}2\pi}{1-r^2}\int_{b_r}^1 \log^{1+\varepsilon}\left(\frac{1}{1-s}\right) s\,ds \\ &\leq c_2\frac{2^{m+1}}{1-r^2}(1-b_r)\log^{1+\varepsilon}\left(\frac{1}{1-b_r}\right). \end{aligned}$$

In view of the definition of b_r the last expression is estimated by

$$c_3\frac{2^{m+1}}{1-r^2}2^{-m}m^{-1-\varepsilon}(1-r) \leq c_4 m^{-1-\varepsilon}.$$

It follows immediately that (3.10) is valid, completing the proof of the theorem.

(3.11) Corollary *Theorem (3.8) remains true if S_r as defined there is replaced by*

$$S'_r = \left\{ z_k : \frac{1-r}{1-|z_k|} \geq \max\left(4, h^2(z_k), h(z_k)p(z_k)\log^{1+\varepsilon} p(z_k)\right)\right\}. \tag{3.12}$$

Proof This is trivial, since we have only made S_r smaller, thus making S_r^c larger, and increasing $\varphi(r,\theta)$ for every r and θ. So if the A^p condition of Theorem (3.8) holds for this enlarged φ, then it certainly holds for the smaller φ described in the statement of the theorem. We mention the corollary only because it provides the appropriate framework for the result of the next section.

4. Random zero sets

In the recent papers of Leblanc [8] and Bomash [1] random A^p zero sets were examined. In this section we extend their work. Our contribution is twofold: first, we verify for all A^p Bomash's sharp condition which he obtained only for $p \leq 2$; second, we do so not by directly estimating canonical products as was done in [8] and [1] but rather by proving that under appropriate conditions on the moduli of zeros, the sufficient condition of Corollary (3.11) holds for almost every choice of arguments, thus adding evidence to the sharpness of the condition.

Our concept of random zero sets will be exactly as in [8] and [1]; i.e., after the moduli of the zeros have been specified, we shall consider their arguments as independent random variables, each one having a uniform distribution on $[0, 2\pi)$. Specifically we have

(4.1) Theorem *Let $\{\lambda_k\}$ denote a sequence in $(0,1)$. Define*

$$\psi(r) = \sum_{\lambda_k \leq r} 1 - \lambda_k, \quad 0 < r < 1.$$

Suppose that for some $\varepsilon > 0$ and for some p, $0 < p < \infty$,

$$\int_0^1 e^{p\psi(r)} \log^{2+2\varepsilon}\left(\frac{1}{1-r}\right) dr < \infty. \tag{4.2}$$

Then for almost all independent choices of $\{\theta_k\}$ the set $\{\lambda_k e^{i\theta_k}\}$ is an A^p zero set, satisfying the condition of Corollary (3.11).

Note that for $p \leq 2$, our result is slightly weaker than Theorem (2.11) in [1]. However, we have:

(4.3) Corollary *In the context of the theorem, if for some p, $0 < p < \infty$,*

$$\overline{\lim}_{\varepsilon \to 0} \frac{\psi(1-\varepsilon)}{\log(1/\varepsilon)} < \frac{1}{p},$$

then $\{\lambda_k e^{i\theta_k}\}$ is almost surely an A^p zero set.

This is an extension of Theorem 1 in [1]. It follows immediately from Theorem (4.1).

Proof of the theorem For $0 < r < 1$ and $0 \le \theta < 2\pi$, let S'_r and $\varphi(r,\theta)$ be defined as indicated in Corollary (3.11). We shall prove that almost surely

$$\int_0^{2\pi} \int_0^1 e^{p\varphi(r,\theta)} dr d\theta < \infty,$$

which certainly implies the sufficient condition of Corollary (3.11). By Fubini's theorem it suffices to show that

$$\int_0^{2\pi} \int_0^1 E(e^{p\varphi(r,\theta)}) dr d\theta < \infty,$$

where $E(e^{p\varphi(r,\theta)})$ is the expected or average value of $e^{p\varphi(r,\theta)}$ over all possible choices of the arguments $\{\theta_k\}$. Now

$$\varphi(r,\theta) = \sum_{n=1}^{\infty} \varphi_k(r,\theta),$$

where $\varphi_k(r,\theta)$ is the contribution to $\varphi(r,\theta)$ from the point $\lambda_k e^{i\theta_k}$. From the definition of φ in Theorem (3.8) and Corollary (3.11) we see that if $(1-\lambda_k)h^2(\lambda_k) > 1 - r$, then

$$\varphi_k(r,\theta) = \begin{cases} h^{-1}(\lambda_k) & |\theta - \theta_k| < \pi(1-\lambda_k)h(\lambda_k), \\ 0 & \text{otherwise.} \end{cases}$$

