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BOUNDEDNESS OF COMMUTATORS RELATED TO MARCINKIEWICZ INTEGRALS ON HARDY TYPE SPACES*

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Abstract

In this paper, the authors study the boundedness of the operator $[\mu_\Omega, b]$, *the commutator generated by a function* $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)(0 < \beta \le 1)$ *and the Marcinkiewicz integrals* μ_Ω, *on the classical Hardy spaces and the Herz-type Hardy spaces in the case* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \le 1)$.

Key words *Marcinkiewicz integral, commutator, Lipschitz space, Hardy space, Herz space, atom*

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1 Introduction

Suppose that S^{n-1} denotes the unit sphere of $\mathbb{R}^n(n \ge 2)$ with Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. The higher-dimentional Marcinkiewicz integral μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y.$$

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The operator μ_Ω was first defined by Stein[1]. Meanwhile, Stein proved the following result.

Theorem A. *If Ω is continuous and satisfies a $Lip_\alpha(S^{n-1})(0<\alpha\le 1)$ condition on S^{n-1}, then*

$$\|\mu_\Omega(f)(x)\|_p\le c_p\|f\|_p,\ 1<p\le 2;$$

when $p=1$,

$$\lambda|\{\mu_\Omega(f)>\lambda\}|\le c\|f\|_1,\qquad \forall\lambda>0.$$

In [2], Benedek, Calderón and Panzone proved that if $\Omega\in C^1(S^{n-1})$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Ding, Fan and Pan [3] proved the weighted $L^p(\mathbb{R}^n)$ boundedness with A_p weights for a class of rough Marcinkiewicz integrals. Recently, Ding, Fan and Pan [4] improved the results mentioned above and showed that if $\Omega\in H^1(S^{n-1})$, the Hardy space on the unit sphere(see[5]), then μ_Ω is still a bounded operator on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. In [6], Chen, Xu and Ying proved the same result as [4] a using different method.

On the other hand, in 1990, Torchinsky and Wang[7] considered the boundedness for the commutator of μ_Ω. Let $b\in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[\mu_\Omega,b]$ is defined by

$$[\mu_\Omega,b](f)(x)=\left(\int_0^\infty |F_{\Omega,b,t}(f)(x)|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,b,t}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f(y)\mathrm{d}y.$$

In [7], Torchinsky and Wang proved that if Ω is continuous and satisfies a $Lip_\alpha(S^{n-1})$ condition for $0<\alpha\le 1$, then the commutator $[\mu_\Omega,b]$ is bounded on $L^p(\omega)$ for $1<p<\infty$, $\omega\in A_p$, the Muckenhoupt weight class. Recently, in [8], the authors improved the above result and proved the weighted $L^p\,(1<p<\infty)$ boundedness of the commutator $[\mu_\Omega,b]$. In [9], Wang considered the commutator generated by a $\mathrm{Lip}_\beta(\mathbb{R}^n)$ $(0<\beta\le 1)$ function and μ_Ω, and proved the following theorem.

Theorem B. *Suppose $1<p<\infty, 0<\beta<1, 1/q=1/p-\beta/n$. If $b\in Lip_\beta(\mathbb{R}^n)$, then*

$$\|[\mu_\Omega,b](f)\|_q\le c\|b\|_{\mathrm{Lip}_\beta}\|f\|_p.$$

Here, for $\beta>0$, the Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is the space of functions f satisfying

$$\|f\|_{\mathrm{Lip}_\beta}=\sup_{x,y\in\mathbb{R}^n,x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\beta}<\infty.$$

Obviously, $\mathrm{Lip}_\beta(\mathbb{R}^n)$ contains only constant if $\beta>1$ and $[\mu_\Omega,b]\equiv 0$ in this case, so we will only concentrate our discussion to the cases $0<\beta\le 1$ in what follows.

The organization of the paper is as follows. In Section 2, we will study the boundedness of the operator $[\mu_\Omega,b]$ formed by the Marcinkiewicz integral μ_Ω and a $\mathrm{Lip}_\beta(\mathbb{R}^n)$ function b on Hardy spaces. In Section 3, we consider the boundedness of the commutator $[\mu_\Omega,b]$ on Herz-type Hardy spaces.

2. Boundedness on Hardy Spaces

In order to study the boundedness of $[\mu_\Omega, b]$ on Hardy spaces, let us first introduce the atomic decomposition characteritions of Hardy spaces.

Definition 2.1. *For $0 < p \leq 1$, a function a on $\mathbb{R}^n$ is called a $(p,2)$ atom if it satisfies*

1) $\operatorname{supp} a \subset B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ *for some* $x_0 \in \mathbb{R}^n$ *and some* $r > 0$;

2) $\|a\|_2 \leq |B(x_0, r)|^{1/2-1/p}$;

3) $\int_{\mathbb{R}^n} a(x)\mathrm{d}x = 0.$

The following lemma can be found in [10, Chapter 3] or [11, Chapter 2].

Lemma 2.1. *Let $0 < p \leq 1$. A distribution f on $\mathbb{R}^n$ is in $H^p(\mathbb{R}^n)$ if and only if f can be written as*

$$f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$$

in distributional sense, where each a_j is a $(p,2)$ atom and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{H^p} \sim \inf\{(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p}\}$$

with the infimum taken over all decompositions of f as above.

Our result concerning the boundedness on Hardy space for $\mu_{\Omega,b}$ can be stated as follows.

