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ON THE RATE OF CONVERGENCE OF BEZIER VARIANT OF SZASZ-DURRMEYER OPERATORS

Vijay Gupta
(*Netaji Subhas Institute of Technology, New Delhi, India*)
Ulrich Abel
(*University of Applied Sciences, Friedberg, Germany*)

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Abstract

In the present paper, we introduce Szász-Durrmeyer-Bézier operators $M_{n,\alpha}(f,x)$, which generalize the Szász-Durrmeyer operators. Here we obtain an estimate on the rate of convergence of $M_{n,\alpha}(f,x)$ for functions of bounded variation. Our result extends and improves that of Sahai and Prasad[9] *and Gupta and Pant*[3].

Key Words *Szász-Durrmeyer operator, total variation, asymtotic order*

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1 Introduction

For a functions defined on the infinite interval $[0,\infty)$, the Szász-Mirakyan operators $S_n (n\in N)$ are defined by

$$S_n(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x)f(k/n),\quad \text{where}\quad p_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}.$$

The rate of convergence of these operators S_n was discussed in [1].

Recently X. M. Zeng[10] defined for each $\alpha\geqslant 1$, a Bézier variant of Szász-Mirakyan operators by

$$S_{n,\alpha}(f,x)=\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)f(k/n), \tag{1.1}$$

where $Q_{n,k}^{(\alpha)}(x)=J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)$ and $\sum_{j=k}^{\infty}p_{n,j}(x)=J_{n,k}(x)$ is the sum of Szász basis functions.

Kasana et al.[7] and Mazhar and Totik[8] independently introduced the Szász Durrmeyer operators to approximate Lebesgue integrable functions on the interval $[0,\infty)$, as

$$M_n(f,x)=n\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)\mathrm{d}t,\qquad x\in[0,\infty).\tag{1.2}$$

We now introduce the Bézier variant of these Szász-Durrmeyer operators, for $\alpha\geqslant 1$, we define the operators

$$M_{n,\alpha}(f,x)=n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_0^{\infty}p_{n,k}(t)f(t)\mathrm{d}t.\tag{1.3}$$

Obviously, $M_{n,\alpha}(1,x)=1$ and particularly when $\alpha=1$, the operators (1.3) reduce to the usual Szász Durrmeyer operators defined by (1.2) and are studied in [2],[3],[4],[5],[6],[8] and [9] etc. For further properties of $Q_{n,k}^{(\alpha)}(x)$ and $J_{n,k}(x)$, we refer the readers to [10].

In [10] Zeng studied the operators (1.1) and estimate the rate of convergence for bounded variation function. In the present paper, we obtain the rate of convergence for the generalized Szász-Durrmeyer-Bézier operators $M_{n,\alpha}(f,x)$ on functions of bounded variation.

2 Auxiliary Results

In this section we give certain results, which are necessary to prove the main result.

It is well known that the basis function $p_{n,k}(x)$ corresponds with the Poisson distribution in the probability theory. Gupta and Pant[3, Lemma 2.1] obtained the inequality:

$$p_{n,k}(x)\leqslant\frac{32x^2+2x+5}{2\sqrt{nx}},\quad\text{for all}\quad x\in(0,\infty)\quad\text{and}\quad k,n\in N.$$

Gupta et al.[4] derived the improved inequality

$$p_{n,k}(x)\leqslant\frac{\sqrt{3}}{2\sqrt{\pi nx}},\qquad\text{for } x\in(0,\infty).$$

In [10, Lemma 3] Zeng established a sharp estimate as follows:

$$p_{n,k}(x)\leqslant\frac{1}{\sqrt{2\pi[nx]}},\quad\text{for}\quad x>\frac{1}{n}\quad\text{and all}\quad k,n\in N.$$

Very recently Zeng and Zhao[12] improved these results and obtained the exact bound as follows:

Lemma 2.1[12]. *Let j be a fixed non negative integer and*

$$H(j) = \frac{(j+1/2)^{j+1/2}}{j!}e^{-(j+1/2)}.$$

Then for all $k \geqslant j$ and $x \in (0,\infty)$ there holds

$$p_{n,k}(x) \leqslant H(j)\frac{1}{\sqrt{nx}}.$$

Moreover, the coefficient $H(j)$ and the asymptotic order $n^{-1/2}$ (for $n \to \infty$) are the best possible.

