MEAN CONVERGENCE OF HERMITE-FEJÉR TYPE INTERPOLATION ON AN ARBITRARY SYSTEM OF NODES*

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Received Jan. 21, 2002, Revised Dec. 28, 2003

Abstract

In this paper sufficient conditions for mean convergence and rate of convergence of Hermite-Fejér type interpolation in the L^p norm on an arbitrary system of nodes are presented.

Key words *Hermite-Fejér interpolation, Mean convergence, Hermite interpolation, Rate of convergence*

AMS(2000)subject classification 41A05

1 Introduction

In engineering applications, data collected from the field are usually discrete and the physical meanings of the data are not always well known. To estimate the outcomes and, eventually, to have a better understanding of the physical phenomenon, a more analytically controllable function that fits the field data is desirable. In application and theory, it is very important to investigate the convergence of Hermite-Fejér type interpolation on an arbitrary system of nodes.

* Project 19671082 supported by National Natural Science Foundation of China, I acknowledge endless help from Prof. Shi Ying-Guang during finishing this paper.

Although there have been many papers on mean convergence of Hermite interpolation of higher order, almost all of them discuss only interpolation based on special system of nodes and special measure. Only recently, after J. Szabados[1], Y. G. Shi, in the paper [6] , gave a very important result dealing with the Hermite interpolation of higher order on general nodes. In this paper, we distinguish two cases: about arbitrary measures and special measures, the corresponding mean convergence of Hermite-Fejér interpolation will be discussed in the following two sections, respectively. For more recent papers on these topics we refer the reader to [3], [5], [7], [8], [10].

Let $\mu(x)$ be a nondecreasing function on $[-1,1]$ with infinitely many points of increase such that all moments with respect to $d\mu(x)$ are finite. We call $d\mu$ a measure . $d\mu$ is said to be the Jacobi measure if $d\mu(x) = w^{(\alpha,\beta)}(x)dx$, where $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ and $(\alpha, \beta > -1)$.

Let $\mathrm{N}_2=\{2,4,6,\cdots\}$ and $\mathrm{N}_0=\mathrm{N}_2\bigcup\{0\}$. Let $m_{kn} \geq 1$ be integers, $k = 1,2,...,n$, $n = 1,2,...$, and X denote a triangular matrix of nodes

$$1 = x_{0n} \geq x_{1n} > x_{2n} > ... > x_{nn} \geq x_{n+1,n} = -1, \quad n = 1,2,\cdots. \tag{1.1}$$

Throughout this paper let $N := \sum_{k=1}^{n} m_{kn} - 1$, $m = \max_{1\leq k\leq n} m_{kn} < +\infty$. In the following discussions, we denote $x_{kn}, (k = 1,2,...,n)$ and $m_{kn}, (k = 1,2,...,n)$ by $x_k, (k = 1,2,...,n)$ and $m_k, (k = 1,2,...,n)$. Denote by $\boldsymbol{P}_N$ the set of polynomials of degree at most N and by $A_{jk}(x)$ the fundamental polynomials for Hermite interpolation, i.e., $A_{jk} \in P_{\mathrm{N}}$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \quad p = 0,1,...,m_q-1, \quad j = 0,1,...,m_k-1, \quad k,q = 1,2,...,n. \tag{1.2}$$

Then for a fixed integer r, $0 \leq r \leq m-1$, and for $f \in C^r[-1,1]$, the unique truncated Hermite interpolatory polynomial is defined by

$$H_{nmr}(f,x) = \sum_{k=1}^{n}\sum_{j=0}^{r} f^{(j)}(x_k)A_{jk}(x), \quad r = 0,1,2,.... \tag{1.3}$$

Here we agree $A_{jk} = 0$ if $j \geq m_k$. In particular, when $r = 0$ and $r = m-1$, H_{nmr} will be denoted by H_{nm} and H^*_{nm}, respectively. We recognize that H_{n1} is the classical Lagrange interpolation and H_{n2} is the classical Hermite-Fejér interpolation. H_{nmr} is called Lagrange type interpolation if m_k is odd and Hermite-Fejér type interpolation if m_k is even, respectively.