Thus the expected value of $\exp(p\varphi_k(r,\theta))$ when θ_k is chosen randomly is

$$\begin{aligned} &1 + (1-\lambda_k)h(\lambda_k)(\exp[ph^{-1}(\lambda_k)] - 1) \\ =&1 + (1-\lambda_k)h(\lambda_k)\left[ph^{-1}(\lambda_k) + 0\left(p^2h^{-2}(\lambda_k)\right)\right] \\ =&1 + p(1-\lambda_k) + 0[p^2(1-\lambda_k)h^{-1}(\lambda_k)]. \end{aligned}$$

Moreover, the contributions of all these λ_k are mutually independent, and independent of the other λ_k, so using $1 + x \le e^x$ we have

$$E\left(\exp \sum_{(1-\lambda_k)h^2(\lambda_k) > 1-r} p\varphi_k(r,\theta)\right) \le$$

$$\exp\left[\sum_{(1-\lambda_k)h^2(\lambda_k)>1-r} p(1-\lambda_k)+cp^2(1-\lambda_k)h^{-1}(\lambda_k)\right].$$

However, it follows easily from (4.2) (see e.g. [1], formula 2.30) that

$$\sum_{k=1}^{\infty}(1-\lambda_k)h^{-1}(\lambda_k)\le c\sum_{k=1}^{\infty}(1-\lambda_k)\log^{-1-\epsilon}\left(\frac{1}{1-\lambda_k}\right)<\infty.$$

So the above expectation is bounded by

$$c\exp\sum_{(1-\lambda_k)h^2(\lambda_k)>1-r} p(1-\lambda_k). \tag{4.4}$$

The harder part of the proof is to show that the expected contribution of the remaining $\lambda_k e^{i\theta_k}$ is uniformly bounded. One difficulty is that when $(1-\lambda_k)h^2(\lambda_k)\le 1-r$, the functions $\varphi_k(r,\theta)$ are no longer independent; rather a specific $\varphi_k(r,\theta)$ depends on the number of $\lambda_s e^{i\theta_s}$ in the dyadic rectangle containing $\lambda_k e^{i\theta_k}$. However, the contributions of z_k in different dyadic annuli

$$B_n=\left\{z:1-\frac{1}{2^n}\le|z|<1-\frac{1}{2^{n+1}}\right\}\text{ are independent.}$$

So let us consider a B_n in the range where $(1-\lambda_k)h^2(\lambda_k)\le 1-r$; i.e. $2^{-n}n^{2+2\epsilon}\le 1-r$. By (3.12) a given dyadic rectangle in $B_n\cap G_\theta$ will contribute to $\varphi(r,\theta)$ only if the number of $\lambda_k e^{i\theta_k}$ in the rectangle, say N, satisfies $N\log^{1+\epsilon}N>(1-r)2^n n^{-1-\epsilon}\ge n^{1+\epsilon}$, since $2^{-n}n^{2+2\epsilon}<1-r$. In that case the rectangle will contribute at most $Nn^{-1-\epsilon}$ to $\varphi(r,\theta)$. Now let M_n denote the total number of $\{\lambda_k\}$ in B_n and let N_0 be the largest integer such that $N_0\log^{1+\epsilon}N_0\le n^{1+\epsilon}$.

We note that from the assumption that

$$\int_0^1 e^{p\psi(r)}\log^{2+2\epsilon}\left(\frac{1}{1-r}\right)dr<\infty,$$

we must have

$$\psi(r)=0\left(\log\frac{1}{1-r}\right)$$

which implies that $M_n\le cn2^n$ for some $c>1$ independent of n. Moreover, if r is large enough, forcing n to be large (since $2^{-n}n^{2+2\epsilon}\le 1-r$) we have

$$n^{1+\epsilon}\ge N_0>2ce^{p+1}n, \tag{4.5}$$

where c is as in the upper bound on M_n.

