Analysis in Theory and Applications **20:3, 2004, 231-241**

FRAME MULTIRESOLUTION ANALYSIS AND INFINITE TREES IN BANACH SPACES ON LOCALLY COMPACT ABELIAN GROUPS*

S. S. Panday
(R. D. University, India)

Received Oct. 12, 2003 Revised: July 5, 2004

Abstract

We extend the concept of frame multiresolution analysis to a locally compact abelian group and use it to define certain weighted Banach spaces and the spaces of their antifunctionals. We define analysis and synthesis operators on these spaces and establish the continuity of their composition. Also, we prove a general result to characterize infinite trees in the above Banach spaces of antifunctionals.This paper paves the way for the study of corresponding problems associated with some other types of Banach spaces on locally compact abelian groups including modulation spaces.

Key words *frame multiresolution analysis, infinite trees in Banach spaces*

AMS(2000)subject classification 22B99, 43A15, 46E15

1 Introduction

The concept of multiresolution analysis was originally introduced by Mallat [Mal 89] in 1989.This technique has provided a new tool for the study of many problems in the space variables associated with frequency changes. Such problems usually take the form of the approximation of general functions by the sequences of simple functions generated by the multiresolution analysis. Mallat (loc.cit.) and Meyer [Mey 92] have studied in detail various applications of multiresolutions analysis in Euclidean spaces.

* "This work is supported by the financial grant of DST/MS/150 2K".

In a recent paper Galindo and Sanz [GS 01] have generalized the idea of multiresolution analysis to a fairly general locally compact abelian group. They have demonstrate that the spaces of integrable functions $L^p(\mathcal{G})$, $1 \leq p \leq \infty$, and the space of complex Radon measures $M(\mathcal{G})$ can be constructed in terms of multiresolution analysis using the concept of infinite trees.

Benedetto and Li [BL 98], on the other hand, have developed the theory of frame multiresolution analysis (F M R A), on the real line, which provides a new mathematical tool for the study of signal analysis.

In the present paper our aim is to extend the concept of FMRA on a locally compact abelian group G with a special structure and use it for the study of infinite trees in certain Banach spaces. In section 2 we present the notations and basic definition for use in the sequel. In section 3 we define weighted Lebesgue space on G with moderate weight functions and point out some of their basic properties.

In section 4, motivated by the work of Benedetto and Li [BL 98], we develop the notation of frame multiresolution analysis (FMRA) on a locally compact abelian group having a special structure and in the next section we use it to define a coefficient mapping on the lines of Galindo and Valladolid [GV 01,p .863]. Also, we define Banach Spaces $\mathcal{H}^p_w(G)$, $1 \leq p \leq \infty$, where w being a moderate function on G, so that the continuous embedding property hold

$$\mathcal{H}^p_w(G) \hookrightarrow L^2(G) \hookrightarrow \tilde{\mathcal{H}}^p_w(G),$$

hold true, where $\tilde{\mathcal{H}}^p_w(G)$ is the space of continuous conjugate linear functionals (anti-functionals) on the space $\tilde{\mathcal{H}}^p_w(G)$.

Theorem 5.1 provides a result about the continuity of an operator on the space $\tilde{\mathcal{H}}^p_w(G)$, while Theorem 6.1 deals with the continuity of its adjoint operator. Section 6 is devoted to the study of the continuity property of the composite operator of sections 5 and 6. In section 8, we prove a theorem to characterize on infinite tree in the space $\tilde{\mathcal{H}}^p_w(G), 1 \leq p \leq \infty$ while in section 9 imposing additional convergence conditions on the coefficients we obtain the corresponding results for the space $\tilde{\mathcal{H}}^1_w(G)$.Our result in fact, are more general than the corresponding results of Galindoand Valladodlid [GV 01].This paper paves the way for the study of corresponding problems associated with some other types of Banach spaces on locally compact abelian groups.

2 Notations and Basic Concepts

Let G be a locally compact abelian group composed of a sequence $(G_n)_{n \in Z}$ of subgroups satisfying the following conditions:

(i) G_n is open and compact for all $n \in Z$,

(ii) $G_n \subset G_{n+1}$,

(iii) $\cup_{n \in Z} G_n = G$,

(iv) $\bigcap G_n = \{0\}$,

where Z is the set of all integers.