Theorem 2.1. *Suppose that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \leq 1), b \in \mathrm{Lip}_\beta(\mathbb{R}^n)(\ 0 < \beta \leq \alpha/2)$. If $\frac{n}{n+\beta} < p \leq 1$ and $1/q = 1/p - \beta/n$, then $[\mu_\Omega, b]$ maps $H^p(\mathbb{R}^n)$ continuously into $L^q(\mathbb{R}^n)$.*

Proof. By Lemma 2.1, we only need to prove that for any $(p,2)$ atom a, $\|[\mu_\Omega, b]a\|_q \leq C$ with the constant C independent of a. Suppose that $\operatorname{supp} a \subset B = B(x_0, r)$. Write

$$\|[\mu_\Omega, b]a\|_q \leq (\int_{|x-x_0|<2r} |[\mu_\Omega, b]a(x)|^q \mathrm{d}x)^{1/q} + (\int_{|x-x_0|>2r} |[\mu_\Omega, b]a(x)|^q \mathrm{d}x)^{1/q} = I_1 + I_2.$$

Choose $1 < p_1 < \min(2, n/\beta)$ and q_1 satisfying $1/q_1 = 1/p_1 - \beta/n$. By the (L^{p_1}, L^{q_1}) boundedness of $[\mu_\Omega, b]$, the size condition of a and Hölder's inequality, we get

$$I_1 \leq C\|[\mu_\Omega, b]a\|_{q_1} r^{n(1/q-1/q_1)} \leq C\|a\|_{p_1} r^{n(1/q-1/q_1)} \leq C\|a\|_2 r^{n(1/p-1/2)} \leq C.$$

For I_2, since $|x - x_0| > 2r$, we have

$$\begin{aligned} |[\mu_\Omega, b]a(x)| &\leq \left(\int_0^{|x-x_0|+2r} \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x) - b(y)]a(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\ &\quad + \left(\int_{|x-x_0|+2r}^{\infty} \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x) - b(y)]a(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\ &:= J_1 + J_2. \end{aligned}$$

Observe that from $|x-x_0|>2r$ and $y\in B=B(x_0,r)$, it follows that $|x-y|\sim|x-x_0|\sim|x-x_0|+2r$. By the Minkowski inequality and $\mathrm{Lip}_\alpha(S^{n-1})\subset L^\infty(S^{n-1})$, we obtain

$$\begin{aligned}
J_1 &\le C\int_{\mathbf{R}^n}\left(\int_{|x-y|}^{|x-x_0|+2r}\frac{\mathrm{d}t}{t^3}\right)^{1/2}\frac{|a(y)|}{|x-y|^{n-1}}|b(x)-b(y)|\mathrm{d}y &(1)\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\int_{\mathbf{R}^n}|x-y|^\beta\frac{|a(y)|}{|x-y|^{n-1}}\frac{|y|^{1/2}}{|x-x_0|^{3/2}}\mathrm{d}y &(2)\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\int_{\mathbf{R}^n}|x-x_0|^\beta\frac{|a(y)|}{|x-x_0|^{n-1}}\frac{|y|^{1/2}}{|x-x_0|^{3/2}}\mathrm{d}y &(3)\\
&= C\|b\|_{\mathrm{Lip}_\beta}|x-x_0|^{\beta-n-1/2}\int_B|a(y)||y|^{1/2}\mathrm{d}y &(4)\\
&\le C\|b\|_{\mathrm{Lip}_\beta}|x-x_0|^{\beta-n-1/2}r^{1/2+n(1-1/p)} &(5)\\
&\le C\|b\|_{\mathrm{Lip}_\beta}|x-x_0|^{-n}r^{\beta+n(1-1/p)}|x-x_0|^{\beta-1/2}r^{1/2-\beta} &(6)\\
&\le C\|b\|_{\mathrm{Lip}_\beta}|x-x_0|^{-n}r^{\beta+n(1-1/p)}. &(7)
\end{aligned}$$

In the third inequality from the bottom above we used Hölder's inequality and the size condition of a.

Notice that from $t\ge|x-x_0|+2r$ and $y\in B$, it follows $t\ge|x-x_0|+|y-x_0|\ge|x-y|$, and by the vanishing condition of a, we obtain

$$\begin{aligned}
J_2 &\le \left[\int_{|x-x_0|+2r}^{\infty}\left|\int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right]^{1/2}\\
&= \left|\int_B\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a(y)\mathrm{d}y\right|\left(\int_{|x-x_0|+2r}^{\infty}\frac{\mathrm{d}t}{t^3}\right)^{1/2}\\
&\le C\left|\int_B\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a(y)\mathrm{d}y\right|\frac{1}{|x-x_0|+2r}\\
&\le C|[b(x)-b(x_0)]\int_B[\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}]a(y)\mathrm{d}y|\frac{1}{|x-x_0|+2r}\\
&\quad +C|\int_B\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(y)-b(x_0)]a(y)\mathrm{d}y|\frac{1}{|x-x_0|+2r}\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\left(|x-x_0|^\beta\int_B\frac{|y-x_0|^\alpha|a(y)|}{|x-x_0|^{n-1+\alpha}}\mathrm{d}y+\int_B\frac{|y-x_0|^\beta|a(y)|}{|x-y|^{n-1}}\mathrm{d}y\right)\frac{1}{|x-x_0|+2r}\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\left(|x-x_0|^\beta\int_B\frac{|y-x_0|^\alpha|a(y)|}{|x-x_0|^{n+\alpha}}\mathrm{d}y+\int_B\frac{|y-x_0|^\beta|a(y)|}{|x-y|^{n}}\mathrm{d}y\right)\\
&\le C\|b\|_{\mathrm{Lip}_\beta}(|x-x_0|^{\beta-n-\alpha}r^{\alpha+n(1-1/p)}+|x-x_0|^{-n}r^{\beta+n(1-1/p)})\\
&\le C\|b\|_{\mathrm{Lip}_\beta}|x-x_0|^{-n}r^{\beta+n(1-1/p)}.
\end{aligned}$$

So when $|x-x_0|>2r$, we have

$$|[\mu_\Omega,b]a(x)|\le C\|b\|_{\mathrm{Lip}_\beta}|x-x_0|^{-n}r^{\beta+n(1-1/p)}.$$

Therefore,

$$I_2\le C\|b\|_{\mathrm{Lip}_\beta}r^{\beta+n(1-1/p)}(\int_{|x-x_0|>2r}|x-x_0|^{-nq}\mathrm{d}x)^{1/q}\le C\|b\|_{\mathrm{Lip}_\beta}.$$

Combining the estimates for I_1 and I_2, then leads to the desired result.

It is well-known that the dual space of $H^1(\mathbb{R}^n)$ is BMO($\mathbb{R}^n$). From this and Theorem 2.1, by a dual argument, we easily deduce the following conclusion.