Now we need to estimate the expected contribution to $\exp(p\varphi(r,\theta))$ from B_n, which we call $E(r,\theta,n)$. Clearly this expectation grows with M_n, and so to obtain an upper bound on $E(r,\theta,n)$ we can assume that $M_n = cn2^n$. Let us further assume that $B_n \cap G_\theta$ intersects $s(\approx n^{1+\varepsilon})$ dyadic rectangles $R_1 \cdots R_s$. Denote by $x_1 \cdots x_s$ random variables whose values equal the number of $\lambda_k e^{i\theta_k}$ in the corresponding $R_1 \cdots R_s$. With P denoting probability, we have

$$E(r,\theta,n) \le \sum_{\substack{0\le a_i \\ \Sigma a_i = M_n}} P(x_i = a_i)_{i=1}^s \exp \sum_{a_i \ge N_0} p a_i n^{-1-\varepsilon}$$

$$\le \sum_{\substack{\text{subsets} \\ \{x_{i_k}\}_1^{k_0} \subset \{x_i\}}} \sum_{\substack{a_{i_k} \ge N_0 \\ \Sigma a_{i_k} \le M_n}} P(x_{i_k} = a_{i_k})_{k=1}^{k_0} \exp \sum_{k=1}^{k_0} p a_{i_k} n^{-1-\varepsilon}.$$

However, we note that in the latter sum we have always

$$P(x_{i_k} = a_{i_k})_{k=1}^{k_0} \le \prod_{k=1}^{k_0} P(x_{i_k} = a_{i_k}),$$

because we are dealing only with $a_{i_k} \ge N_0 > cn = M_n 2^{-n}$, which is the average of the $\{x_i\}$. Thus the events $\{x_{i_k} = a_{i_k}\}$ are inversely corrolated. It follows that

$$E(r,\theta,n) \le \prod_{i=1}^{s} 1 + \sum_{a_i \ge N_0} P(x_i = a_i) \exp(p a_i n^{-1-\varepsilon}).$$

Thus

$$E^{1/s}(r,\theta,n) \le 1 + \sum_{a \ge N_0} P(x_1 = a) \exp(pan^{-1-\varepsilon})$$

$$= 1 + \sum_{N \ge N_0} \binom{M_n}{N} \left(\frac{1}{2^n}\right)^N \left(1 - \frac{1}{2^n}\right)^{M_n - N} \exp(pNn^{-1-\varepsilon}).$$

Since $M_n = cn2^n$, the ratio of successive terms in the last sum is

$$\frac{cn2^n - N}{N+1} \frac{1}{2^n} \left(1 - \frac{1}{2^n}\right)^{-1} e^{pn^{-1-\varepsilon}}.$$

But $N+1 > N_0$, and so by (4.5) this ratio is less than 1/2. Therefore we can bound the sum by twice its first term. A rough estimate of that first term using (4.5) shows that it is bounded by

$$\frac{(M_n)^{N_0}}{N_0!} \left(\frac{1}{2^n}\right)^{N_0} e^{pN_0 n^{-1-\varepsilon}} = \frac{(cn)^{N_0}}{N_0!} e^{pN_0 n^{-1-\varepsilon}} < c_1 \frac{(cn)^{N_0} e^{p}}{\left(\frac{N_0}{e}\right)^{N_0}}$$

$$< c_1 e^{p} \left(\frac{1}{2}\right)^{N_0} \le c_2 \left(\frac{1}{2}\right)^n.$$

We conclude that

$$E^{1/s}(r,\theta,n) < 1 + c_2 \left(\frac{1}{2}\right)^n \le \exp\left[c_2 \left(\frac{1}{2}\right)^n\right],$$

and since $s \le cn^{1+\varepsilon}$, we have

$$E(r,\theta,n) \le \exp\left[c_3 n^{1+\varepsilon} \left(\frac{1}{2}\right)^n\right]. \tag{4.6}$$

Now the quantities $E(r,\theta,n)$ for various n represent the expected contribution to $\exp(p\varphi(r\theta))$ from the corresponding annuli B_n in the range where $(1-|z|)h^2(z) \le 1-r$. Since these contributions are independent random variables, we can multiply the inequalities (4.6) for various n to obtain a uniform bound

$$E\left(\sum_{(1-\lambda_k)h^2(\lambda_k)\le 1-r} p\varphi_k(r,\theta)\right) \le \exp \sum_n c_3 n^{1+\varepsilon} \left(\frac{1}{2}\right)^n < \infty. \tag{4.7}$$

In fact, the above estimates were made only for "large" r so that (4.5) would hold. However, for "small" r one can trivially obtain an analogue of (4.7), so this together with the bound (4.4) leads to the conclusion that

$$E(e^{p\varphi(r,\theta)}) \le ce^{p\psi(s)}$$

uniformly in U, where s satisfies $(1-s)h^2(s) = 1-r$. Now

$$h(s) = \max\left\{1, \left[\log_2\left(\frac{1}{1-s}\right)\right]^{1+\varepsilon}\right\}$$

and so we can choose c_1 such that

$$h^2(s) \le c_1 \log^{2+2\varepsilon}\left(\frac{1}{1-s}\right), \quad s \ge 1/2.$$