As pointed out by the Galindo and Valladolid [GV 01, p.860], the condition (iv) can be replaced by the equivalent property:

(iv') $(G_n)_{n\in Z}$ is a base of neighbourhoods at 0.

By virtue of the condition (i) each quotient group G_{n+1}/G_n is finite, which implies that G/G_n is countable for all $n \in Z$. Let $(G_{n,j})_{j\in J}$ be the cosets of G_n, J being an index set.

3 Weighted Lebesgue Space on G

A function $u : G \to R_+$ is called a submultiplicative weight function on G provided the following conditions hold :

(i) $u(0) = 1$.

(ii) $u(x+y) \leq u(x)\, u(y)$, for all $x, y \in G$, where R_+ is the set of all positive numbers .

A locally integrable function $w : G \to R_+$ is known as a moderate weight provided there exists a submultipliacative weight function u on G such that

$$w(x+y) \leq u(x)w(y) \quad \text{for all} \quad x, y \in G.$$

Without loss of generality, we may assume that w is symmetric and continuous.

We denote by $L^p_w(G), 1 \leq p < \infty$, the Banach space of all measurable functions f with respect to the norm

$$\| f \|_{p,w} = \| f \mid L^p_w \| = (\int_G \mid f(x) \mid^p w^p(x) dx)^{1/p} < \infty. \tag{4.1}$$

In case $p = \infty$, we define the Banach space $L^\infty_w(G)$ as the space of all measurable functions f such that

$$\| f \|_{\infty,w} = \text{ess sup}\{\mid f(x) \mid w(x) : x \in G\} < \infty. \tag{4.2}$$

It can be easily seen that $L^p_w(G), 1 < p < \infty$, is a reflexive Banach space and $L^1_w(G)$ is a Banach algebra under convolution, usually known as Beurling algebra . Also, it is well known that $L^p_w(G)$ is a convolution module over $L^1_w(G)$. Also, since w is moderate, $L^p_w(G)$ is translation invariant.

We denote by $l^p_w(J)$ the sequence space associated with $L^p_w(J), J$ being an index set.

4 Frame Multiresolution Analysis of $L^2(G)$

Following Galindo and Valladolid [GV 01, p.861], we say that a complex- valued function

f *on* G is G_n-periodic provided

$$f(x+y)=f(x) \quad \text{for all} \quad x\in G \text{ and } y\in G_n.$$

Let P_n be the set of all G_n-periodic functions. Then, on account of the above definitions, it is clear that a function f in P_n is constant on the cosets $G_{n,j}, j\in J$, and we may express f in the form

$$f = \sum_{j\in J} a_j \chi_{\mathcal{G}_{n,j}},$$

where $\chi_{G_n,j}$ is the characteristic function of $G_{n,j}$ and J is an index set.

We write $P=\bigcup_n P_n$. It can be easily seen that the space $P\cap L^p(G)$ is dense in $L^p(G)$, $1\leq p<\infty$ (cf [GV 01] p.862).

We assume that V_n is the linear subspace of $\mathcal{G}_n$-periodic in $L^2(G)$, i.e.,

$$V_n = P_n\cap L^2(G).$$

On this lines of Benedetto and Li [BL 98, p.398], we say that a family $(V_n)_{n\in Z}$ is a frame multiresolution analysis (FMRA) of $L^2(G)$ provided the following conditions hold :

(i) V_n is a closed linear subspace of $L^2(G)$ such that $\bigcap_{n\in Z} V_n = \{0\}$. and $\cup_n V_n$ is dense in $L^2(G)$.

(ii) $\phi\in V_n \Rightarrow \tau_j\,\phi\in V_n, \forall j\in J$ being the translation operator.

(iii) The collection $(\tau_j\,\phi)_{j\in J}$ is a frame for V_n.

By virtue of the above construction, it is clear that the collection of translates $(\tau_j\,\phi_n)_{j\in J}$, where

$$\phi_n=\frac{1}{m(G_n)}\,\chi_{G_n}, \tag{4.1}$$

is a frame for the space V_n.