Since $\max_{j \geqslant 0} H(j) = H(0) = 1/\sqrt{2e}$, Lemma 2.1 implies $p_{n,k}(x) \leqslant 1/\sqrt{2enx}$, for each integer $k \geqslant 0$. This results and application of the mean value theorem yields.

Lemma 2.2. *For each integer $k \geqslant 0$, there holds the inequality*

$$Q_{n,k}^{(\alpha)}(x) \leqslant \alpha p_{n,k}(x) \leqslant \frac{\alpha}{\sqrt{2enx}}.$$

Lemma 2.3. *For each integer $k \geqslant 0$, we have*

$$n\int_x^{\infty} p_{n,k}(t)\mathrm{d}t = \sum_{j=0}^{k} p_{n,j}(x).$$

The proof of this lemma is elementary.

Lemma 2.4[7]. *Let the m-th order central moment be defined by*

$$M_{n,1}((t-x)^m, x) \equiv T_{n,m}(x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)(t-x)^m \mathrm{d}t.$$

Then we have

$$T_{n,0}(x) = 1,\ T_{n,1}(x) = \frac{1}{n},$$

$$nT_{n,m+1}(x) = xT_{n,m}(x) + (m+1)T_{n,m}(x) + 2mxT_{n,m-1}(x), \quad m \geqslant 1.$$

From the above recurrence relation, we have

$$T_{n,2}(x) = \frac{2}{n}\left(x + \frac{1}{n}\right).$$

Moroever, for $n \geqslant 1/x$, we have

$$M_{n,1}((t-x)^2,\ x) \equiv T_{n,2}(x) \leqslant \frac{4x}{n}.$$

Throughout the paper, let

$$K_{n,\alpha}(x,t) = n\sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)p_{n,k}(t),$$

and

$$\lambda_{n,\alpha}(x,t) = \int_0^t K_{n,\alpha}(x,u)\mathrm{d}u.$$

In particular

$$\lambda_{n,\alpha}(x,\infty)=\int_0^\infty K_{n,\alpha}(x,u)\mathrm{d}u=1.$$

Lemma 2. 5. *Let* $n\geqslant 1/x$, *then*

(i) *For* $0\leqslant y<x$, *we have*

$$\int_0^y K_{n,\alpha}(x,t)\mathrm{d}t\leqslant\frac{4\alpha\cdot x}{n(x-y)^2}$$

(ii) *For* $x<z<\infty$, *we have*

$$\int_z^\infty K_{n,\alpha}(x,t)\mathrm{d}t\leqslant\frac{4\alpha\cdot x}{n(z-x)^2}.$$

Proof. Using Lemmas 2. 2 and Lemma 2. 4, we have

$$\int_0^y K_{n,\alpha}(x,t)\mathrm{d}t\leqslant\int_0^y\frac{(x-t)^2}{(x-y)^2}K_{n,\alpha}(x,t)\mathrm{d}t=\frac{1}{(x-y)^2}M_{n,\alpha}((t-x)^2,x)$$
$$\leqslant\frac{\alpha}{(x-y)^2}M_{n,1}((t-x)^2,x)\leqslant\frac{4\alpha\cdot x}{n(x-y)^2}.$$

The proof of (ii) is similar.

3 Main Result

In this section we prove the following main theorem.