If then t is defined by

$$c_1(1-t)\log^{2+2\varepsilon}\left(\frac{1}{1-t}\right) = 1-r = (1-s)h^2(s),$$

we must have $t \ge s$ for all $s \ge 1/2$. Thus for $r \ge 1/2$, and for all θ,

$$E(e^{p\varphi(r,\theta)}) \le ce^{p\psi(s)} \le ce^{p\psi(t)}.$$

By a change of variable,

$$dr = \left[c_1 \log^{2+2\varepsilon}\left(\frac{1}{1-t}\right) - c_1(2+2\varepsilon)\log^{1+2\varepsilon}\left(\frac{1}{1-t}\right)\right] dt,$$

and we obtain that for all θ

$$\int_{1/2}^{1} E(e^{p\varphi(r,\theta)})dr \le c\int_{1/2}^{1} e^{p\psi(t)}dr \le c_2\int_{a}^{1} e^{p\psi(t)}\log^{2+2\varepsilon}\left(\frac{1}{1-t}\right)dt$$

for some a, $\frac{1}{2} < a < 1$. By (4.2) the last integral is finite, and this completes the proof of the theorem.

5. Transformation of zero sets

In this section we present an alternative method of dealing with the problem of non-thin zero sets, apart from the method of section 3. The idea is to move overly dense concentrations of zeros to "thinner" areas. What is needed is a theorem, like those of Korenblum in [7], which will guarantee that, under appropriate conditions, if the transformed sequence is an A^p zero set, then so is the original sequence.

(5.1) Definition Let $\{z_k\}$ be a sequence of not necessarily distinct points in U, and m a natural number. An admissible transformation of degree m on these points is made by replacing them by a single point $w_0 \neq 0$ such that

(5.2) $1 - |w_0| = \sum 1 - |z_k|$ and $|\arg z_k - \arg w_0| \le m\pi(1 - |w_0|)$ for all k.

Two such transformations are called disjoint if the sets $\{z_k\}$ which they transform are disjoint.

(5.3) Theorem *Let $\{z_n\}$ be a sequence in U, and let $\{w_n\}$ be a sequence obtained from $\{z_n\}$ by means of finitely or infinitely many disjoint admissible transformations of fixed degree m. Then if $\{w_n\}$ satisfies an A^p condition of Theorem (1.3), (3.2), or (3.8) for some p, $0 < p < \infty$, it follows that $\{z_n\}$ is an A^p zero set.*

Proof We concentrate on the effects of a single admissible transformation. Thus suppose that $\{z_k\}$ is a subsequence of $\{z_n\}$ which is transferred to the point w_0 as in (5.2). Now in all of our theorems, infinite products are constructed in which the zero at w_0 is included by a factor of the form

$$B_{w_0}(z)S_{w_0}^{(N)}(z)$$

as in (1.4) and (1.5). The proofs are all built on the basic estimate from Lemma (1.6), with $\xi_0 = w_0/|w_0|$

$$\left|B_{w_0}(z)S_{w_0}^{(N)}(z)\right| \le \begin{cases} \exp\left[cN\left(\frac{1-|w_0|}{|\xi_0-z|}\right)^2\right]; & \frac{1-|w_0|}{1-|z|} \le \min\left(\frac{1}{N},\frac{1}{4}\right), \\ \left|S_{w_0}^{(N)}(z)\right| & \text{otherwise.} \end{cases}$$

Thus, if we can replace $B_{w_0}(z)S_{w_0}^{(N)}(z)$ by a function which vanishes at $\{z_k\}$ and satisfies exactly the same estimates, and if we do likewise for all the $\{w_n\}$, then any of our theorems which apply to $\{w_n\}$ will automatically apply to $\{z_n\}$. In fact we shall replace $B_{w_0}(z)S_{w_0}^{(N)}(z)$ by

$$F(z) = \left(\prod_k B_{z_k}(z)\right) S_{w_0}^{(N)}(z).$$

Of course, the estimate $|F(z)| \le \left|S_{w_0}^{(N)}(z)\right|$ holds everywhere, so it remains only to prove that

$$|F(z)| \le \exp\left[cN\left(\frac{1-|w_0|}{|\xi_0-z|}\right)^2\right]; \quad \frac{1-|w_0|}{1-|z|} \le \min\left(\frac{1}{N},\frac{1}{4}\right)$$

where c will depend only on m.