To obtain an explicit formula of $A_{jk}(x)$ we set

$$L_k(x) = \prod_{q=1,q\neq k}^{n}\left(\frac{x-x_q}{x_k-x_q}\right)^{m_q}, \qquad k = 1,2,...,n,$$

$$b_{vk} = \frac{1}{v!}\left[\frac{1}{L_k(x)}\right]^{(v)}_{x=x_k}, \qquad v = 0,1,...,m_k-1, \quad k = 1,2,...,n,$$

$$B_{jk}(x) = \sum_{v=0}^{m_k-j-1} b_{vk}(x-x_k)^v, \qquad j = 0,1,...,m_k-1, \quad k = 1,2,...,n.$$

Then by [5,(1.5)] we have

$$A_{jk}(x) = \frac{1}{j!}(x-x_k)^j B_{jk}(x)L_k(x), \quad 0 \le j \le m_k - 1, \quad 1 \le k \le n. \tag{1.4}$$

The most special case is $m_k \equiv m$. In this case we have the simple formulas for $k = 1, 2, \ldots, n$, that is

$$L_k(x) = l_k(x)^m,$$

where

$$l_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}, \qquad \omega_n(x) = \prod_{k=1}^{n}(x-x_k).$$

We also need the following notations ($0 < p < +\infty$)

$$\begin{aligned}
&\|f\|_{d\mu,p} := \|f\|_{L^p_{d\mu,p}} := \left\{\int_{-1}^{1} |f(x)|^p \mathrm{d}\mu(x)\right\}^{1/p}, \\
&\|f\|_r := \max_{0\le j\le r} \|f^{(j)}\|, \quad f \in C^r[-1,1], \\
&\|H_{nmr}(X)\|_{d\mu,p} := \sup_{\|f\|_r \le 1} \|H_{nmr}(X,f)\|_{d\mu,p}, \\
&R_{nmr}(X,f,x) := |H_{nmr}(X,f,x) - f(x)|, \\
&S_{nmr}(X,x) := \sum_{k=1}^{n} |(x-x_k)A_{rk}(x)|, \\
&f_i(x) := x^i, \quad i = 1, 2, \ldots.
\end{aligned}$$

In the following the sequence $c, c_1, c_2, \ldots$ will stand for positive constants , being independent of variables and indices unless otherwise indicated, their value may be different at different occurrences , even in the subsequent formulas.

2 An Arbitrary Measure

First we list some known results needed later. Let $d_k = \max\{|x_k - x_{k+1}|, |x_k - x_{k-1}|\}, k = 1, 2, \ldots, n$, and $D_n = \max_{1\le k\le n} d_k$.

Lemma A[6,Theorem2.1]. *If $m_k - j$ is odd and $j \le i \le m_k - 1$, then*

$$B_{jk}(x) \ge c\left|\frac{x-x_k}{d_k}\right|^{i-j} |B_{ik}(x)|,$$

$$|A_{ik}(x)| \le c_1 d_k^{i-j} |A_{jk}(x)|, \quad x \in R, \quad 1 \le k \le n,$$

where c, c_1 are positive constants depending only on m.

Lemma B[9,Lemma1]. *Let $m_k \equiv m$. If $m - j$ is odd and*

$$\|B_{jk}(x)L_k(x)\| \le \mu_n, \quad k = 1, 2, \ldots, n,$$

then

$$d_k \le c\Delta_n(x_k)\ln^4(n\mu_n), \quad k = 1, 2, ..., n,$$

where $\Delta_n(x) = \dfrac{(1-x^2)^{1/2}}{n} + \dfrac{1}{n^2}$.

Lemma C[6,Lemma2.3]. *For $A_{ij}(x)$ we have*

$$\sum_{i=0}^{r}\sum_{k=1}^{n}\frac{(-1)^i}{(j-i)!}(x-x_k)^{j-i}A_{ik}(x) = \frac{1}{j!}\sum_{i=0}^{j}(-1)^i\binom{j}{i}x^{j-i}H_{nmr}(f_i,x)$$

$$= \frac{1}{j!}\sum_{i=0}^{j}(-1)^i\binom{j}{i}x^{j-i}R_{nmr}(f_i,x), \quad 1 \le j \le \mathbf{N}.$$

Here we let $1/r!$ *by* 0 *if* $r<0$.

Lemma D[5,Theorem4.1]. *Let both m and $m-r$ be even integers. Then for an arbitrary system X of nodes,*

$$R_{nmr}(X, f_{r+1}, x) + R_{nmr}(X, f_{r+2}, x) \ge c_2 S_{n,m,r+1}(X, x),$$

where c_2 depends only on m.

From Lemma D, if

$$\lim_{n\to\infty}\|H_{nmr}(X,f) - f\|_{d\mu,p} = 0 \tag{2.1}$$

holds for every $f \in C^r[-1,1]$, then

$$\|H_{nmr}(X)\|_{d\mu,p} \le c_3 < \infty \tag{2.2}$$

and

$$\lim_{n\to\infty}\|S_{n,m,r+1}(X)\|_{d\mu,p} = 0. \tag{2.3}$$

This shows that $S_{n,m,r+1}(X,x)$ plays an important role in the mean convergence of Hermite-Fejér type interpolation.

Now let us prove some lemmas. Following [6, Lemma 2.3], the first lemma below is easily proven.