Corollary 2.1. *Suppose that* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \le 1), b \in \mathrm{Lip}_\beta(\mathbb{R}^n)(\, 0 < \beta \le \alpha)$. *Then* $[\mu_\Omega, b]$ *maps* $L^{n/\beta}(\mathbb{R}^n)$ *continuously into* BMO($\mathbb{R}^n$).

In general, the (H^p, L^q) boundedness of $[\mu_\Omega, b]$ fails when $p = \dfrac{n}{n+\beta}$. To be precise, we have the following characterization theorem.

Theorem 2.2. *Suppose that* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \le 1), b \in \mathrm{Lip}_\beta(\mathbb{R}^n)(\, 0 < \beta \le \alpha/2)$. *Then the following two statements are equivalent:*

(i) $[\mu_\Omega, b]$ *maps* $H^{n/(n+\beta)}(\mathbb{R}^n)$ *continuously into* $L^1(\mathbb{R}^n)$;

(ii) $\Omega \equiv 0$ *or* $\int_{\mathbb{R}^n} b(y)a(y)\mathrm{d}y = 0$ *holds for any* $(n/(n+\beta), 2)$ *atom* a.

Proof. By Lemma 2.1, $[\mu_\Omega, b]$ is bounded from $H^{n/(n+\beta)}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ if and only if $\|[\mu_\Omega, b](a)\|_1 \le C$ holds for any $(n/(n+\beta), 2)$ atom a with C independent of a. We only need to consider the behavior of $[\mu_\Omega, b]$ on $(n/(n+\beta), 2)$ atom.

Let a be such an atom with the support $B = B(x_0, r)$. For $u \in B$, let

$$v_1(x) = \chi_{2B}(x)[\mu_\Omega, b]a(x),$$
$$v_2(x) = \chi_{(2B)^c}(x)\left\{ \int_0^{|x-x_0|+2r} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x) - b(y)]a(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right\}^{1/2},$$

and

$$v_3(x) = \chi_{(2B)^c}(x)\left\{ \int_{|x-x_0|+2r}^{\infty} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x) - b(y)]a(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right\}^{1/2}.$$

Then

$$[\mu_\Omega, b]a(x) = v_1(x) + v_2(x) + v_3(x).$$

By a method similar to the estimate for I_1 in proof of Theorem 2.1, we can prove $\|v_1\|_1 \le C$. To estimate v_2, we note that $|x - x_0| > 2r$ and $y \in B = B(x_0, r)$ implies $|x - y| \sim |x - x_0| \sim |x - x_0| + 2r$. For v_2, by a method similar to the estimate for J_1 in Theorem 2.1, we get

$$\begin{aligned} &\chi_{(2B)^c}(x)\left(\int_0^{|x-x_0|+2r} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x) - b(y)]a(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\ \le\ & C\|b\|_{\mathrm{Lip}_\beta}\chi_{(2B)^c}(x)|x - x_0|^{\beta-n-1/2}r^{1/2-\beta}. \end{aligned}$$

Here we have used $\Omega \in \mathrm{Lip}_\alpha(S^{n-1}) \subset L^\infty(S^{n-1})(0 < \alpha \le 1)$. So we have

$$\begin{aligned} \|v_2\|_1 &\le C\|b\|_{\mathrm{Lip}_\beta} r^{1/2-\beta} \int_{(2B)^c} |x - x_0|^{\beta-n-1/2}\mathrm{d}x \\ &\le C\|b\|_{\mathrm{Lip}_\beta} r^{1/2-\beta} \int_{2r}^\infty \rho^{\beta-n-1/2}\rho^{n-1}\mathrm{d}\rho \\ &\le C\|b\|_{\mathrm{Lip}_\beta}. \end{aligned}$$

Thus we see that $\|[\mu_\Omega, b](a)\|_1 \le C$ is equivalent to $\|v_3\|_1 \le C$.

Note that $t \ge |x-x_0| + 2r$ and $y \in B$ implies that $t \ge |x-x_0| + |y-x_0| \ge |x-y|$. Thus we get

$$\begin{aligned} v_3(x) &= \chi_{(2B)^c}(x)\left\{\int_{|x-x_0|+2r}^{\infty} \left|\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^3}\right\}^{1/2} \\ &= \chi_{(2B)^c}(x)\left|\int_B \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a(y)\mathrm{d}y\right|\left(\int_{|x-x_0|+2r}^{\infty} \frac{\mathrm{d}t}{t^3}\right)^{1/2} \\ &\ge C\chi_{(2B)^c}(x)\left|\int_B \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a(y)\mathrm{d}y\right|\frac{1}{|x-x_0|+2r} \\ &= C\Bigg|\chi_{(2B)^c}(x)[b(x)-b(x_0)]\int_B \frac{\Omega(x-y)}{|x-y|^{n-1}}a(y)\mathrm{d}y\frac{1}{|x-x_0|+2r} \\ &\quad -\chi_{(2B)^c}(x)\int_B\left[\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-u)}{|x-u|^{n-1}}\right][b(y)-b(x_0)]a(y)\mathrm{d}y\frac{1}{|x-x_0|+2r} \\ &\quad -\chi_{(2B)^c}(x)\frac{\Omega(x-u)}{|x-u|^{n-1}}\int_B b(y)a(y)\mathrm{d}y\frac{1}{|x-x_0|+2r}\Bigg| \\ &:= C|u_1(x)-u_2(x)-u_3(x)| \ge C[u_1(x)-u_2(x)-u_3(x)]. \end{aligned}$$

Write $K(x) = \Omega(x)/|x|^{n-1}$. By the vanishing condition of a, we have

$$\begin{aligned} |u_1(x)| &\le C\|b\|_{\mathrm{Lip}_\beta}\chi_{(2B)^c}(x)|x-x_0|^\beta \\ &\quad \times \int_B |K(x-y)-K(x-x_0)||a(y)|\mathrm{d}y\frac{1}{|x-x_0|+2r} \\ &\le C\|b\|_{\mathrm{Lip}_\beta}\chi_{(2B)^c}(x)|x-x_0|^{\beta-n-1-\alpha}r^{\alpha-\beta}\frac{1}{|x-x_0|+2r} \\ &\le C\|b\|_{\mathrm{Lip}_\beta}\chi_{(2B)^c}(x)|x-x_0|^{\beta-n-\alpha}r^{\alpha-\beta}. \end{aligned}$$