Now

$$S_{w_0}^{(N)}(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)\right\}$$

where $d\mu(\theta) = \frac{1}{N}d\theta$ on the interval $\{e^{i\theta} = |\theta - \arg w_0| \le N\pi(1-|w_0|)\}$ and zero elsewhere. Since $1-|w_0| = \sum 1-|z_k|$,

$$S_{w_0}^{(N)}(z) = \prod S_k(z),$$

where $S_k(z)$ corresponds to the measure

$$d\mu_k = \frac{1-|z_k|}{1-|w_0|}d\mu.$$

We now turn to the estimation of $S_k(z)B_{z_k}(z)$ for a given k. For simplicity let $z_k = \beta$; $z_k/|z_k| = \xi$ and define

$$S(z) = \exp\left\{(1-|\beta|)\frac{\xi+z}{\xi-z}\right\}.$$

As in the proof of Lemma (1.6) we have

$$(5.4) \qquad \left|\log B_{z_k}(z) + \log S(z)\right| \le c\left(\frac{1-|\beta|}{|\xi - z|}\right)^2 \quad \text{when } 1-|\beta| \le \frac{1}{4}(1-|z|).$$

We continue by estimating

$$|\log S_k(z) - \log S(z)| \quad \text{when } \frac{1-|w_0|}{1-|z|} \le \min\left(\frac{1}{N}, \frac{1}{4}\right).$$

However,

$$\log S_k(z) - \log S(z) = \frac{\alpha}{2\pi}\int_{\alpha_1}^{\alpha_2}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z} - \frac{\xi+z}{\xi-z}\right)d\theta,$$

where

$$\alpha_{1,2} = \arg w_0 \pm N\pi(1-|w_0|) \quad \textit{and} \quad \alpha = \frac{1}{N}\frac{1-|\beta|}{1-|w_0|}.$$

Thus

$$\log S_k(z) - \log S(z) = \frac{\alpha}{2\pi}\int_{\alpha_1}^{\alpha_2}\frac{2z(\xi - e^{i\theta})d\theta}{(e^{i\theta}-z)(\xi - z)},$$

and so

$$|\log S_k(z) - \log S(z)| \le \frac{c\alpha|\alpha_2 - \alpha_1|^2}{\min\limits_{\alpha_1 \le \theta \le \alpha_2} |e^{i\theta} - z||\xi - z|}.$$

However, since $1-|z| \ge N(1-|w_0|)$, which is of the order of magnitude of $|\alpha_i - \arg w_0|$, $i = 1, 2$, and since (5.2) holds, we can deduce that

$$\min_{\alpha_1 \le \theta \le \alpha_2} |e^{i\theta} - z||\xi - z| \ge c|\xi_0 - z|^2, \quad \text{for a constant } c$$

which depends only on m. Since we are working with a fixed m, we can conclude that

$$|\log S_k(z) - \log S(z)| \le c\frac{\alpha|\alpha_2-\alpha_1|^2}{|\xi_0 - z|^2} = c\frac{N(1-|\beta|)(1-|w_0|)}{|\xi_0 - z|^2}.$$

Combining with (5.4) we obtain a similar estimate for $|\log B_{z_k}(z) + \log S_k(z)|$. Finally, we have that whenever

$$\frac{1-|w_0|}{1-|z|} \le \min\left(\frac{1}{N}, \frac{1}{4}\right),$$

then

$$\left|\prod_k B_{z_k}(z)\right|\left|S_{w_0}^{(N)}(z)\right| \le \exp\sum_k \left|\log B_{z_k}(z) + \log S_k(z)\right|$$
$$\le \exp\sum_k cN\frac{(1-|z_k|)(1-|w_0|)}{|\xi_0 - z|^2}$$
$$= \exp\left[cN\left(\frac{1-|w_0|}{|\xi_0 - z|}\right)^2\right],$$

which is exactly the estimate we needed to prove.

6. Necessary conditions

In this section we wish to develop some strong necessary conditions on A^p zero sets. In fact, the main results are not new, as they appeared in the author's doctoral thesis [5]. However, they were never published in journal form. So we will bring them here, with outlines of the proofs. Then we shall add some new conclusions from these results.

First let f be analytic in the unit disc or in a sector thereof, and assume that $f(0) \neq 0$. If θ_1 is such that $f(re^{i\theta_1})$ is defined and nonzero for, say, $0 \le r < r_0$ then for $0 \le r_1 < r_0$ we define $g(r_1, \theta_1) =$ the net change in $\arg(f)$ along the ray $\theta = \theta_1$, between $r = 0$ and $r = r_1$. Thus

$$(6.1)\qquad g(r_1,\theta_1) = \int_0^{r_1} \frac{\partial[\arg f(re^{i\theta_1})]}{\partial r}dr = -\int_0^{r_1}\left\{\frac{\partial[\log|f(re^{i\theta})|}{\partial\theta}\right\}_{\theta=\theta_1}\frac{dr}{r}.$$

Finally, let

$$(6.2)\qquad h(r_1,\theta_1) = \frac{1}{2\pi}\int_0^{r_1}\frac{g(r,\theta_1)}{r}dr \quad (0 \le r_1 < r_0).$$

With this notation we present a generalized Jensen formula for a sector.