Lemma 1. *For $A_{jk}(x)$ we have*

$$i!\sum_{k=1}^{n}\sum_{j=1}^{m_k-1}\frac{(-1)^{j+1}}{(i-j)!}(x-x_k)^{i-j}A_{jk}(x) = \sum_{k=1}^{n}(x-x_k)^iA_{0k}(x). \tag{2.4}$$

Lemma 2. *Let $m_k \in \mathbf{N}_2$, $1 \le k \le n$, $r \in \mathbf{N}_0$, $r \le q \le m-1$, $0<p<\infty$ and let $d\mu$ be a measure on $[-1,1]$. Assume that (2.1) holds for $f = f_j$ with $j = r+1, r+2, ..., M$, where*

$$M = \begin{cases} m-2, & r \le m-4, \\ m, & r \ge m-2, \end{cases}$$

then (2.1) holds for every polynomial f.

Proof. If $q = m-1$ then $H_{nmr} = H^*_{nm}$ is a projector operator and the lemma is trivial.

Now we assume that $q \le m-2$ and hence $r \le m-2$, since $r \in \mathbf{N}_0$ and $m \in \mathbf{N}_2$.

First we observe that $R_{nmr}(f_j) = 0, \quad j = 0,1,2,...,r$, and thus (2.1) holds for $f = f_j, \quad j = 0,1,...,M$. Hence by Lemma C

$$\lim_{n\to\infty} \left\| \sum_{i=0}^{r}\sum_{k=1}^{n} \frac{(-1)^i}{(j-i)!}(x-x_k)^{j-i}A_{ik}(x) \right\|_{d\mu,p} = 0 \tag{2.5}$$

holds for every $j \le M$.

Now for $j \ge M+1$ let us estimate the term

$$S(x) = \left| \sum_{i=r+1}^{j}\sum_{k=1}^{n} \frac{(-1)^i}{(j-i)!}(x-x_k)^{j-i}A_{ik}(x) \right|. \tag{2.6}$$

Clearly

$$S(x) \le \sum_{i=r+1}^{j}\sum_{k=1}^{n} \left|(x-x_k)^{j-i}A_{ik}(x)\right| \le \sum_{i=r+1}^{j-1}\sum_{k=1}^{n} |(x-x_k)A_{ik}(x)| + \sum_{k=1}^{n}|A_{jk}(x)|. \tag{2.7}$$

We separate two cases. Recall that $j \ge M+1$, implies $j \ge m-1$.

Case 1. $j = m-1$. In this case $M+1 \le j = m-1$ implies that $M \le m-2$ and so we have $M = m-2$ and hence $r \le m-4$. By Lemma A and (2.7)

$$\begin{aligned} S(x) &\le c\left[\sum_{i=r+1}^{m-2}\sum_{k=1}^{n} \left|(x-x_k)^{m-1-i}A_{ik}(x)\right| + \sum_{k=1}^{n}\left|(x-x_k)^2 A_{m-3,k}(x)\right| \right] \\ &\le c_4(1 + D_n + ... + D_n^{m-3-r})S_{n,m,r+1}(X,x). \end{aligned}$$

Case 2. $j \ge m$. By (2.7), Lemma A and recalling that $A_{jk}(x) = 0$, we have

$$S(x) \le c_5 \sum_{i=r+1}^{j-1}\sum_{k=1}^{n} d_k^{i-r-1}\left|A_{r+1,k}(x)(x-x_k)\right| \le cS_{n,m,r+1}(X,x).$$

By Lemma D and condition of this lemma, this leads to

$$\lim_{n\to\infty} \|S(x)\|_{d\mu,p} = 0, \tag{2.8}$$

which coupled with (2.4) implies that (2.5) holds for every $j \ge M+1$ and hence holds for every $j \ge 0$.

Now (2.8) with $j = r+1$ gives

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} A_{r+1,k} \right\|_{d\mu,p} = 0. \tag{2.9}$$

By Lemma A and (2.3) we see that

$$\lim_{n\to\infty}\left\|\sum_{k=1}^{n}(x-x_k)^v A_{ik}(x)\right\|_{d\mu,p}=0 \tag{2.10}$$

holds for every $v\geq 1$ and every $i\geq r+1$, since

$$\left|\sum_{k=1}^{n}(x-x_k)^v A_{ik}(x)\right|\leq c\sum_{k=1}^{n}|(x-x_k)A_{r+1,k}(x)|=cS_{n,m,r+1}(X,x). \tag{2.11}$$

Again applying Lemma C, (2.8) and (2.9) and using induction on j we conclude that (2.10) holds for every $v\geq 0$ and $i\geq r+1$. Using the inequality $|a+b|^p\leq 2^p(|a|^p+|b|^p)$ we have

$$\|f+g\|_{d\mu,p}\leq 2^{1+1/p}(\|f\|_{d\mu,p}+\|g\|_{d\mu,p}),\quad 0<p<\infty. \tag{2.12}$$