Therefore, $\|u_1\|_1 \le C$. Using $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})(0<\alpha\le 1)$, we also get

$$\begin{aligned} \|u_2(.,u)\|_1 &\le C\int_{(2B)^c}\int_B \frac{|y-u|^\alpha}{|x-y|^{n+\alpha}}|b(y)-b(x_0)||a(y)|\mathrm{d}y\mathrm{d}x\frac{1}{|x-x_0|+2r} \\ &\le C\int_{(2B)^c}\int_B \frac{|y-u|^\alpha}{|x-y|^{n+\alpha}}|b(y)-b(x_0)||a(y)|\mathrm{d}y\mathrm{d}x \\ &\le C\|b\|_{\mathrm{Lip}_\beta}r^\alpha\int_{(2B)^c}|x-x_0|^{-(n+\alpha)}\mathrm{d}x = C\|b\|_{\mathrm{Lip}_\beta} \end{aligned}$$

The estimates above shows that the estimate $\|v_3\|_1 \le C$ is equivalent to the estimate $\|u_3(.,u)\|_1 \le C$. Set

$$L(b,a) = \int_B b(y)a(y)\mathrm{d}y.$$

We see that

$$\begin{aligned} C &\ge \|u_3(.,u)\|_1 \ge |L(b,a)|\int_{3r<|x-u|<Nr}\frac{|\Omega(x-u)|}{|x-y|^{n-1}}\frac{1}{|x-x_0|+2r}\mathrm{d}x \\ &\ge C_1|L(b,a)|\int_{3r<|x-u|<Nr}\frac{|\Omega(x-u)|}{|x-y|^n}\mathrm{d}x \\ &= C_1|L(b,a)|\int_{3r}^{Nr}\rho^{-1}\int_{S^{n-1}}|\Omega(x)|\mathrm{d}\sigma(x)\mathrm{d}\rho = C_1\log(N/3)|L(b,a)|\|\Omega\|_{L^1(S^{n-1})}, \end{aligned}$$

where $N > 3$ can be any large positive integer. Letting $N \to \infty$, we obtain

$$|L(b,a)|\|\Omega\|_{L^1(S^{n-1})} = 0.$$

Thus, $\Omega \equiv 0$ or $L(b,a) = 0$, that is , $\Omega \equiv 0$ or $\int_{\mathbb{R}^n} b(y)a(y)dy = 0$.

Hence, it follows from the above that (i) is equivelant to (ii). This finishes the proof.

Though $(H^{n/(n+\beta)}, L^1)$ boundedness fails except for the trivial cases, we can prove an estimate of weak type.

Theorem 2.3. *Suppose that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})(0 < \alpha \le 1), b \in \mathrm{Lip}_\beta(\mathbb{R}^n), 0 < \beta \le \alpha/2$, then $[\mu_\Omega, b]$ maps $H^{n/(n+\beta)}(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$, that is, $[\mu_\Omega, b]$ satisfies*

$$|\{x \in \mathbb{R}^n : |[\mu_\Omega, b]f(x)| > \lambda\}| \le C\lambda^{-1}\|f\|_{H^{n/(n+\beta)}},$$

where C is independent of f.

Proof. By Lemma 2.1, we write $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ with each a_j an $(n/(n+\beta), 2)$ atom and $\sum_{j=-\infty}^{\infty} |\lambda_j|^{n/(n+\beta)} < \infty$. Suppose that supp $a_j \subset B_j = B(x_j, r_j)$. Write

$$\begin{aligned}
|[\mu_\Omega, b]f(x)| &\le C|\mu_\Omega(\sum_{j=-\infty}^{\infty} \lambda_j(b(\cdot) - b(x_j))a_j)(x)| + C\sum_{j=-\infty}^{\infty} |\lambda_j||\chi_{2B_j}||b(x) - b(x_j)||\mu_\Omega(a_j)(x)| \\
&+ C\sum_{j=-\infty}^{\infty} |\lambda_j||\chi_{(2B_j)^c}||b(x) - b(x_j)|\left(\int_0^{|x-x_j|+2^{j+1}} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}(a_j)(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&+ C\sum_{j=-\infty}^{\infty} |\lambda_j||b(x) - b(x_j)||\chi_{(2B_j)^c}\left(\int_{|x-x_j|+2^{j+1}}^{\infty} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}(a_j)(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&:= J_1(x) + J_2(x) + J_3(x) + J_4(x).
\end{aligned}$$

A trivial computation shows that

$$\begin{aligned}
\|\chi_{2B_j}|b(x) - b(x_j)|\mu_\Omega(a_j)(x)\|_1 &= \int_{\mathbb{R}^n} \chi_{2B_j}|b(x) - b(x_j)||\mu_\Omega(a_j)(x)|dx \\
&\le \|b\|_{\mathrm{Lip}_\beta} \int_{|x-x_j|<2^{j+1}} |x - x_j|^\beta |\mu_\Omega(a_j)(x)|dx \\
&\le C\|b\|_{\mathrm{Lip}_\beta} 2^{j\beta}\|\mu_\Omega(a_j)\|_2 (\int_{|x-x_j|<2^{j+1}} dx)^{1/2} \\
&\le C\|b\|_{\mathrm{Lip}_\beta} 2^{j\beta}|B_j|^{1/2-\frac{n+\beta}{n}} 2^{jn/2} = C,
\end{aligned}$$

where in the last inequality, we use the (L^2, L^2) boundedness of μ_Ω and the size condition of a_j. By a method similar to the estimate for J_1 in the proof of Theorem 2.1 and the estimate for u_1 in the proof of Theorem 2.2, we get

$$\left\|\chi_{(2B_j)^c}|b(x) - b(x_j)|\left(\int_0^{|x-x_j|+2^{j+1}} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}(a_j)(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2}\right\|_1 \le C.$$