(6.3) Lemma *Let f be analytic in the closed sector $\theta_1 \le \arg z \le \theta_2$, $|z| \le r_1$ with*

$$0 < \beta = \frac{\theta_2 - \theta_1}{2\pi} < 1.$$

Assume that $f(z) \neq 0$ on the boundary rays of the sector and let $\{z_k\}_1^N$, be the zeros of f in the interior, repeated according to their multiplicity. Define g and h as above. Then

$$(6.4)\quad \beta\log|f(0)| + \sum_{k=1}^N \log\frac{r_1}{|z_k|} = h(r_1,\theta_1) - h(r_1,\theta_2) + \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\log|f(r_1e^{i\theta})|d\theta.$$

The proof is a direct adaptation of the classical proof of Jensen's formula using the argument principle.

In order to apply this lemma to A^p functions we need estimates on the function h. The next two lemmas provide these estimates.

(6.5) Lemma *Let $f \in N$, the Nevanlinna class; i.e., let f be analytic in the disc with*

$$\|f\|_N = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta < \infty. \tag{6.5}$$

Suppose that $f(0) \neq 0$ and assume that there is some value $\theta = \theta_0$ such that $f(re^{i\theta_0}) \neq 0$ $(0 \leq r < 1)$ and a constant M with

$$|h(r, \theta_0)| \leq M \quad \text{for } 0 \leq r < 1 \quad (h \text{ as in } [6.2]).$$

Then for all θ and r for which h is defined

$$h(r, \theta) \leq M + 2\|f\|_N + 2\log^- |f(0)|.$$

Proof Let $\{z_k\}$ denote the zero set of f, with repetition according to multiplicity. Now if for some θ, $f(re^{i\theta}) \neq 0$ for $0 \leq r \leq r_0$, and if $\beta = (\theta - \theta_0)/2\pi$, then by Lemma (6.3),

$$h(r, \theta) = h(r, \theta_0) - \beta \log |f(0)| - \sum_{\substack{0<|z_k|<r \\ 0<\arg z_k<\theta_0}} \log \frac{r}{|z_k|} + \frac{1}{2\pi} \int_{\theta_0}^{\theta} \log |f(re^{i\theta})| d\theta.$$

Thus

$$|h(r, \theta)| \leq M + |\log |f(0)|| + \sum_{|z_k|<r} \log \frac{r}{|z_k|} + \frac{1}{2\pi} \int_0^{2\pi} |\log |f(re^{i\theta})|| d\theta.$$

But by the ordinary Jensen formula

$$\sum_{|z_k|<r} \log \frac{r}{|z_k|} = -\log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta,$$

and so we obtain that

$$|h(r, \theta)| \leq M + 2\log^- |f(0)| + 2\|f\|_N.$$

(6.6) Lemma *Let $f \in A^p$, $0 < p < \infty$, and assume that $f(z) \neq 0$ for $|z| \leq 1/2$. Then there exists a constant M depending only on f such that*

$$|h(r,\theta)| \leq M \text{ on all rays where } f \text{ does not vanish.}$$

Proof For $0 \leq \theta < 2\pi$, let D_θ denote the disc $|z - \frac{1}{2}e^{i\theta}| < \frac{1}{2}$, and let f_θ denote the restriction of f to D_θ. It is well known, and easily seen, that each f_θ belongs to the Nevanlinna class for D_θ, and that

$$\|f_\theta\|_N \leq M_0, \quad M_0 \text{ independent of } \theta. \tag{6.7}$$

In D_θ we define our two auxillary functions, denoted by G_θ and H_θ, and we identify points in D_θ by their polar coordinates in the full unit disc. Note that where defined $G_\theta(r,\theta)$ equals the net change in $\arg f_\theta$ along the segment between $\frac{1}{2}e^{i\theta}$ (the origin in D_θ) and $re^{i\theta}$. Thus

$$G_\theta(r,\theta) = g(r,\theta) - g(\tfrac{1}{2},\theta).$$

Similarly

$$H_\theta(r,\theta) = \int_{1/2}^{r} \frac{g(t,\theta) - g(1/2,\theta)}{t - 1/2}\,dt. \tag{6.8}$$