5 Continuity of Coefficient Operators On Banach Spaces

On the lines of Galindo and Valladolid [GV 01, p.863], we write

$$\alpha_j(g,\phi_n)=\langle g,\tau_i\,\phi_n\rangle_{j\in J}, \qquad \forall\, g\in L^2(G),$$

where $\phi_n\in V_n,\ \forall\, n\in Z$.

We now define a space $\mathcal{H}^p_w(G)$ by

$$\mathcal{H}^p_w(G)=\{g: g\in L^2(G) \quad \text{and} \quad \|\,\alpha_j(g,\phi_n)\mid l^p_w(J)\,\|<\infty\} \tag{5.1}$$

and endow it with the norm

$$\| g \mid \mathcal{H}^p_w(G) \| = \sup_{n \in Z} (m(G_n))^{1/p} \| \alpha_j(g, \phi_n) \mid l^p_w(J) \|, \qquad \forall j \in J. \tag{5.2}$$

We denote by $\tilde{\mathcal{H}}^p_w(G)$ the space of all continuous conjugate linear functionals on $\mathcal{H}^p_w(G)$. On account of these definitions, it is obvious that the continuous embeddings

$$\mathcal{H}^p_w(G) \hookrightarrow L^2(G) \hookrightarrow \tilde{\mathcal{H}}^p_w(G)$$

hold true and the innerproduct $L^2(G) \times L^2(G)$ extends to the sesquilinear form $\mathcal{H}^p_w(G) \times \tilde{\mathcal{H}}^p_w(G)$. The norm on the space $\tilde{\mathcal{H}}^p_w(G)$ takes the form

$$\| f \mid \tilde{\mathcal{H}}^p_w(G) \| = \sup (m(G_n))^{1/p} \| \alpha_j.(f, \phi_n) \mid l^q_{1/w}(J) \|, \tag{5.3}$$

where $\forall j \in J, \dfrac{1}{p} + \dfrac{1}{q} = 1$.

We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators $T : X \to Y$ between two Banach spaces X and Y with the operator norm $|||\, T \,|||_{\mathcal{L}}$. In case $X = Y$, we denote by $\mathcal{L}(X)$ the space of all bounded linear operators form X onto itself.

Our first result is the following:

Theorem 5.1. *If* $f \in \tilde{\mathcal{H}}^p_w(G)$, $g \in \mathcal{H}^p_w(G)$ *and*

$$T_g \equiv T_{g,J} : f \to (\alpha_j(f, g))_{j \in J} \tag{5.4}$$

is a linear operator associated with g , *then*

$$T_g \in \mathcal{L}(\tilde{\mathcal{H}}^p_w(G), l^p_w(J))$$

with

$$|||\, T_g \,|||_{\mathcal{L}(\tilde{\mathcal{H}}^p_w, l^p_w)} \le C_J \| \alpha_j(g, \phi_n) \mid l^p_w(J) \|, \qquad \forall n \in Z,$$

C_J *being a positive constant not necessarily the same at each occurrence.*

Proof. From the definitions of $\mathcal{H}^p_w(G)$ and $\tilde{\mathcal{H}}^p_w(G)$, we see that

$$\begin{aligned} | T_g f | &= | \alpha_j(f, g) | \\ &= | \langle f, \tau_j g \rangle | \\ &\le \| \tau_j g \mid \mathcal{H}^p_w \| \| f \mid \tilde{\mathcal{H}}^p_w \| \\ &\le C_J \| g \mid \mathcal{H}^p_w \| \\ &\le C_j \| \alpha_j(g, \phi_n) \mid l^p_w(J) \|, \end{aligned}$$

which completes the proof.

6 Continuity of Synthesis Operators

In this section we study the continuity property of synthesis operators T_h^* associated with T_g as discussed in § 4.