Now for any $f\in P_{\rm N}$, applying the mean value theorem for the derivatives it follows from (2.10) and (2.12) that as $n\to\infty$

$$\begin{aligned}&\|H_{nmq}(f)-f\|_{d\mu,p}\\&=\left\|\sum_{i=q+1}^{m-1}\sum_{k=1}^{n}f^{(i)}(x_k)A_{ik}(x)\right\|_{d\mu,p}=\left\|\sum_{i=q+1}^{m-1}\sum_{k=1}^{n}\left\{f^{(i)}(x)-[f^{(i)}(x)-f^{(i)}(x_k)]\right\}A_{ik}(x)\right\|_{d\mu,p}\\&\leq c\sum_{i=q+1}^{m-1}\left\{\left\|f^{(i)}\right\|\left\|\sum_{k=1}^{n}A_{ik}(x)\right\|_{d\mu,p}+\|f^{(i+1)}\|\left\|\sum_{k=1}^{n}|(x-x_k)A_{ik}(x)|\right\|_{d\mu,p}\right\}\to 0.\end{aligned}$$

Lemma 3. *Let m_k, $k=1,2,...,n$ and r be even integers. Then for an arbitrary system of nodes* (1.1), *we have*

$$\|S_{n,m,r+1}(X)\|_{d\mu,p}\leq c\|H_{nmr}(X)\|_{d\mu,p},\quad n\geq 1, \tag{2.13}$$

where $c>0$ depends only on m and p .

Proof. By Lemma C and (2.4) for $j=r+2$ we obtain

$$\sum_{k=1}^{n}\{(x-x_k)A_{r+1,k}(x)-A_{r+2,k}(x)\}=\frac{1}{(r+2)!}\sum_{i=0}^{r+2}(-1)^i\binom{r+2}{i}x^{r+2-i}H_{nmr}(f_i,x). \tag{2.14}$$

On the other hand owing to $B_{r+1,k}(x)-B_{r+2,k}(x)=b_{m_k-r-2,k}(x-x_k)^{m_k-r-2}\geq 0$, we have

$$\begin{aligned}&\sum_{k=1}^{n}[(x-x_k)A_{r+1,k}(x)-A_{r+2,k}(x)]\\&\quad=\sum_{k=1}^{n}(x-x_k)^{r+2}L_k(x)\left[\frac{1}{(r+1)!}B_{r+1,k}(x)-\frac{1}{(r+2)!}B_{r+2,k}(x)\right]\\&\quad=\sum_{k=1}^{n}\frac{1}{(r+1)!}(x-x_k)^{r+2}L_k(x)\left[B_{r+1,k}(x)-\frac{1}{r+2}B_{r+2,k}(x)\right]\\&\quad\geq\frac{r+1}{r+2}\sum_{k=1}^{n}(x-x_k)A_{r+1,k}(x),\end{aligned}$$

By Lemma A, $S_{n,m,r+1}(X,x) = \left|\sum_{k=1}^{n}(x-x_k)A_{r+1,k}(x)\right|$, so by Lemma C and (2.14)

$$\begin{aligned}\left\|\sum_{k=1}^{n}(x-x_k)A_{r+1,k}(x)\right\|_{d\mu,p} &\le c\left\|\sum_{k=1}^{n}\{(x-x_k)A_{r+1,k}(x)-A_{r+2,k}(x)\}\right\|_{d\mu,p}\\ &\le c_0\sum_{i=0}^{r+2}\|H_{nmr}(f_i)\|_{d\mu,p}\le c_6\|H_{nmr}(X)\|_{d\mu,p}\end{aligned}$$

since

$$\|H_{nmr}(f_i)\|_{d\mu,p}\le\|f_i\|_r\|H_{nmr}(X)\|_{d\mu,p}\le i!\|H_{nmr}(X)\|_{d\mu,p}.$$

This proves the lemma.

An immediate consequence of Lemma 2 is the following theorem.

Theorem 1. *Let $m_k\in\mathrm{N}_2$, $1\le k\le n$, $r\in\mathrm{N}_0$, $0<p<\infty$ and $d\mu$ be a measure on $[-1,1]$. Then (2.1) is true for every $f\in C^r[-1,1]$ if and only if (2.2) is true and (2.1) holds for $f=f_i,\quad i=r+1,r+2,...,M$.*

Lemma 4. *Let $m_k\in\mathrm{N}_2$, $1\le k\le n$, $m\ge 4$, $r\in\mathrm{N}_0$, $0<p<\infty$ and $d\mu$ be a measure in $[-1,1]$. Then*

$$\|H^*_{nm}(X)\|_{d\mu,p}\le c\|H_{nmr}(X)\|_{d\mu,p},\tag{2.15}$$

where $c>0$ depends only on m and p.