Theorem 3. 1. *Let f be a function of bounded variation on every finite subinterval of* $[0,\infty)$. *Satisfying the growth condition* $|f(t)|\leqslant Ke^{\beta\cdot t}$ *on* $(0,\infty)$, *for some constants* $K,\beta>0$. *Then for* $x\in(0,\infty)$, $\alpha\geqslant 1$ *and* $n\geqslant\max\left\{\frac{1}{x},4\beta\right\}$, *we have*

$$\left|M_{n,\alpha}(f,x)-\left\{\frac{1}{\alpha+1}f(x+)+\frac{\alpha}{\alpha+1}f(x-)\right\}\right|\leqslant\frac{\alpha}{\sqrt{2enx}}|f(x+)-f(x-)|$$
$$+\frac{x+12\alpha}{nx}\sum_{k=1}^{n}V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)+\frac{4\sqrt{2}K\alpha}{\sqrt{nx}}e^{2\beta x},\tag{3.1}$$

where

$$g_x(t)=\begin{cases}f(t)-f(x-), & 0\leqslant t<x,\\ 0, & t=x,\\ f(t)-f(x+), & x<t<\infty,\end{cases}$$

and $V_a^b(g_x)$ *is the total variation of* g_x *on* $[a,b]$.

Proof. Following [11], we obtain

$$\left|M_{n,\alpha}(f,x)-\left\{\frac{1}{\alpha+1}f(x+)+\frac{\alpha}{\alpha+1}f(x-)\right\}\right|$$
$$\leqslant|M_{n,\alpha}(g_x,x)|+\frac{1}{2}\left|M_{n,\alpha}(\mathrm{sign}(t-x),x)+\frac{\alpha-1}{\alpha+1}\right||f(x+)-f(x-)|.\tag{3.2}$$

First, we estimate $M_{n,\alpha}(\mathrm{sign}(t-x),x)$. There holds

$$M_{n,\alpha}(\mathrm{sign}(t-x),x)=n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(\int_x^{\infty}p_{n,k}(t)\mathrm{d}t-\int_0^x p_{n,k}(t)\mathrm{d}t\right)$$

$$=n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(2\int_0^{\infty}p_{n,k}(t)\mathrm{d}t-2\int_0^x p_{n,k}(t)\mathrm{d}t\right)$$

$$=2n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_x^{\infty}p_{n,k}(t)\mathrm{d}t-1.$$

Using Lemma 2.3 and the methods as in [3], we have

$$M_{n,\alpha}(\mathrm{sign}(t-x),x)=2\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(\sum_{j=0}^{k}p_{n,j}(x)\right)-1$$

$$=2\sum_{j=0}^{\infty}p_{n,j}(x)\sum_{k=j}^{\infty}Q_{n,k}^{(\alpha)}(x)-1$$

$$=2\sum_{j=0}^{\infty}p_{n,j}(x)J_{n,j}^{\alpha}(x)-1.$$

Thus

$$M_{n,\alpha}(\mathrm{sign}(t-x),x)+\frac{\alpha-1}{\alpha+1}=2\sum_{j=0}^{\infty}p_{n,j}(x)J_{n,j}^{\alpha}(x)-\frac{2}{\alpha+1}\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha+1)}(x),$$

since $\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha)}(x)=1$. Using the mean value theorem, we obtain

$$Q_{n,j}^{(\alpha+1)}(x)=J_{n,j}^{\alpha+1}-J_{n,j+1}^{\alpha+1}(x)=(\alpha+1)p_{n,j}(x)\xi_{n,j}^{\alpha}(x),$$

where $J_{n,j}^{\alpha}(x)<\xi_{n,j}^{\alpha}(x)<J_{n,j}^{\alpha}+1(x)$. Hence

$$\left|M_{n,\alpha}(\mathrm{sign}(t-x),x)+\frac{\alpha-1}{\alpha+1}\right|=2\sum_{j=0}^{\infty}p_{n,j}(x)\left(J_{n,j}^{\alpha}(x)-\xi_{n,j}^{\alpha}(x)\right)$$

$$\leqslant 2\sum_{j=0}^{\infty}p_{n,j}(x)\left(J_{n,j}^{\alpha}(x)-J_{n,j+1}^{\alpha}(x)\right)$$

$$\leqslant 2\alpha\sum_{j=0}^{\infty}p_{n,j}(x)\left(J_{n,j}(x)-J_{n,j+1}(x)\right)=2\alpha\sum_{j=0}^{\infty}p_{n,j}^2(x).$$