Precisely, we prove the following:

Theorem 6.1. *If the linear operator T_h^* is defined by*

$$T_h^* \equiv T_{h,j}^* : (\alpha_j(f,g))_{j\in J} \to \sum_{j\in J} \alpha_j(f,g)\tau_j h, \tag{6.1}$$

then, for all $h \in \mathcal{H}_w^p(G)$, we have

$$T_h^* \in \mathcal{L}(l_w^p(J), \mathcal{H}_w^p(G))$$

with

$$\||\, T_h^* \,\||_{\mathcal{L}(l_w^p(J),\mathcal{H}_w^p)} \le C_J \,\| h \mid \mathcal{H}_w^p(G) \,\| .$$

and the series on the right- hand side of (6.1) *is absolutely convergent in the norm topology of $\mathcal{H}_w^p(G)$.*

Proof. On account of the relation (5.2), we see that

$$\sum_{j\in J} \alpha_j(f,g) \le C_J \,\| g \mid \mathcal{H}_w^p(G) \,\|, \qquad \forall f \in \tilde{\mathcal{H}}_w^p(G).$$

This ensures that the series $\sum_{j\in J} \alpha_j(f,g)\tau_j h$ is absolutely convergent in the norm topology of $\mathcal{H}_w^p(G)$.

Hence we have

$$\begin{aligned} |\, T_h^*(\alpha_j(f,g)) \,| &= |\sum_{j\in J} \langle f, \tau_j g\rangle \tau_j h \,| \\ &\le \sup\{\sum_{j\in J} \langle f, \tau_j g\rangle\} \,\| \tau_j h \mid \mathcal{H}_w^p \,\| \\ &\le C_J \,\| h \mid \mathcal{H}_w^p(G) \,\| \\ &\Rightarrow T_h^* \in \mathcal{L}(l_w^p(J), \mathcal{H}_w^p(G)) \end{aligned}$$

with

$$\||\, T_h^* \,\||_{\mathcal{L}(l_w^p,\mathcal{H}_w^p)} \le C_J \,\| h \mid \mathcal{H}_w^p \,\| .$$

Thus the theorem holds true.

7 Continuity of the Composition Operator

In this section our aim is to study the continuity property of the composition operator

$$S_{h,g} \equiv S_{h.g,J} : f \to \sum_{j\in J} \alpha_j(f,g)\tau_j h. \tag{7.1}$$

We prove the following:

Theorem 7.1. *If $g, h \in \mathcal{H}^p_w(G)$, then $S_{h,g} \in \mathcal{L}(\tilde{\mathcal{H}}^p_w(G))$ with*

$$||| S_{h,g} |||_{\mathcal{L}(\tilde{\mathcal{H}}^p_w(G))} \leq C_J \| g \mid \mathcal{H}^p_w \| \| h \mid \mathcal{H}^p_w \|$$

and the series in (7.1) is unconditionally convergent in the norm topology of $l^q_{1/w}(J)$.

Proof. Since $\mathcal{H}^p_w(G) \subseteq \tilde{\tilde{\mathcal{H}}}^p_w(G)$ isometrically, we have

$$\begin{aligned}
| S_{h,g} f | &= | \sum_{j\in J} \alpha_j(f,g)\tau_j h | \\
&= | \sum_{j\in J} \langle f, \tau_j g\rangle \tau_j h | \\
&= \leq \sup\{\sum_{j\in J} \langle f, \tau_j g\rangle\} \| \tau_j h \mid \mathcal{H}^p \| \\
&\leq C_j \| f \mid \tilde{\mathcal{H}}^p_w \| \| \tau_j g \mid \tilde{\tilde{\mathcal{H}}}^p_w \| \| \tau_j h \mid \mathcal{H}^p_w \| \\
&\leq C_J \| f \mid \tilde{\mathcal{H}}^p_w \| g \mid \mathcal{H}^p_w \| \| h \mid \mathcal{H}^P_w \| \\
&\Rightarrow S_{h,g} \in \mathcal{L}(\tilde{\mathcal{H}}^p_w(G))
\end{aligned}$$

with

$$||| S_{h,g} |||_{\mathcal{L}(\tilde{\mathcal{H}}^p)} \leq C_J \| g \mid \mathcal{H}^p_w \| \| h \mid \mathcal{H}^p_w \|$$

and the series on the right - hand side of (6.1) is unconditionally convergent in $l^q_{1/w}(J)$.