Proof. It is easy to see that

$$\begin{aligned}\|H^*_{nm}\|_{d\mu,p} &\le c\sup_{\|f\|_{m-1}\le1}\left[\left\|\sum_{j=0}^{r}\sum_{k=1}^{n}f^{(j)}(x_k)A_{jk}(x)+\sum_{j=r+1}^{m-1}\sum_{k=1}^{n}f^{(j)}(x_k)A_{jk}(x)\right\|_{d\mu,p}\right]\\ &\le c\left[\sup_{\|f\|_{m-1}\le1}\left\|\sum_{j=0}^{r}\sum_{k=1}^{n}f^{(j)}(x_k)A_{jk}(x)\right\|_{d\mu,p}+\sup_{\|f\|_{m-1}\le1}\left\|\sum_{j=r+1}^{m-1}\sum_{k=1}^{n}f^{(j)}(x_k)A_{jk}(x)\right\|_{d\mu,p}\right]\\ &:=c\left[S+\sum_{j=r+1}^{m-1}S_j\right].\end{aligned}$$

Clearly

$$S\le\sup_{\|f\|_r\le1}\left\|\sum_{j=0}^{r}\sum_{k=1}^{n}f^{(j)}(x_k)A_{jk}(x)\right\|_{d\mu,p}\le\|H_{nmr}(X)\|_{d\mu,p}.$$

In order to estimate S_j for $r+1\le j\le m-1$, we separate the cases for $r+1\le j<m-1$ and $j=m-1$.

Case 1 . $r+1 \le j < m-1$. By the mean value theorem for derivatives one has

$$S_j = \sup_{\|f\|_{m-1}\le 1} \left\| \sum_{k=1}^{n} \left\{ f^{(j)}(x) - [f^{(j)}(x) - f^{(j)}(x_k)] \right\} A_{jk}(x) \right\|_{d\mu,p}$$
$$\le c \left[\left\| \sum_{k=1}^{n} A_{jk} \right\|_{d\alpha,p} + \left\| \sum_{k=1}^{n} |(x-x_k)A_{jk}(x)| \right\|_{d\mu,p} \right].$$

Also by Lemma A, we have

$$\left\| \sum_{k=1}^{n} |(x-x_k)^i A_{jk}(x)| \right\|_{d\mu,p} \le c \left\| \sum_{k=1}^{n} |(x-x_k)A_{r+1,k}(x)| \right\|_{d\mu,p} \le c\|S_{n,m,r+1}(X)\|_{d\mu,p}. \quad (2.16)$$

So by (2.14) and Lemma A we can get

$$\left\| \sum_{k=1}^{n} A_{jk}(x) \right\|_{d\mu,p} = \left\| \sum_{i=0}^{j-1} \sum_{k=1}^{n} \frac{(-1)^i}{(j-i)!} (x-x_k)^{j-i} A_{ik}(x) \right\|_{d\mu,p}$$
$$\le c \left\{ \left\| \sum_{i=0}^{r} \sum_{k=1}^{n} \frac{(-1)^i}{(j-i)!} (x-x_k)^{j-i} A_{ik}(x) \right\|_{d\mu,p} + \left\| \sum_{i=r+1}^{j-1} \sum_{k=1}^{n} \frac{(-1)^i}{(j-i)!} (x-x_k)^{j-i} A_{ik}(x) \right\|_{d\mu,p} \right\}$$
$$\le c \left[\sum_{i=0}^{j} \|H_{nmr}(f_i)\|_{d\mu,p} + \left\| \sum_{k=1}^{n} |(x-x_k)A_{r+1,k}(x)| \right\|_{d\mu,p} \right]$$
$$\le c \left[\sum_{i=0}^{j} \|H_{nmr}(f_i)\|_{d\mu,p} + \|S_{n,m,r+1}(X)\|_{d\mu,p} \right].$$

Since

$$\|H_{nmr}(f_i)\|_{d\mu,p} \le \|f_i\|_r \|H_{nmr}(X)\|_{d\mu,p} \le i! \|H_{nmr}(X)\|_{d\mu,p}$$

it follows that

$$S_j \le c(\|H_{nmr}(X)\|_{d\mu,p} + \|S_{n,m,r+1}(X)\|_{d\mu,p}).$$

Case 2. $j = m-1$. By Lemma A and (2.16)

$$S_{m-1} \le \left\| \sum_{k=1}^{n} |A_{m-1,k}(x)| \right\|_{d\mu,p} \le c \left\| \sum_{k=1}^{n} (x-x_k)A_{r+1,k}(x) \right\|_{d\mu,p} \le c\|S_{n,m,r+1}(X)\|_{d\mu,p}.$$

Therefore the estimate (2.15) follows from Lemma 3.

By this lemma it is easy to obtain the following theorem.