Using Lemma 2.1, we get

$$\left|M_{n,\alpha}(\mathrm{sign}(t-x),x)+\frac{\alpha-1}{\alpha+1}\right|=\frac{2\alpha}{\sqrt{2enx}}. \tag{3.3}$$

In the second part of the proof we estimate $M_{n,\alpha}(g_x,x)$. We represent it by four integrals

$$M_{n,\alpha}(g_x,x)=\int_0^{\infty}K_{n,\alpha}(x,t)g_x(t)\mathrm{d}t=\left(\int_{I_1}+\int_{I_2}+\int_{I_3}+\int_{I_4}\right)K_{n,\alpha}(x,t)g_x(t)\mathrm{d}t$$

$$=E_1+E_2+E_3+E_4,\text{ say}, \tag{3.4}$$

where $I_1=[0,x-x/\sqrt{n}]$, $I_2=[x-x/\sqrt{n},x+x/\sqrt{n}]$, $I_3=[x+x/\sqrt{n},2x)$ and $I_4=[2x,\infty)$. We first estimate E_1, Writing $y=x-x/\sqrt{n}$ and using Stieltjes integration by parts, we have

$$E_1=\int_0^y g_x(t)d_t(\lambda_{n,a}(x,t))=g_x(y)\lambda_{n,a}(x,y)-\int_0^y\lambda_{n,a}(x,t)d_t(g_x(t)).$$

Since $|g_x(y)|\leqslant V_y^x(g_x)$, it follows that

$$|E_1|\leqslant V_y^x(g_x)\lambda_{n,a}(x,y)+\int_0^y\lambda_{n,a}(x,t)d_t(-V_t^x(g_x)).$$

By using Lemma 2.5, we obtain

$$|E_1|\leqslant V_{y+}^x(g_x)\frac{4ax}{n(x-y)^2}+\frac{4ax}{n}\int_0^y\frac{1}{(x-t)^2}d_t(-V_t^x(g_x)).$$

Integrating by parts the last term, we have after simple computation

$$|E_1|\leqslant\frac{4a.x}{n}\left[\frac{V_0^x(g_x)}{x^2}+2\int_0^y\frac{V_t^x(g_x)}{(x-t)^3}\right].$$

Now replacing the variable y in the last integral by $x-x/\sqrt{n}$, we obtain

$$|E_1|\leqslant\frac{8a}{nx}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x).\tag{3.5}$$

We proceed with E_2. For $t\in[x-x/\sqrt{n},x+x/\sqrt{n}]$, we have

$$|g_x(t)|=|g_x(t)-g_x(x)|\leqslant V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)$$

and therefore

$$|E_2|\leqslant V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}d_t(\lambda_{n,a}(x,t)).$$

Since $\int_a^b d_t(\lambda_{n,a})(x,t))\leqslant 1$ for $(a,b)\subset[0,\infty)$, we conclude

$$|E_2|\leqslant V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)\leqslant\frac{1}{n}\sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x).\tag{3.6}$$

Next, we estimate E_3, by setting $z=x+x/\sqrt{n}$, we have

$$E_3=\int_z^{2x}K_{n,a}(x,t)g_x(t)dt=-\int_z^{2x}g_x(t)d_t(1-\lambda_{n,a}(x,t))$$

$$=-g_x(2x)(1-\lambda_{n,a}(x,2x))+g_x(z)(1-\lambda_{n,a}(x,z))+\int_z^{2x}(1-\lambda_{n,a}(x,t))d_tg_x(t).$$

Since $|g_x(t)|=|g_x(t)-g_x(x)|\leqslant V_x^t(g_x)$ it follows, by Lemma 2.5,

$$|E_3|\leqslant\frac{4ax}{n}\left\{x^{-2}V_x^{2x}(g_x)+(z-x)^{-2}V_x^z(g_x)+\int_z^{2x}(t-x)^{-2}d_tV_x^t(g_x)\right\}.$$