8 Infinite Trees in $\tilde{\mathcal{H}}^p(G)$

On the lines of Following Galindo and Valladolid [GV 01,p.864], we say that a family $(\alpha_j(f,\phi_n))_{j\in J}$ of complex numbers forms an infinite tree provided it satisfies the cascade conditions

$$\alpha_k(f, \phi_s) = \frac{m(G_r)}{m(G_s)} \sum_{j:G_{r,j}\subset G_{s,k}} \alpha_j(f,\phi_r) \tag{8.1}$$

for all $r \geq s$ and $f \in \tilde{\mathcal{H}}^p, 1 \leq p \leq \infty$.

Motivated by the work of Galindo and Valladolid (loc.cit), we prove the following theorem to characterize an infinite tree in the space $\tilde{\mathcal{H}}^p_w(G)$, which includes the corresponding results of Galindo and Valladolid [GV 01 ,Th 3.4] as a particular case for $w = 1\xi$. Precisely, we prove the following:

Theorem 8.1. *If $f \in \tilde{\mathcal{H}}^p_w(G), 1 < p < \infty$, then the family $(\alpha_j(f, \phi_n))_{j\in J}$ is an infinite tree in the space $\tilde{\mathcal{H}}^p_w(G)$. and*

$$\| f \mid \tilde{\mathcal{H}}^p_w(G) \| = \sup_{n\in Z}(m(G_n))^{1/q} \| \alpha_j(f,\phi_n) \mid l^q_{1/w}(J) \| \tag{8.2}$$

Conversely, if $\{\alpha_j(n)\}$ is an infinite tree such that

$$\sup_{n\in Z}(m(G_n))^{1/q} \| \alpha_j(n) \mid l^q_{1/w}(J) \| < \infty \tag{8.3}$$

then there exists a unique $f \in \tilde{\mathcal{H}}^p_w(G)$ *such that*

$$\alpha_j(n) =< f, \tau_j\phi_n > . \tag{8.4}$$

Proof. For $r \geq s$, we have the equation

$$\chi_{G_s,k} = \sum_{j\in G_{r,j}} \subset G_{s,k}\chi_{G_r,j}.$$

Now, using the relation (4.1), we have

$$\begin{aligned} m(Gs)\phi_{s,k} &= \sum_{j\in G_{r,j}\subset G_{s,k}} m(G_r)\phi_{r,j} \\ \text{i.e.,} \langle f, \tau_k\phi_s\rangle &= \frac{m(G_r)}{m(G_s)} \sum_{j:G_{r,j}\subset G_{s,k}} \langle f, \tau_j.\phi_r\rangle \\ \text{i.e.,}\alpha_k(f,\phi_s) &= \frac{m(G_r)}{m(G_s)} \sum_{j:G_{r,j}\subset G_{s,k}} \alpha_j(f,\phi_r) \end{aligned}$$

$\Rightarrow$ the collection $\{\alpha_j(f,\phi_n)\}_{j\in J}$ satisfies the cascade condition (8.1).

$\Rightarrow$ $\{\alpha_j(f,\phi_n)\}_{j\in J}$ frames on an infinite tree in the space $\tilde{\mathcal{H}}^p_w(G)$.

On the lines of Galindo and Valladolid (loc . cit.), next we suppose that

$$\begin{aligned} S_n(f) &= \sum_{j\in J} \alpha_j(f,\phi_n)\chi_{G_n,j} \\ &= \sum_{j\in J} < f, \tau_j\phi_n > \chi_{G_n,j} \qquad (8.5) \\ \Rightarrow S_n f &= f * \phi_n. \end{aligned}$$

Since $\{\phi_n\}_{n\in z}$ is an approximate identity in $L^q(G)$, we claim that it is an approximate identity in $\tilde{\mathcal{H}}^p_w(G)$ too. In order to verify our assertion, it is enough to show that $\{\phi_n\}_{n\in z}$ is an approximate identity in $L^q_{1/w}(G)$.

We suppose that $C_c(G)$ denotes the space of continuous functions on G with compact support. Since w is moderate, it is locally bounded . This ensures that w^{-1} also moderate and locally bounded (cf.CH.90,pp.26-27).