Theorem 2. *Let $m_k \in \mathrm{N}_2$, $1 \le k \le n$, $r \in \mathrm{N}_0$, $0 < p < \infty$ and let $d\mu$ be a measure on $[-1,1]$. If (2.1) holds for every $f \in C^r[-1,1]$ then*

$$\lim_{n\to\infty} \|H^*_{nm}(f) - f\|_{d\mu,p} = 0$$

holds for every $f \in C^{m-1}[-1,1]$.

3 The Special Measure

For the special measure the main results in section 2 can be improved . To do this, we first list some known results and prove an auxiliary lemma needed later. Let

$$\Gamma := \max\{\alpha,\beta\},\quad \gamma := \min\{\alpha,\beta\},\quad \Gamma^* := \max\{2\Gamma+2,1\},\quad \gamma^* := \min\{\gamma,0\}.$$

Lemma E[4, Lemma 3.1]. *Let $0<p<+\infty$ and $d\mu$ be a measure on $[-1,1]$ satisfying*

$$\frac{d\mu(x)}{dx} \ge C_0 w^{(\alpha,\beta)}(x), \quad \text{a. e.} \quad x\in[-1,1], \tag{3.1}$$

Then

$$\|P\| \le cn^{\Gamma^*/p}\|P\|_{d\mu,p}, \quad P\in \boldsymbol{P}_n, \tag{3.2}$$

where $c=c(w^{(\alpha,\beta)},C_0,p)>0$.

Lemma F[4, Lemma 3.3]. *Let $d=\max\limits_{k=0,1,\dots,n-1}\{|x_k-x_{k+1}|\}$. If*

$$c_1 \le \frac{w}{w^{(\alpha,\beta)}} \le c_2 \tag{3.3}$$

and $\delta_n=(\xi-h,\xi+h)$, where $h=dn^{-\Gamma^*/(\gamma^*+1)}$ or $h=d(1-\xi^2)^{-\gamma^*}n^{-\Gamma^*}$, then

$$\|P\|_{w,p} \le \begin{cases} cn^{\Gamma^*/(\gamma^*+1)}\|(x-\xi)P(x)\|_{w,p}, \\ c(1-\xi^2)^{\gamma^*}n^{\Gamma^*}\|(x-\xi)P(x)\|_{w,p}. \end{cases} \tag{3.4}$$

where c_1 and c_2 are positive constants.

Lemma G[4,Lemma3.4]. Let $\xi\in[-1,1]$, $\xi=\cos\theta$, and $P\in\boldsymbol{P}_n$. Then

$$\|P\| \le \frac{cn[\|(x-\xi)P(x)\|+|P(\xi)|]}{\sin\theta}, \quad |\xi|<1, \tag{3.5}$$

and

$$\|P\| \le cn^2\|(x-\xi)P(x)\|. \tag{3.6}$$

For simplicity let $R(\mu,p,r) = \|R_{nmr}(f_{r+1})\|_{d\mu,p} + \|R_{nmr}(f_{r+2})\|_{d\mu,p}$. To give the main result of this section, we prove following lemma.

Lemma 5. *Let $m_k\equiv m\in \mathrm{N}_2$, $r\in\mathrm{N}_0$, $P\in\boldsymbol{P}_n$, $0<p<\infty$ and $d\mu$ be a measure on $[-1,1]$. If (3.1) is true , then*

$$\|R_{nmr}(P)\|_{d\mu,p} \le c\|P\|_m R(\mu,p,r)\{1+n^{\Gamma^*/p-1}\ln^{10}[n(1+n^{\Gamma^*/p}R(\mu,p,r))]\}. \tag{3.7}$$

If $d\mu(x)=w(x)dx$ and (3.3) is true , then

$$\|R_{nmr}(P,x)\|_{d\mu,p} \le c\|P\|_m R(\mu,p,r)\{1+n^{\max\{\Gamma^*-2,\Gamma^*/(1+\gamma^*)-4\}}\ln^{10}[n(1+n^{\Gamma^*/p}R(\mu,p,r))]\},$$

where $c = c(d\mu, m, p) > 0$.

Proof. By definition of $R_{nmr}(P)$

$$\|R_{nmr}(P)\|_{d\mu,p} \le c \sum_{j=r+1}^{m-1} \left\| \sum_{k=1}^{n} P^{(j)}(x_k) A_{jk}(x) \right\|_{d\mu,p} := c \sum_{j=r+1}^{m-1} S_j.$$

We separate the three cases for $j = r+1$, $j = r+2$ and $j \ge r+3$.