Now by Lemma (6.5), if $r > \frac{1}{2}$

$$|H_\theta(r,\theta)| \leq \sup_{0 \leq r \leq 1/2} |h(r,\theta)| + 2\|f_\theta\|_N + 2\log^{-1}|f(\frac{1}{2}e^{i\theta})|. \tag{6.9}$$

Since f is nonvanishing for $|z| \leq 1/2$, it follows from the definition (6.2) that $h(r,\theta)$ is uniformly bounded for all $r < \frac{1}{2}$ and all θ, and clearly $\log^- |f(\frac{1}{2}e^{i\theta})|$ is uniformly bounded. Together with Lemma (6.5) and (6.7), this shows that there exists M_1 such that

$$|H_\theta(r,\theta)| \leq M_1 \quad \text{for all } r > 1/2 \text{ and } \theta, \tag{6.10}$$

where f does not vanish on the ray $\arg z = \theta$.

Now we turn to a bound on h, using our bound on H_θ. However, for $r \leq \frac{1}{2}$, h is trivially bounded, and if $r > 1/2$

$$h(r,\theta) = h(1/2,\theta) + \int_{1/2}^{r} \frac{g(t,\theta)}{t}\,dt,$$

so we need only bound the last integral. However,

$$\int_{1/2}^{r} \frac{g(t,\theta)}{t} dt = \int_{1/2}^{r} \frac{g(t,\theta) - g(1/2,\theta)}{t - 1/2} \frac{t - 1/2}{t} d\theta + \int_{1/2}^{r} \frac{g(1/2,\theta)}{t} dt.$$

By (6.8) and (6.10) the first integral is uniformly bounded, and since g is continuous for $|z| \leq 1/2$ the second integral is clearly bounded. Thus the proof of the lemma is complete.

(6.11) Theorem *Let $f \in A^p$ $0 < p < \infty$, and let $f(0) \neq 0$. Then there exists a constant M depending only on f and p such that the following conditions hold: If $\{z_k\}$, $0 < |z_1| \leq |z_2| \leq \cdots$ denote the zeros of f, counting multiplicity, in a sector $\{z : \theta_1 \leq \arg z \leq \theta_2\}$ with $0 \leq (\theta_2 - \theta_1)/2\pi = \beta < 1$, then for $0 < r < 1$,*

(i) $\sum_{|z_k| \leq r} \log(r/|z_k|) \leq M + \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \log|f(re^{i\theta})| d\theta$.

(ii) $\prod_{|z_k| \leq r} r/|z_k| \leq M \left\{ \int_{\theta_1}^{\theta_2} |f(re^{i\theta})|^p \right\}^{\beta/p} \leq M \left\{ 1/(1-r)^{\beta/p} \right\}$.

(iii) *For every N,* $\prod_{k=1}^{N} 1/(|z_k|) \leq M N^{\beta/p}$.

(iv) *If the full zero set of f is thin, then the expressions $r/|z_k|$ in* (i) *and* (ii) *can be replaced by $1/(|z_k|)$. Moreover, with φ as in Theorem* (1.3) *one has for each m,*

$$\frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \varphi(r,\theta,m) d\theta \leq c_m + \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \log|f(re^{i\theta})| d\theta$$
$$\leq c'_m + \log \left\{ \frac{1}{(1-r)^{\beta/p}} \right\},$$

where c_m and c'_m depend only on f, p and m.

Proof If f has no zeros on the rays $\theta = \theta_1$ and $\theta = \theta_2$ for $|z| \leq r$, and no zeros in the disc $|z| \leq 1/2$, (i) follows immediately from Lemmas (6.3) and (6.6). For general f divide by a finite Blaschke product of f's zeros of modulus $r \leq 1/2$, and use a limit process to take care of zeros on the bounding rays; namely approximate the given sector by sectors in which f is nonzero on the bounding rays. To obtain (ii) from (i) simply write

$$\frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \log|f(re^{i\theta})| d\theta = \frac{\beta}{p} \int_{\theta_1}^{\theta_2} \log|f(re^{i\theta})|^p \frac{d\theta}{\theta_2 - \theta_1}$$

and exponentiate (i) using Jensen's inequality together with the fact that $\left(\frac{1}{\theta_2 - \theta_2}\right)^{\beta}$ is bounded as $\beta \to 0$. To prove (iii) we note that (ii) actually implies a more general inequality; namely, for every N and every r,

$$\prod_{k=1}^{N} \frac{r}{|z_k|} \leq M \left\{ \int_{\theta_1}^{\theta_2} |f(re^{i\theta})|^p \right\}^{\beta/p}. \tag{6.12}$$