By a method similar to the estimate for J_2 in the proof of Theorem 2.1, we have

$$\left\| |b(x)-b(x_j)|\chi_{(2B_j)^c}\left(\int_{|x-x_j|+2^{j+1}}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(a_j)(y)\,dy\Big|^2\frac{dt}{t^3}\right)^{1/2}\right\|_1 \le C.$$

Thus, we obtain

$$|\{x\in\mathbb{R}^n:|J_i(x)|>\lambda/4\}|\le C\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|, \qquad i=2,3,4.$$

Noting that

$$\|(b-b(x_j))a_j\|_1\le C\int_{B_j}r_j^{\beta}|a(y)|dy\le C,$$

by the weak (L^1,L^1) boundedness of μ_Ω, we obtain

$$|\{x\in\mathbb{R}^n:|J_1(x)|>\lambda/4\}|\le C\lambda^{-1}\|\sum_{j=-\infty}^{\infty}|\lambda_j|(b-b(x_j))a_j\|_1\le C\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|.$$

Therefore,

$$\begin{aligned}|\{x\in\mathbb{R}^n:|[\mu_\Omega,b]f(x)|>\lambda\}| &\le C\sum_{j=1}^{4}|\{x\in\mathbb{R}^n:|J_i(x)|>\lambda/4\}|\\ &\le C\lambda^{-1}\sum_{j=-\infty}^{\infty}|\lambda_j|\\ &\le C\lambda^{-1}(\sum_{j=-\infty}^{\infty}|\lambda_j|^{n/(n+\beta)})^{(n+\beta)/n},\end{aligned}$$

where in the last inequality, we have used the fact that $n/(n+\beta)<1$, and the constant C is independent of f. Taking the infimum over all the decompositions of f as in Lemma 2.1, we obtain the desired estimate. This finishes the proof.

3 Boundedness on Herz-type Hardy Spaces

In this section, we consider the boundedness of $[\mu_\Omega,b]$ on Herz-type Hardy Spaces. Let us now begin with some notations.

Definition 3.1. Let $B_k=\{x\in\mathbb{R}^n:|x|<2^k\}, E_k=B_k/B_{k-1}$ and $\chi_k=\chi_{E_k}$ for $k\in\mathbb{Z}$. Let $\alpha\in\mathbb{R}$ and $0<p,q\le\infty$.

(i) The homogeneous Herz space is defined by

$$\dot K_q^{\alpha,p}(\mathbb{R}^n)=\{f:f\in L^q_{loc}(\mathbb{R}^n)/\{0\}\text{ and }\|f\|_{\dot K_q^{\alpha,p}}<\infty\},$$

where

$$\|f\|_{\dot K_q^{\alpha,p}}=(\sum_{k=-\infty}^{\infty}2^{\alpha p/n}\|f\chi_k\|_q^p)^{1/p}$$

with usual modifications made when $p=\infty$.

(ii) The inhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n)=\{f: f\in L_{loc}^q(\mathbb{R}^n) \text{ and } \|f\|_{K_q^{\alpha,p}}<\infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}}=(\sum_{k=1}^{\infty}2^{\alpha p/n}\|f\chi_k\|_q^p+\|f_{\chi_{B_0}}\|_q^p)^{1/p}$$

with usual modifications made when $p=\infty$.

Definition 3.2. Let $\alpha\in\mathbb{R}$ and $0<p,q<\infty$.

(i) The homogeneous Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)=\{f: f\in S'(\mathbb{R}^n): G(f)\in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}$$

and

$$\|f\|_{H\dot{K}_q^{\alpha,p}}=\|G(f)\|_{\dot{K}_q^{\alpha,p}}.$$

(ii) The inhomogeneous Herz-type Hardy space $HK_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n)=\{f: f\in S'(\mathbb{R}^n): G(f)\in K_q^{\alpha,p}(\mathbb{R}^n)\}$$

and

$$\|f\|_{HK_q^{\alpha,p}}=\|G(f)\|_{K_q^{\alpha,p}}.$$

Here, $S'(\mathbb{R}^n)$ is the space of the tempered distributions on $\mathbb{R}^n$ and $G(f)$ is the grand maximal function of f; see [10,p.90] for the definition of $G(f)$.

Obviously, $H\dot{K}_p^{0,p}(\mathbb{R}^n)=HK_p^{0,p}(\mathbb{R}^n)=H^p(\mathbb{R}^n)$. Thus, the Herz-type Hardy spaces is an extension of the classical Hardy spaces. In addition, Herz-type Hardy spaces link closely with Herz spaces. If $1<q<\infty$, it can be proved that $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)=\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n)=K_q^{\alpha,p}(\mathbb{R}^n)$ when $-n/q<\alpha<n(1-1/q)$, but $HK_q^{\alpha,p}(\mathbb{R}^n)\subset K_q^{\alpha,p}(\mathbb{R}^n)$ when $\alpha\geq n(1-1/q)$; see [12], [13].

The Herz-type Hardy spaces have central atomic decomposition characterizations stated as follows.

Definition 3.3. Let $\alpha\in\mathbb{R}^n$ and $1<q<\infty$. A function a on $\mathbb{R}^n$ is called a central(α,q) atom if a satisfies

1) supp $a\subset B(0,r)$ for some $r>0$;

2) $\|a\|_q\leq|B(0,r)|^{-\alpha/n}$;

3) $\int_{\mathbb{R}^n}a(x)dx=0$.

Lemma 3.1[13]. *Let $0<p<\infty, 1<q<\infty$ and $\alpha\geq n(1-1/q)$. A distribution f on $\mathbb{R}^n$ belongs to $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if it can be written as $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$ in distributional sense,* *where each a_j is a central (α,q) atom on B_j and* $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$. *Moreover,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}}\sim\inf\{(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p}\}$$

with the infimum taken over all decompositions of f as above.

In what follows, we only give the results for homogeneous Herz-type Hardy spaces. It should be pointed out that the results for inhomogeneous spaces is similar. And we omit it.