Clearly

$$\left\| \sum_{k=1}^{n} A_{r+1,k} \right\|_{d\mu,p} = \|R_{nmr}(f_{r+1})\|_{d\mu,p}$$

by Lemma A, $\left| \sum_{k=1}^{n} (x - x_k) A_{r+1,k}(x) \right| = S_{n,m,r+1}(X)$, so we have

$$\left\| \sum_{k=1}^{n} (x - x_k) A_{r+1,k}(x) \right\|_{d\mu,p} = \|S_{n,m,r+1}(X)\|_{d\mu,p}.$$

Thus by the mean value theorem for derivatives one has

$$\begin{aligned} S_{r+1} &= \left\| \sum_{k=1}^{n} \left\{ P^{(r+1)}(x) - [P^{(r+1)}(x) - P^{(r+1)}(x_k)] \right\} A_{r+1,k}(x) \right\|_{d\mu,p} \\ &\le c\|P\|_m \left[\|R_{nmr}(f_{r+1})\|_{d\mu,p} + \|S_{n,m,r+1}(X)\|_{d\mu,p} \right]. \end{aligned} \tag{3.8}$$

Now let us estimate S_{r+2}. We have

$$\begin{aligned} S_{r+2} &= \left\| \sum_{k=1}^{n} \left\{ P^{(r+2)}(x) - [P^{(r+2)}(x) - P^{(r+2)}(x_k)] \right\} A_{r+2,k}(x) \right\|_{d\mu,p} \\ &\le c\|P\|_m \left[\left\| \sum_{k=1}^{n} A_{r+2,k}(x) \right\|_{d\mu,p} + \|S_{n,m,r+1}(X)\|_{d\mu,p} \right]. \end{aligned} \tag{3.9}$$

Let us estimate $\|\sum_{k=1}^{n} A_{r+2,k}(x)\|_{d\mu,p}$. By Lemma C and (2.4) (for $j = r+2$) we have

$$\begin{aligned} \sum_{k=1}^{n} A_{r+2,k}(x) &= \sum_{k=1}^{n} (x - x_k) A_{r+1,k}(x) \\ &\quad + \frac{1}{(r+1)!} x R_{nmr}(f_{r+1}) - \frac{1}{(r+2)!} R_{nmr}(f_{r+2}), \end{aligned}$$

so by Lemma D and (2.12)

$$\begin{aligned} &\left\| \sum_{k=1}^{n} A_{r+2,k}(x) \right\|_{d\mu,p} \\ &\le c \left\| \sum_{k=1}^{n} (x - x_k) A_{r+1,k}(x) \right\|_{d\mu,p} + c \left\| \frac{1}{(r+1)!} x R_{nmr}(f_{r+1}) - \frac{1}{(r+2)!} R_{nmr}(f_{r+2}) \right\|_{d\mu,p} \\ &\le c\{ \|R_{nmr}(f_{r+1})\|_{d\mu,p} + \|R_{nmr}(f_{r+2})\|_{d\mu,p} \}. \end{aligned}$$

For $j \geq r+3$, Lemma A gives

$$\begin{aligned} S_j &\leq \|P^{(j)}\| \left\| \sum_{k=1}^{n} |A_{jk}(x)| \right\|_{d\mu,p} \leq c\|P^{(j)}\| \left\| \sum_{k=1}^{n} d_k^2 |A_{r+1,k}(x)| \right\|_{d\mu,p} \\ &= c\|P^{(j)}\| \left\| \sum_{|x-x_k|\geq d_k} d_k^2 |A_{r+1,k}(x)| + \sum_{|x-x_k|<d_k} d_k^2 |A_{r+1,k}(x)| \right\|_{d\mu,p} \\ &\leq c\|P^{(j)}\| \left[\|S_{n,m,r+1}(X)\|_{d\mu,p} + \left\| \sum_{|x-x_k|<d_k} d_k^2 |A_{r+1,k}(x)| \right\|_{d\mu,p} \right]. \end{aligned} \tag{3.10}$$

Now let us estimate the term

$$S = \left\| \sum_{|x-x_k|<d_k} d_k^2 |A_{r+1,k}(x)| \right\|_{d\mu,p}.$$

Let $x_k = \cos\theta_k, k = 0, 1, ..., n+1$. Then By Lemma G and [6 Lemma 4.1],we get

$$\|A_{r+1,k}(x)\| \leq \frac{cn[\|(x-x_k)A_{r+1,k}(x)\| + |A_{r+1,k}(x_k)|]}{\sin\theta_k} \leq \frac{cn\|S_{n,m,r+1}(X)\|}{\sin\theta_k}. \tag{3.11}$$

$$\|A_{r+1,k}(x)\| \leq cn^2\|S_{n,m,r+1}(X)\|. \tag{3.12}$$

On the other hand using

$$S_{n,m,r+1}(X,x) \geq c\sum_{k=1}^{n} d_k^{2+r-m}|(x-x_k)l_k(x)|^m \geq c\sum_{k=1}^{n} |(x-x_k)l_k(x)|^m$$

we have $|(x-x_k)l_k(x)| \leq c\|S_{n,m,r+1}(X)\|^{1/m}$. Hence by (3.6)

$$\|l_k(x)\| \leq cn^2\|S_{n,m,r+1}(X)\|^{1/m}, \quad k = 1, 2, ..., n. \tag{3.13}$$

