THE SPECTRUM OF COMPACT HYPERSURFACE IN SPHERE

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Abstract

Let M be a compact minimal hypersurface of sphere $S^{n+1}(1)$. Let $\overline{M}$ be $H(r)$-torus of sphere $S^{n+1}(1)$. Assume they have the same constant mean curvature H, the result in [1] is that if $\mathrm{Spec}^0(M,g)=\mathrm{Spec}^0(\overline{M},g)$, *then for $3\le n\le 6, r^2\le \frac{n-1}{n}$ or $n\ge 6, r^2\ge \frac{n-1}{n}$, then M is isometric to $\overline{M}$. We improved the result and prove that: if* $\mathrm{Spec}^0(M,g)=\mathrm{Spec}^0(\overline{M},g)$, *then M is isometric to $\overline{M}$. Generally, if* $\mathrm{Spec}^p(M,g)=\mathrm{Spec}^p(\overline{M},g)$, *here p is fixed and satisfies that $n(n-1)\ne 6p(n-p)$, then M is isometric to $\overline{M}$.*

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Let (M,g) be a compact oriented n dimensional Riemann manifold, Δ^p be the Laplace operator acting on p-forms on M. Then we have its discrete spectrum, which is denoted it by

$$\mathrm{Spec}^p(M,g)=\{0\le\lambda_1^p\le\lambda_2^p\le\cdots<+\infty\}.$$

One important problem on spectrum is the following: Let (M,g) and $(\overline{M},g)$ be a compact oriented n dimensional Riemann manifolds with $\mathrm{Spec}^p(M,g)=\mathrm{Spec}^p(\overline{M},g)$, then is (M,g) iso-

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metric to $(\overline{M},g)$? This problem is proved to be wrong by J.Milnor in 1964. But when (M,G) and $(\overline{M},g)$ are some special Riemann manifolds, under certain conditions, we can prove that (M,g) is isometric to $(\overline{M},g)$.

A recent result in [1] is as follows:

Theorem A. *Let M be an $H(r)-$ torus and $\overline{M}$ be a compact hypersurface. Assume that they have the same constant mean curvature H and*

$$\mathrm{Spec}^0(M,g)=\mathrm{Spec}^0(\overline{M},g).$$

Then under the following conditon: $3\le n\le 6, r^2\le \frac{n-1}{n}$ or $n\ge 6, r^2\ge \frac{n-1}{n}$, then M is isometric to $\overline{M}$.

We improve the result and obtain the followign conclusions.

Theorem 1. *Let M be an $H(r)$-torus and $\overline{M}$ be a compact hypersurface. Assume that they have the same constant mean curvature H and*

$$\mathrm{Spec}^p(M,g)=\mathrm{Spec}^p(\overline{M},g),$$

here p is fixed and satisfies that $n(n-1)\ne 6p(n-p)$. Then M is isometric to $\overline{M}$.

Theorem 2. *Let M be an $H(r)$-torus and $\overline{M}$ be a compact hypersurface. Assume that they have the same constant mean curvature H and $\mathrm{Spec}^0(M,g)=\mathrm{Spec}^0(\overline{M},g)$. Then M is isometric to $\overline{M}$.*

Lemma 1[3]**.** *Let $\lambda_i(i=1,2,\cdots,n)$ be real numbers such that $\sum_{i=1}^{n}\lambda_i=0$. Then*

$$\left|\sum_{i=1}^{n}\lambda_i^3\right|\le \frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{i=1}^{n}\lambda_i^2\right)^{\frac{3}{2}},$$

and equality holds, if and only if $\lambda_1=\sqrt{\frac{n-1}{n}\sum_{i=1}^{n}\lambda_i^2}, \lambda_2=\cdots=\lambda_n=-\sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}\lambda_i^2}$ in addition to the order or the opposite orientation.

Lemma 2. *Let $\lambda_i(i=1,2,\cdots,n)$ be real numbers such that*

$$\sum_{i=1}^{n}\lambda_i=0,\qquad \sum_{i=1}^{n}\lambda_i^2=A.$$

Then

$$\sum_{i=1}^{n}\lambda_i^4\le \frac{n^2-3n+3}{n(n-1)}A^2,$$

and equality holds if and only if $(n-1)$ of the λ_i's are equal.

We have the Minakshisundaram formula:

$$\sum_{i=0}^{\infty} e^{-\lambda_i^p t} \sim (4\pi t)^{\frac{n}{2}} \sum_{i=0}^{n} a_i^p t^i, \quad t \to 0^+,$$

$$a_0^p = \begin{pmatrix} n \\ p \end{pmatrix} \mathrm{Vol}(M),$$

$$a_1^p = \int_M c(n,p) r * 1,$$

$$a_2^p = \int_M \left(c_1(n,p)\rho^2 + c_2(n,p)|\mathrm{Ric}|^2 + c_3(n,p)|R|^2\right) * 1,$$

where ρ, Ric, R are Scalar curvature, Ricci curvature tensor, and Riemann curvature tensor respectively.