Theorem 3.1. *Suppose that $\Omega\in\mathrm{Lip}_\gamma(S^{n-1})(0<\gamma\le 1), b\in\mathrm{Lip}_\beta(\mathbb{R}^n), 0<\beta\le\gamma/2$. If $0<p<\infty, 1<q_1,q_2<\infty, 1/q_2=1/q_1-\beta/n$ and $n(1-1/q_1)\le\alpha<n(1-1/q_1)+\beta$, then $[\mu_\Omega,b]$ maps $H\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)$ continuously into $\dot{K}_{q_2}^{\alpha,p}(\mathbb{R}^n)$.*

Proof. By Lemma 3.1, we write $f=\sum_{k=-\infty}^{\infty}\lambda_k a_k$, where each a_k is a central (α,q) atom supported on B_k and $\sum_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$. Then we have

$$\begin{aligned}\|[\mu_\Omega,b]f\|_{\dot{K}_{q_2}^{\alpha,p}}^p &\le \sum_{l=-\infty}^{\infty}2^{l\alpha p}\left(\sum_{k=l-1}^{\infty}|\lambda_k|\|[\mu_\Omega,b](a_k)\cdot\chi_l\|_{q_2}\right)^p\\ &\quad+\sum_{l=-\infty}^{\infty}2^{l\alpha p}\left(\sum_{k=-\infty}^{l-2}|\lambda_k|\|[\mu_\Omega,b](a_k)\cdot\chi_l\|_{q_2}\right)^p\\ &:= I_1+I_2.\end{aligned}$$

By the (L^{q_1},L^{q_2}) boundedness of $[\mu_\Omega,b]$, it is easy to verify that

$$\begin{aligned}I_1 &\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{l=-\infty}^{\infty}2^{l\alpha p}\left(\sum_{k=l-1}^{\infty}|\lambda_k|\|a_k\|_{q_1}\right)^p\\ &\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{l=-\infty}^{\infty}\left(\sum_{k=l-1}^{\infty}|\lambda_k|2^{(l-k)\alpha}\right)^p.\end{aligned}$$

When $0<p\le 1$,

$$I_1\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p\sum_{l=-\infty}^{k+1}2^{(l-k)\alpha p}\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.$$

If $p>1$, by Hölder's inequality, we get

$$\begin{aligned}I_1 &\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{l=-\infty}^{\infty}\left(\sum_{k=l-1}^{\infty}|\lambda_k|^p2^{(l-k)\alpha p/2}\right)\left(\sum_{k=l-1}^{\infty}2^{(l-k)\alpha p'/2}\right)^{p/p'}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p\end{aligned}$$

Let us now estimate I_2. Note that

$$\begin{aligned}|[\mu_\Omega,b](a_k)(x)| &\le \left[\int_0^{|x|+2^{k+1}}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a_k(y)\,\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right]^{1/2}\\ &\quad+\left[\int_{|x|+2^{k+1}}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]a_k(y)\,\mathrm{d}y\right|^2\frac{\mathrm{d}}{t^3}\right]^{1/2}\\ &:= J_1+J_2.\end{aligned}$$

When $x\in E_l$ and $|x-y|<t$ with $t<|x|+2^{k+1}$, it follows from $l\ge k+2$ that $|x-y|\sim|x|\sim|x|+2^{k+1}$. Then by the Minkowski inequality,

$$\begin{aligned}J_1 &\le C\int_{\mathbf{R}^n}\left(\int_{|x-y|}^{|x|+2^{k+1}}\frac{\mathrm{d}t}{t^3}\right)^{1/2}\frac{|b(x)-b(y)||a_k(y)|}{|x-y|^{n-1}}\mathrm{d}y\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\int_{\mathbf{R}^n}\frac{|x|^\beta|a_k(y)|}{|x|^{n-1}}\frac{|y|^{1/2}}{|x|^{3/2}}dy\\ &\le C\|b\|_{\mathrm{Lip}_\beta}|x|^{-n}2^{k(\beta+n(1-1/q_1)-\alpha)}.\end{aligned}$$

By a method similar to the estimate for J_2 in the proof of Theorem 2.1, we also get

$$J_2\le C\|b\|_{\mathrm{Lip}_\beta}|x|^{-n}2^{k(\beta+n(1-1/q_1)-\alpha)}.$$

This gives

$$\|[\mu_\Omega,b]a_k\chi_l\|_{q_2}\le C\|b\|_{\mathrm{Lip}_\beta}2^{-l\alpha}2^{(l-k)(\alpha-n(1-1/q_1)-\beta)}:=C\|b\|_{\mathrm{Lip}_\beta}2^{-l\alpha}W(l,k).$$

Thus,

$$I_2\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{l=-\infty}^{\infty}\left(\sum_{k=-\infty}^{l-2}|\lambda_k|W(l,k)\right)^p.$$

When $p\le 1$,

$$I_2\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{l-2}|\lambda_k|^pW(l,k)^p\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.$$

If $p>1$, then

$$\begin{aligned}I_2 &\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{l=-\infty}^{\infty}\left(\sum_{k=-\infty}^{l-2}|\lambda_k|^pW(l,k)^{p/2}\right)\left(\sum_{k=-\infty}^{l-2}W(l,k)^{p'/2}\right)^{p/p'}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.\end{aligned}$$

The estimates for I_1 and I_2 lead to

$$\|[\mu_\Omega,b]f\|_{\dot K_{q_2}^{\alpha,p}}\le C\|b\|_{\mathrm{Lip}_\beta}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p}$$

and the desired estimate follows from taking infimum over all decompositions of f.

This kind of boundedness fails when $\alpha = n(1-1/q_1)+\beta$.

Theorem 3.2. *Suppose that* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0<\alpha\le 1), b\in \mathrm{Lip}_\beta(\mathbb{R}^n), 0<\beta\le\alpha/2$. *Let* $0<p<\infty, 1<q_1,q_2<\infty, 1/q_2 = 1/q_1-\beta/n$. *Then the following two statements are equivalent:*

(i) $[\mu_\Omega, b]$ *maps* $H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}(\mathbb{R}^n)$ *continuously into* $\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}(\mathbb{R}^n)$;

(ii) $\Omega\equiv 0$ *or for any central* $(n(1-1/q_1)+\beta, q_1)$ *atom* a, *there is* $\int_{\mathbb{R}^n} b(y)a(y)\mathrm{d}y = 0$.