We now observe that $C_c(G)$ is dense $L^q_{1/w}(G)$. In fact, it is well known that $C_c(G)$ is dense in $L^q(G)$. Hence for any given $\epsilon > 0$ there exists a function $f_c \in C_c(G)$ such that

$$\| f_c - fw^{-1} \|_q < \epsilon$$

putting $f_c = g_c w$, we see that

$$\| g_c - f \|_{q,w^{-1}} < \epsilon$$

$\Rightarrow$ $C_c(G)$ is dense in $L^q_{1/w}(G)$.

At first we prove that

$k * \phi_n \to k$ in $L^q_{1/w}(G)$ for all $k \in C_c(G)$.

We have

$$
\begin{aligned}
(k * \phi_n)(x) - k(x) &= \int_G (k(x-y) - k(x))\phi_n(y)\mathrm{d}y \\
\| g * \phi_n - g \|_{q,1/w} &\le (\int_G w^{-q}(x)\mathrm{d}x \mid \int_G (k(x-y) - k(x))\phi_n(y)\mathrm{d}y \mid^q)^{1/q} \\
&\le \int_G \phi_n(y)\mathrm{d}y \int_G (w^{-q}(x) \mid k(x-y) - k(x) \mid^q \mathrm{d}x)^{1/q}
\end{aligned}
$$

by Minkowsky's inequality.

Since k is uniformly continuous on supp k + supp ϕ_n and w^{-q} locally integrable, the right hand side can be made arbitrary small.

Now , let $f \in L^q_{1/w}(G)$. Then there exists $f_c \in C_c(G)$ such that

$$\| f - f_c \|_{q,w^{-1}} < \epsilon.$$

Thus we have

$$
\begin{aligned}
\| f * \phi_n - f \|_{q,w^{-1}} &\le \| (f - f_c) * \phi_n \|_{q,w^{-1}} + \| f_c * \phi_n - f_c \|_{q,w^{-1}} + \| f_c - f \|_{q,w^{-1}} \\
&\le (\| \phi_n \|_q + 1) \| f - f_c \|_{q,w^{-1}} + \| f_c * \phi_n - f_c \|_{q,w^{-1}}
\end{aligned}
$$

since $\| \phi_n \|_q$ is bounded, the right- hand side can be made arbitrarily small by f_c and ϕ_n conveniently.

Hence, by virtue of the definition of $\tilde{\mathcal{H}}^p_w(G)$, $\{\phi_n\}_{n \in z}$ is an approximate identity in it. Therefore,using (8.5), we obtain

$$
\begin{aligned}
\| S_n f \mid \tilde{\mathcal{H}}^p_w(G) \| &\le \| f \mid \tilde{\mathcal{H}}^p_w(G). \\
\Rightarrow \lim_{n \to \infty} \| S_n f - f \mid \tilde{\mathcal{H}}^p_w(G) \| &= 0 \\
\Rightarrow \| f \mid \tilde{\mathcal{H}}^p_w(G) &= \sup_{n \in Z} \| S_n f \mid \tilde{\mathcal{H}}^p_w(G) \| \\
&= \sup_{n \in Z} (m(G_n))^{1/q} \| \alpha_j(f, \phi_n) l^q_{1/w} \|,
\end{aligned}
$$

which proves (8.2).

Conversely, we Suppose that $\{\alpha_j(n)\}_{j \in J}$ is an infinite tree such that the condition (8.3) holds true.

Now on the lines of Galindo and Valladolid [GV 01,p.865], we define a sequence of functions $\{f_n\}_{n \in Z}$ such that

$$f_n = \sum_j \alpha_j(n) \chi_{G_{n,j}} \qquad (8.6).$$

Thus, on account of the hypothesis (8.3), the Sequence $\{f_n\}_{n\in Z}$ is bounded in $L^q_{1/w}(G)$. Hence, by Alaoglu theorem, there exists a subsequence $\{f_{n_m}\}_{m\in Z}$, which converges weakly to a function f (say) in $L^q_{1/w}(G)$. Thus we see that

$$\begin{aligned}\alpha_k(f,\phi_n) &= \langle f,\tau_k\phi_n\rangle \\ &= \lim_{m\to\infty}\langle f_{n_m},\tau_k\phi_n\rangle \\ &= \lim_{m\to\infty}\frac{m(G_{n_m})}{m(G_n)}\sum_{j:G_{n_m},j\subset G_{n,k}}\alpha_j(n_m) \\ &= \alpha_k(n)\end{aligned}$$

which proves the validity of (8.4).