Again integrating by parts, we derive

$$|E_3|\leqslant\frac{4ax}{n}\left\{2x^{-2}V_x^{2x}(g_x)+2\int_z^{2x}V_x^t(g_x)(t-x)^{-3}dt\right\}.$$

Thus arguing similarly as in the estimate of E_1, we obtain

$$|E_3|\leqslant\frac{12a}{nx}\sum_{k=1}^n V_x^{x+x/\sqrt{k}}(g_x).\tag{3.7}$$

Finally, we estimate E_4. By assumption, we have $|g_x(t)|\leqslant 2Ke^{\beta\cdot t}$, for $t\geqslant x$. Application of Lemma 2.2 yields

$$|E_4| = \left|n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_{2x}^{\infty}p_{n,k}(t)g_x(t)\mathrm{d}t\right| \leqslant 2K\alpha\cdot n\sum_{k=0}^{\infty}p_{n,k}(x)\int_{2x}^{\infty}p_{n,k}(t)e^{\beta\cdot t}\mathrm{d}t.$$

Furthermore, for $t\geqslant 2x>0$, we have $(t-x)/x\geqslant 1$ and application of the Schwarz inequality yields

$$|E_4|\leqslant \frac{2K\alpha}{x}n_{n,k}p_{n,k}(x)\int_{2x}^{\infty}p_{n,k}(t)(t-x)e^{\beta\cdot t}\mathrm{d}t$$

$$\leqslant \frac{2K\alpha}{x}\left(n\sum_{k=0}^{\infty}p_{n,k}(x)\int_{2x}^{\infty}p_{n,k}(t)(t-x)^2\mathrm{d}t\right)^{1/2}\left(n\sum_{k=0}^{\infty}p_{n,k}(x)\int_{2x}^{\infty}p_{n,k}(t)e^{2\beta\cdot t}\mathrm{d}t\right)^{1/2}$$

$$\leqslant \frac{2K\alpha}{x}(M_{n,1}((t-x)^2,x))^{1/2}(M_{n,1}(e^{2\beta\cdot t},x))^{1/2}.$$

For $n\geqslant 4\beta>0$, we have

$$M_{n,1}(e^{2\beta\cdot t},x) = \sum_{k=0}^{\infty}\left(\frac{n}{n-2\beta}\right)^{k+1}p_{n,k}(x) \leqslant \left(\frac{n}{n-2\beta}\right)\exp\left(2\beta\frac{nx}{n-2\beta}\right)\leqslant 2e^{4\beta\cdot x}.$$

Therefore by Lemma 2.4, we obtain

$$|E_4|\leqslant\frac{4\sqrt{2}K\alpha}{\sqrt{nx}}e^{2\beta\cdot x} \tag{3.8}$$

Collecting the estimates of (3.5) to (3.8), we have

$$|M_{n,\alpha}(g_x,x)|\leqslant\frac{x+12\alpha}{nx}\sum_{k=1}^{n}V_x^{x+x/\sqrt{k}}(g_x)+\frac{4\sqrt{2}K\alpha}{\sqrt{nx}}e^{2\beta\cdot x}. \tag{3.9}$$

Finally, taking advantage of (3.2), the estimates (3.3) and (3.9) imply the required result (3.1). This completes the proof of the theorem.

Remark 2. We can estimate the rate of convergence for the operators $M_{n,\alpha}$ in terms of Chanturiya's modulus of variation as obtained by the authors [4] for modified Szász-Mirakyan operators.

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Vijay Gupta
School of Applied Sciences
Netaji Subhas Institute of Technology
Sector 3 Dwarka
New Delhi 110045, India
e-mail: vijay@nsit. ac. in

Ulrich Abel
Fachbereich MND
Fachhochschule Giessen-Friedberg
University of Applied Sciences
Wilhelm-Leuschner-Strabbe 13, 61169 Friedberg, Germany
e-mail: Ulrich. Abel@mnd. fh-friedberg. de