Proof. Suppose that a is a central $(n(1-1/q_1)+\beta,q_1)$ atom. Without loss of generality, we may assume that a is supported on some B_k. For $u\in B_k$, let v_1,v_2,v_3,u_1,u_2 and u_3 be the same as in the proof of Theorem 2.2, but with the ball B_k instead of B. It follows from the (L^{q_1},L^{q_2}) boundedness of $[\mu_\Omega,b]$ and the size condition of a that

$$\begin{aligned}\|v_1\|_{\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}} &\le \left\{\sum_{j=-\infty}^{k+1} 2^{j(n(1-1/q_1)+\beta)p}\|\chi_j([\mu_\Omega,b]a)\|_{q_2}^p\right\}^{1/p}\\ &\le \left\{\sum_{j=-\infty}^{k+1} 2^{j(n(1-1/q_1)+\beta)p}\|a\|_{q_1}^p\right\}^{1/p}\le C.\end{aligned}$$

For v_2, by Theorem 2.2, we see that

$$|v_2|\le C\|b\|_{\mathrm{Lip}_\beta}\chi_{(2B)^c}(x)|x-x_0|^{\beta-n-1/2}2^{k(1/2-\beta)}.$$

If $|x|\ge 2^{k+1}$, then

$$\begin{aligned}\|v_2\|_{\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}} &\le C\|b\|_{\mathrm{Lip}_\beta}\left\{\sum_{j=k+2}^{\infty} 2^{j(n(1-1/q_1)+\beta)p}2^{k(1/2-\beta)p}\left(\int_{E_j}|x|^{(\beta-n-1/2)q_2}\mathrm{d}x\right)^{p/q_2}\right\}^{1/p}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left\{\sum_{j=k+2}^{\infty}2^{(k-j)(\beta-1/2)p}\right\}^{1/p} = C\|b\|_{\mathrm{Lip}_\beta}.\end{aligned}$$

From Theorem 2.2, it follows that

$$|u_1|\le C\|b\|_{\mathrm{Lip}_\beta}|x|^{\beta-n-\alpha}2^{k(\alpha-\beta)}$$

if $|x|\ge 2^{k+1}$. Thus,

$$\begin{aligned}\|u_1\|_{\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}} &\le C\|b\|_{\mathrm{Lip}_\beta}\left\{\sum_{j=k+2}^{\infty} 2^{j(n(1-1/q_1)+\beta)p}2^{k(\alpha-\beta)p}\left(\int_{E_j}|x|^{(\beta-n-\alpha)q_2}\mathrm{d}x\right)^{p/q_2}\right\}^{1/p}\\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left\{\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha-\beta)p}\right\}^{1/p} = C\|b\|_{\mathrm{Lip}_\beta}.\end{aligned}$$

Concerning the term $u_2(x,u)$, we have

$$\begin{aligned}\|u_2\|_{\dot K_{q_2}^{n(1-1/q_1)+\beta,p}} &\le C\|b\|_{\mathrm{Lip}_\beta} \\ &\quad \left\{\sum_{j=k+2}^{\infty} 2^{j(n(1-1/q_1)+\beta)p}2^{k\beta p}\left[\int_{E_j}\left(\int_{B_k}\frac{|y-u|^\alpha}{|x-y|^{n+\alpha}}|a(y)|\mathrm{d}y\right)^{q_2}\mathrm{d}x\right]^{p/q_2}\right\}^{1/p} \\ &\le C\|b\|_{\mathrm{Lip}_\beta}\left(\sum_{j=k+2}^{\infty}2^{(k-j)p}\right)^{1/p} = C\|b\|_{\mathrm{Lip}_\beta}.\end{aligned} \tag{8}$$

If (ii) holds, we get $\|v_3\|_{\dot K_{q_2}^{n(1-1/q_1)+\beta,p}} \le C$. So (i) holds.

If (i) holds, since $\|a\|_{H\dot K_{q_1}^{n(1-1/q_1)+\beta,p}} \le 1$, we have $\|[\mu_\Omega,b]a\|_{\dot K_{q_2}^{n(1-1/q_1)+\beta,p}} \le C$. This implies $\|v_3\|_{\dot K_{q_2}^{n(1-1/q_1)+\beta,p}} \le C$. Combining the above estimates, we get

$$\|u_3(.,u)\|_{\dot K_{q_2}^{n(1-1/q_1)+\beta,p}} \le C.$$

Let

$$L(b,a) = \int_B b(y)a(y)\mathrm{d}y.$$

Then we see that

$$\begin{aligned} C^p &\ge \sum_{j=k+2}^{\infty} 2^{j(n(1-1/q_1)+\beta)p}\left(\int_{E_j}\left|L(b,a)\frac{\Omega(x-u)}{|x-u|^{n-1}}\frac{1}{|x|+2^{k+1}}\right|^{q_2}\mathrm{d}x\right)^{p/q_2} \\ &\ge \sum_{j=k+3}^{k+N} 2^{j(n(1-1/q_1)+\beta)p}\left(|L(b,a)|^{q_2}\int_{\frac{5}{8}2^j<|x-u|<\frac{7}{8}2^j}\left|\frac{\Omega(x-u)}{|x-u|^n}\right|^{q_2}\mathrm{d}x\right)^{p/q_2} \\ &= \sum_{j=k+3}^{k+N} 2^{j(n(1-1/q_1)+\beta)p}\left(|L(b,a)|^{q_2}\int_{\frac{5}{8}2^j}^{\frac{7}{8}2^j}\rho^{-nq_2+n-1}\int_{S^{n-1}}|\Omega(x)|^{q_2}\mathrm{d}\sigma(x)\mathrm{d}\rho\right)^{p/q_2} \\ &= C(N-2)|L(b,a)|^p\|\Omega\|^p_{L^{q_2}(S^{n-1})}.\end{aligned}$$

It is easy to verify that (ii) holds, that is, $L(b,a)=0$ or $\Omega\equiv 0$.