In particular, if $N = N(r) =$ the number of $\{z_k\}$ in $|z| \leq r$, then (6.12) is (ii). However, for $N > N(r)$ we have simply added to the left side of (ii) factors less than 1, and for $N < N(r)$ we have removed factors greater than 1. Thus we can relate to (6.12) as an inequality valid for a fixed N and variable r. It follows that

$$\int_0^1 \prod_{k=1}^{N} \left(\frac{r}{|z_k|}\right)^{p/\beta} rdr \leq M^{p/\beta} \|f\|_p^p.$$

By integration and simplification we obtain

$$\prod_{k=1}^{N} \frac{1}{|z_k|} \leq M \|f\|_p^\beta (Np/\beta + 1))^{\beta/p} \leq cN^{\beta/p},$$

proving (iii).

The first claim in (iv) follows from the fact that for thin zero sets we have only $0\left(\frac{1}{1-r}\right)$ zeros of modulus less than r. To prove the second claim, in view of the first claim and the result of (i), it is enough to show that the expression

$$\frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \varphi(r, \theta, m) d\theta - \sum_{\substack{|z_k| \leq r \\ \theta_1 \leq \arg z_k \leq \theta_2}} \log \frac{1}{|z_k|} \tag{6.13}$$

is uniformly bounded. However, by (1.13) we have

$$\frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \varphi(r, \theta, m) d\theta = \int_{\theta_1}^{\theta_2} \sum_{|z_\ell| \leq r} d\mu_{\ell,m}(\theta), \tag{6.14}$$

where the sum is over all zeros $\{z_\ell\}$ of f in the disc $|z| \leq r$. Now for those z_ℓ in the sector $S = \{z : |z| \leq r;\ \theta_1 \leq \arg z \leq \theta_2\}$ which are not too close to the bounding rays, i.e., those $z_\ell \in S$ which satisfy

$$|\arg z_\ell - \theta_i| \geq m\pi(1 - |z_\ell|), \quad i = 1, 2,$$

the full support of $d\mu_{\ell,m}$ is contained in $[\theta_1, \theta_2]$, and so the contribution of the corresponding term to the integral on the right side of (6.14) is the full mass of $d\mu_{\ell,m}$, namely $1 - |z_\ell|$. Regarding the remaining z_ℓ in S, their closeness to the boundary rays of S, together with the thinness of the full set $\{z_\ell\}$, guarantees that there can be only boundedly many such z_ℓ in each dyadic annulus $\{z : 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}\}$. Therefore, the sum $\sum \int_{\theta_1}^{\theta_1'} d\mu_{\ell,m} \leq \sum 1 - |z_\ell|$ over these z_ℓ is uniformly bounded.

Similarly, any contribution to the sum in (6.14) which comes from outside S can only come from those z_ℓ satisfying

$$|\arg z_\ell - \theta_i| \leq m\pi(1-|z_\ell|), \quad i=1,2,$$

and as above, their contribution is uniformly bounded. We conclude that

$$\frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\varphi(r,\theta,m)d\theta - \sum_{z_k\in S} 1-|z_k| \leq M,$$

and so the boundedness of (6.13) follows immediately since for the full zero set $\{z_k\}$, which is a thin set, we have

$$\sum \left| \log\frac{1}{|z_\ell|} - (1-|z_\ell|)\right| < \infty.$$

This completes the proof of (iv).

We note that condition (iv) implies that in some sense $\varphi(r,\theta,m)$ is "close" to $\log|f|$ for every $f \in A^p$ whose zero set is thin. Thus one might reasonably conjecture that the sufficient condition of Theorem (1.3) is also necessary. We have so far been unable to resolve that conjecture.

REFERENCES

[1] G. Bomash, *A Blaschke-type product and random zero sets for Bergman spaces*, to appear.

[2] P. L. Duren, D. Khavinson, H. S. Shapiro and C. Sundberg, *Contractive zero divisors in Bergman spaces*, to appear.

[3] H. Hedenmalm, *A factorization theorem for square integrable analytic functions*, to appear.

[4] C. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. **41** (1974), 693–710.

[5] C. Horowitz, *Zeros of Functions in the Bergman Spaces*, Thesis, Univ. of Michigan, 1974.

[6] B. Korenblum, *An extension of the Nevanlinna theory*, Acta Math. **135** (1975), 187–219.

[7] B. Korenblum, *Transformations of zero sets by contractive operators in the Bergman space*, Bull. Sci. Math., 2^e serie **114** (1990), 385–394.

[8] E. Leblanc, *A probabilistic zero condition for the Bergman space*, Mich. Math. J. **37** (1990), 427–436.

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