Later, we will denote $c_i(n,p)$ by $c_i, c(n,p)$ by c,

$$c = \frac{1}{6}\begin{pmatrix} n \\ p \end{pmatrix} - \begin{pmatrix} n-2 \\ p-1 \end{pmatrix},$$

$$c_1 = \frac{1}{72}\begin{pmatrix} n \\ p \end{pmatrix} - \frac{1}{6}\begin{pmatrix} n-2 \\ p-1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} n-4 \\ p-2 \end{pmatrix},$$

$$c_2 = -\frac{1}{180}\begin{pmatrix} n \\ p \end{pmatrix} + \frac{1}{2}\begin{pmatrix} n-2 \\ p-1 \end{pmatrix} - 2\begin{pmatrix} n-4 \\ p-2 \end{pmatrix},$$

$$c_3 = \frac{1}{180}\begin{pmatrix} n \\ p \end{pmatrix} - \frac{1}{12}\begin{pmatrix} n-2 \\ p-1 \end{pmatrix} + 2\begin{pmatrix} n-4 \\ p-2 \end{pmatrix}.$$

Proof of Theorem 1. For M, we choose a local adapted orthonormal frames $\{e_1, e_2, \cdots, e_n\}$ such that the second fundamental form of M is $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ at the point of p. By simple calculating, we get

$$\rho = n(n-1) + nH^2 - S, \tag{1}$$

$$|\mathrm{Ric}|^2 = n(n-1)^2 + 2(n-1)n^2H^2 - 2(n-1)S + \sum_{i=1}^{n}\lambda_i^4 + Sn^2H^2 - 2nH\sum_{i=1}^{n}\lambda_i^3, \tag{2}$$

$$|R|^2 = 2n(n-1) + 2S^2 - 2\sum_{i=1}^{n}\lambda_i^4 + 4n^2H^2 - 4S. \tag{3}$$

We set $\varphi = S - nH^2, u_i = H - \lambda_i (i = 1, 2, \cdots, n)$, then we have

$$\rho = n(n-1) + \varphi, \tag{4}$$

$$\begin{aligned} |\mathrm{Ric}|^2 = n(n-1)^2 + \textstyle\sum_{i=1}^n \lambda_i^4 + (2n-4)H \sum_{i=1}^n \lambda_i^3 \\ +[(n^2-6n+6)H^2 - 2(n-1)]\varphi + (n-1)^2 nH^2(H^2+2), \end{aligned} \tag{5}$$

$$|R|^2 = 2n(n-1) + 2\varphi^2 + (4nH^2 - 12H^2 - 4)\varphi + 8H\sum_{i=1}^n \lambda_i^3 - 2\sum_{i=1}^n \lambda_i^4 + 2n(n-1)H^2(H^2+2). \tag{6}$$

Similarly, we choose a local field of adapted orthonormal frames $\{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n\}$ in $\overline{M}$ such that $\{\bar{e}_1\}$ and $\{\bar{e}_2, \cdots, \bar{e}_n\}$ are local field of orthonormal frames of $S^1(\sqrt{1-r^2})$ and $S^{n-1}(r)$ respectively. By choosing $\{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n\}$ propertly, we can assume that the matrix of the second fundamental form of $\overline{M}$ is $\mathrm{diag}(-r/\sqrt{1-r^2}, r^{-1}\sqrt{1-r^2}I_{n-1})$ at every point of $\overline{M}$. We denote the Riemann curvature tensor, the Ricci curvature tensor and scalar curvature of $\overline{M}$ by $\overline{R}, \overline{\mathrm{Ric}}, \bar{\rho}$ respectively.

We set

$$\bar{\varphi} = \overline{S} - n\overline{H}^2 = \overline{S} - nH^2,$$

where $\overline{S}$ and $\overline{H}$ are the length of the second fundamental form and mean curvature respectively. Since $\mathrm{Spec}^0(M, g) = \mathrm{Spec}^0(\overline{M}, g)$, we have

$$\mathrm{Vol}(M) = \mathrm{Vol}(\overline{M}), \qquad \int_M \rho * 1 = \int_{\overline{M}} \bar{\rho} * 1, \tag{7}$$

$$\int_M (c_1\rho^2 + c_2|R|^2 + c_3|\mathrm{Ric}|^2) * 1 = \int_{\overline{M}} (c_1\bar{\rho}^2 + c_2|\overline{R}|^2 + c_3|\overline{\mathrm{Ric}}|^2) * 1. \tag{8}$$

Using (7) we get

$$\int_M \varphi * 1 = \int_{\overline{M}} \bar{\varphi} * 1. \tag{9}$$

Using (4),(5),(6),(8) and (9), we easily get

$$\begin{aligned} &\int_M [(c_1 + 2c_3)\varphi^2 + (c_2(2n-4) + 8c_3)H\sum_{i=1}^n u_i^3 + (c_2 - 2c_3)\sum_{i=1}^n u_i^4] * 1 \\ &= \int_{\overline{M}} [(c_1 + 2c_3)\bar{\varphi}^2 + (c_2(2n-4) + 8c_3)H\sum_{i=1}^n \bar{u}_i^3 + (c_2 - 2c_3)\sum_{i=1}^n \bar{u}_i^4] * 1. \end{aligned} \tag{10}$$