In the extreme case of Theorem 3.2, we also have an estimate of weak type which is similar to Theorem 2.3. Let us first introduce the weak Herz spaces which was first introduced in [14].

Definition 3.4. Let $\alpha\in\mathbb{R}$ and $0<p,q<\infty$.

(i) A measurable function f is said to belong to homogeneous weak Herz space $W\dot K_q^{\alpha,p}(\mathbb{R}^n)$ if

$$\|f\|_{W\dot K_q^{\alpha,p}} = \sup_{\lambda>0}\lambda\left(\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left|\{x\in E_k : |f(x)|>\lambda\}\right|^{p/q}\right)^{1/p} < \infty.$$

(ii) A measurable function f is said to belong to inhomogeneous weak Herz space $WK_q^{\alpha,p}(\mathbb{R}^n)$

if

$$\|f\|_{WK_q^{\alpha,p}} = \sup_{\lambda>0}\lambda\{|\{x\in B_0:|f(x)|>\lambda\}|^{p/q} + \sum_{k=1}^{\infty}2^{k\alpha p}|\{x\in E_k:|f(x)|>\lambda\}|^{p/q}\}^{1/p}<\infty.$$

Obviously, $W\dot{K}_p^{0,p}(\mathbb{R}^n)=WK_p^{0,p}(\mathbb{R}^n)=WL^p(\mathbb{R}^n)$ and $W\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n)=WL^p_{|x|^\alpha}(\mathbb{R}^n)$ for $0<p<\infty$ and $\alpha\in\mathbb{R}$. That is, weak Herz spaces include weak $L^p(\mathbb{R}^n)$ spaces as special cases and homogeneous weak Herz spaces are a generalization of the weak $L^p(\mathbb{R}^n)$ spaces with power weights.

Theorem 3.3. *Suppose that* $\Omega\in\mathrm{Lip}_\alpha(S^{n-1})\,(0<\alpha\le 1), b\in\mathrm{Lip}_\beta(\mathbb{R}^n),\ 0<\beta\le\alpha/2$. *If* $0<p\le 1,\ 1<q_1,q_2<\infty,\ \frac{1}{q_2}=\frac{1}{q_1}-\frac{\beta}{n}$, *then* $[\mu_\Omega,b]$ *maps* $H\dot{K}_{q_1}^{n(1-1/q_1)+\beta,p}(\mathbb{R}^n)$ *continuously into* $W\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}(\mathbb{R}^n)$.

Proof. By Lemma 3.1, we write $f=\sum_{k=-\infty}^{\infty}\lambda_k a_k$, where each a_k is a central $(n(1-1/q_1)+\beta,q_1)$ atom supported on B_k and $\sum_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$. Write

$$\begin{aligned}&\|[\mu_\Omega,b]f\|_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta,p}}\\ \le\ &\sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta)p}\left|\left\{x\in E_l:\left|[\mu_\Omega,b]\left(\sum_{k=l-3}^{\infty}\lambda_k a_k\right)(x)\right|>\lambda/2\right\}\right|^{p/q_2}\right\}^{1/p}\\ &+\sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta)p}\left|\left\{x\in E_l:\left|[\mu_\Omega,b]\left(\sum_{k=-\infty}^{l-4}\lambda_k a_k\right)(x)\right|>\lambda/2\right\}\right|^{p/q_2}\right\}^{1/p}\\ :=\ &G_1+G_2.\end{aligned}$$

By the (L^{q_1},L^{q_2}) boundedness of $[\mu_\Omega,b]$, and an estimate similar to that for I_1 in Theorem 3.1, we obtain

$$\begin{aligned}G_1^p &\le C\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta)p}\left\|[\mu_\Omega,b]\left(\sum_{k=l-3}^{\infty}\lambda_k a_k\right)(x)\chi_l\right\|_{q_2}^p\\ &\le C\|b\|_{\mathrm{Lip}_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.\end{aligned}$$

That is

$$G_1\le C\|b\|_{\mathrm{Lip}_\beta}\Big(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\Big)^{1/p}.$$

To estimate G_2, let us now use the estimate

$$|[\mu_\Omega,b]a_k(x)|\le C\|b\|_{\mathrm{Lip}_\beta}|x|^{-n}2^{k(\beta+n(1-1/q_1)-\alpha)}$$

which we get in the proof of Theorem 3.1. Note that when $x \in E_l$,

$$\lambda < \sum_{k=-\infty}^{l-4} |\lambda_k| |[\mu_\Omega, b](a_k)(x)| \leq C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{l-4} |\lambda_k| 2^{-ln} \leq C\|b\|_{\mathrm{Lip}_\beta} 2^{-ln} (\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p}.$$

Here we used $\alpha = n(1 - 1/q_1) + \beta$ and $0 < p \leq 1$. For $\forall \lambda > 0$, let l_λ be the maximal positive integer satisfying

$$2^{l_\lambda n} \leq C\|b\|_{\mathrm{Lip}_\beta} \lambda^{-1} (\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p},$$

then if $l > l_\lambda$, we have

$$\left|\left\{ x \in E_l : \left| [\mu_\Omega, b] \left(\sum_{k=-\infty}^{l-4} \lambda_k a_k \right)(x) \right| > \lambda/2 \right\}\right| = 0.$$

So we obtain

$$G_2 \leq \sup_{\lambda > 0} \lambda \left\{ \sum_{l=-\infty}^{l_\lambda} 2^{lnp} \right\}^{1/p} \leq C \sup_{\lambda > 0} \lambda 2^{l_\lambda n} \leq C\|b\|_{\mathrm{Lip}_\beta} (\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p}.$$

Now, combining the above estimates for G_1 and G_2, we obtain

$$\|[\mu_\Omega, b]f\|_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta, p}} \leq C\|b\|_{\mathrm{Lip}_\beta} (\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p},$$

and Theorem 3.3 follows by taking the infimum over all central atomic decompositions of f.

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