Finally, let g be another function in $\tilde{\mathcal{H}}^p_w(G)$ such that

$$\begin{aligned}\alpha_j(n) &= \langle g,\tau_j\phi_n\rangle \\ \Rightarrow \alpha_j(f-g,\phi_n) &= 0 \qquad \text{for all } n,j \\ \Rightarrow \| (f-g) \mid \tilde{\mathcal{H}}^p_w(G) \| &= 0\end{aligned}$$

$\Rightarrow f$ is unique.

9 Infinite Trees in $\tilde{\mathcal{H}}^1_w(G)$

Galindo and Valladolid [GV 01,p.866], by means of an example, have demonstrated that the there Theorems 3.4, which holds for the space $L^p(G), 1 < p < \infty$, does not hold for $p = 1$. However, imposing an additional convergence condition on the sequence $\{\alpha_j(f,\phi_n)\}_{j\in J}$, they have proved the corresponding result for the space $L^1(G)$. Proceeding on the lines of Galindo and Valladolid (loc .cit), we have the following.

Theorem 9.1. *If $f \in \tilde{\mathcal{H}}^1_w(G)$, then the set $\{\alpha_j(f,\phi_n)\}_{j\in J}$ forms an infinite tree such that the following two properties hold true.:*

$$\| f \mid \tilde{\mathcal{H}}^1_w(G) \| = \sup_{n\in Z} m(G_n) \| \alpha_j(f,\phi_n) \mid l^1_{1/w}(J) \| \tag{9.1}$$

and $\forall\epsilon > 0$ there exists $N \in Z$

$$m(G_r)\sum_k \sum_{j:G_{r,j}\subset G_{s,k}} | \alpha_j(f,\phi_r) - \alpha_k(f,\phi_s) | < \epsilon, \tag{9.2}$$

for all $r > s \geq N$.

Conversely, if $\{\alpha_j(n)\}$ is an infinite tree such that

$$\sup_{n\in Z} m(G_n) \| \alpha_j(n) \mid l^1_{1/w} \| < \infty \tag{9.3}$$

and $\forall\epsilon > 0$ *there exists* $N \in Z$ *such that*

$$m(G_r)\sum_k \sum_{j:G_{r,j}\subset G_{s,k}} | \alpha_j(r) - \alpha_k(s) | < \epsilon, \tag{9.4}$$

for all $r > s \geq N$, *then* $\exists$ *a unique* $f \in \tilde{\mathcal{H}}^1_w(G)$. *such that*

$$\alpha_j(n) = \alpha_j(f, \phi_n) \tag{9.5}$$

Proof. Since the condition (9.4) implies that the sequence $\{f_n\}_{n\in z}$ defined by (8.6) is a Cauchy sequence in $L^q_{1/w}(G)$, the proof is now analogous to that of theorem 8.1.

10 Remarks

The corresponding results for the conjugate spaces of $\tilde{\mathcal{H}}^p_w(G), 1 \leq p \leq \infty$, can be easily verified.

References

[1] [BL 98] Benedetto, J. and Li, S., The Theory of Multiresolution Analysis Frames and Applications to Filter Bank. Appl. Comput. Harm. Anal., 5(1998), 389-427.

[2] [GV 01] Felix Galindo and J. Sanz Vallandolid. Multiresolution Analysis and Radon Measures on a Locally Compact Abelian Group., Czechoslovak Math. Journ., 51(2001), 859-871.

[3] [CH 01] Heil, C., Wiener Amalgam in Generalized Harmonic Analysis and Wavelet Theory, Ph.D. Thesis, University of Maryland (U.S.A), 1999.

[4] [Mal 89] Mallat, S.G., Multiresolution Approximations and Wavelet Orthonormal Bases of $\mathcal{L}^2(R)$, Trans. Amer. Math. Soc., 315(1989), 69-87.

[5] [Mey 92] Meyer, Y. Wavelets and Operators, Cambridge University Press, Cambridge, 1992.

Department of Mathematics

R. D. University

Jabalpur

India

e-mail: sheelpandey@hotmail.com