If $(c_2(2n-4) + 8c_3)H \geq 0, (c_2 - 2c_3) \geq 0$, using (9),(10), lemma 1 and lemma 2, we have

$$\begin{aligned} &\int_M \{(c_1 + 2c_3) + (c_2 - 2c_3)\frac{n^2-3n+3}{n(n-1)} + \frac{1}{2}[c_2(2n-4) + 8c_3][(n-1) - nr^2](n-2)\}\varphi^2 * 1 \\ &\leq \int_{\overline{M}} \{(c_1 + 2c_3) + (c_2 - 2c_3)\frac{n^2-3n+3}{n(n-1)} + \frac{1}{2}[c_2(2n-4) + 8c_3][(n-1) - nr^2](n-2)\}\bar{\varphi}^2 * 1. \end{aligned} \tag{11}$$

On the other hand, we have

$$\int_M \{(c_1+2c_3)+(c_2-2c_3)\frac{n^2-3n+3}{n(n-1)}+\frac{1}{2}[c_2(2n-4)+8c_3][(n-1)-nr^2](n-2)\}\varphi^2 * 1$$
$$\geq \int_{\overline{M}} \{(c_1+2c_3)+(c_2-2c_3)\frac{n^2-3n+3}{n(n-1)}+\frac{1}{2}[c_2(2n-4)+8c_3][(n-1)-nr^2](n-2)\}\overline{\varphi}^2 * 1. \tag{12}$$

Using (11),(12), we get that if

$$c_1+2c_3+(c_2-2c_3)\frac{n^2-3n+3}{n(n-1)}+\frac{1}{2}[c_2(2n-4)+8c_3][(n-1)-nr^2](n-2)\neq 0,$$

we have $\int_M \varphi^2 * 1 = \int_{\overline{M}} \overline{\varphi}^2 * 1$; if

$$c_1+2c_3+(c_2-2c_3)\frac{n^2-3n+3}{n(n-1)}+\frac{1}{2}[c_2(2n-4)+8c_3][(n-1)-nr^2](n-2)=0,$$

we have

$$\begin{aligned}
&\int_M (c_2(2n-4)+8c_3)H\frac{n-2}{2nr\sqrt{1-r^2}}\varphi * 1\\
&\leq \int_M [(c_1+2c_3)\varphi^2+(c_2(2n-4)+8c_3)H\sum_{i=1}^n u_i^3+(c_2-2c_3)\sum_{i=1}^n u_i^4] * 1\\
&= \int_{\overline{M}} [(c_1+2c_3)\overline{\varphi}^2+(c_2(2n-r)+8c_3)H\sum_{i=1}^n \overline{u}_i^3(c_2-2c_3)\sum_{i=1}^n \overline{u}_i^4] * 1\\
&= \int_{\overline{M}} (c_2(2n-r)+8c_3)H\frac{n-2}{2nr\sqrt{1-r^2}}\overline{\varphi} * 1.
\end{aligned} \tag{13}$$

Using (9), we know

$$\int_M (c_2(2n-4)+8c_3)H\frac{n-2}{2nr\sqrt{1-r^2}}\varphi * 1 = \int_{\overline{M}} (c_2(2n-4)+8c_3)H\frac{n-2}{2nr\sqrt{1-r^2}}\overline{\varphi} * 1.$$

Note that all the inequalities hold, we get after remuneration if necessary

$$u_1=(n-1)\sqrt{\frac{1}{n(n-1)}\varphi},\quad u_2=\cdots=u_n=-\sqrt{\frac{1}{n(n-1)}\varphi},\quad \varphi=\frac{n-1}{nr^2(1-r^2)},$$

so we have

$$\lambda_1=H-u_1=-\frac{r}{\sqrt{1-r^2}},\quad u_2=\cdots=u_n=\frac{\sqrt{1-r^2}}{r}.$$

If

$$(c_2(2n-4)+8c_3)H\geq 0,\quad (c_2-2c_3)<0,$$

or

$$(c_2(2n-4)+8c_3)H\geq 0,\quad (c_2-2c_3)\geq 0,$$

or

$$(c_2(2n-4)+8c_3)H\geq 0,\quad (c_2-2c_3)\geq 0,$$

the same as the above, we have the same result.

From [3] we know that M is isometric to $\overline{M}$. This completes the proof of Theorem 1.

Proof of Theorem 2. If $n = 1$, we know Theorem 2 is correct from [1]; if $n \neq 1, n(n-1) \neq 0$ satisfies the conditions of Theorem 1, so Theorem 2 is correct. This completes the proof of Theorem 2.

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