By an estimation of Erdös[1] as in [6, proof of Theorem 4.1(Claim 2)] we obtain

$$|\theta_{k+1}-\theta_k| \leq c\frac{(\ln n)\ln(n\|S_{n,m,r+1}(X)\|)}{n} \leq c\frac{\ln^2[n(1+\|S_{n,m,r+1}(X)\|)]}{n}, \quad k = 1, 2, ..., n, \tag{3.14}$$

and the maximum number N_n of the set $\{k : |x-x_k| < d_k\}$ with x running over the interval $[-1, 1]$ satisfies

$$N_n \leq c(\ln n)\ln(n\|S_{n,m,r+1}(X)\|) \leq c\ln^2[n(1+\|S_{n,m,r+1}(X)\|)]. \tag{3.15}$$

Similarly as in [6, proof of Theorem 4.1(Claim 2)], from (3.14) we get

$$d_k \leq c\frac{\ln^2[n(1+\|S_{n,m,r+1}(X)\|)]}{n}\left\{\sin\theta_k + \frac{\ln^2[n(1+\|S_{n,m,r+1}(X)\|)]}{n}\right\}. \tag{3.16}$$

The final preliminary step in the proof is to show that

$$\|S_{n,m,r+1}(X)\| \leq n^{\Gamma^*/p}\|S_{n,m,r+1}(X)\|_{d\mu,p}. \tag{3.17}$$

In fact, since $B_{r+1,k} \geq 0$, by [5, Theorem 2.1], we get

$$S_{n,m,r+1}(X,x) = \sum_{k=1}^{n} |(x-x_k)A_{r+1,k}(x)| = \frac{1}{(r+1)!}\sum_{k=1}^{n} |(x-x_k)^{r+2}B_{r+1,k}(x)l_k^m(x)|$$
$$= \frac{1}{(r+1)!}\sum_{k=1}^{n}\left|\frac{\omega_n(x)}{\omega_n'(x_k)}\right|^{r+2} B_{r+1,k}(x)l_k(x)^{m-r-2}$$
$$= \left|\frac{1}{(r+1)!}\sum_{k=1}^{n}\frac{\omega_n(x)^{r+2}}{|\omega_n'(x_k)|^{r+2}}B_{r+1,k}(x)l_k(x)^{m-r-2}\right|.$$

Here the function

$$\frac{1}{(r+1)!}\sum_{k=1}^{n}\frac{\omega_n(x)^{r+2}}{|\omega_n'(x_k)|^{r+2}}B_{r+1,k}(x)l_k(x)^{m-r-2}$$

is a polynomial. Thus by (3.2) we get (3.17).

Now we distinguish two cases.

Case 1. $\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} \geq cw^{(\alpha,\beta)}$, a.e..

In this case by applying Lemma E , (3.11), (3.12), (3.15) and (3.16) we obtain that

$$\sum_{|x-x_k|<d_k} d_k^2|A_{r+1,k}(x)| \leq cn^{-1}\|S_{n,m,r+1}(X)\|\ln^{10}[n(1+\|S_{n,m,r+1}(X)\|)]$$

and hence by (3.17) we have

$$S \leq c\|S_{n,m,r+1}(X)\|_{d\mu,p}n^{\Gamma^*/p-1}\ln^{10}\left[n(1+n^{\Gamma^*/p}\|S_{n,m,r+1}(X)\|_{d\mu,p})\right]. \tag{3.18}$$

Case 2. $c_1w^{(\alpha,\beta)} \leq w \leq c_2w^{(\alpha,\beta)}$.

If $\sin\theta_k \geq \ln^2[n(1+\|S_{n,m,r+1}(X)\|)]/n$, then by Lemma F, (3.16) we have

$$d_k^2\|A_{r+1,k}(x)\|_{d\mu,p}$$
$$\leq c\left\{\frac{\ln^2[n(1+\|S_{n,m,r+1}(X)\|)]}{n}\sin\theta_k\right\}^2(\sin\theta_k)^{2\gamma^*}n^{\Gamma^*}\ \|(x-x_k)A_{r+1,k}(x)\|_{d\mu,p}$$
$$\leq c\|S_{n,m,r+1}(X)\|_{d\mu,p}(\sin\theta_k)^{2(1+\gamma^*)}n^{\Gamma^*-2}\ln^4[n(1+\|S_{n,m,r+1}(X)\|)]$$
$$\leq c\|S_{n,m,r+1}(X)\|_{d\mu,p}n^{\Gamma^*-2}\ln^4[n(1+\|S_{n,m,r+1}(X)\|)];$$

and if $\sin\theta_k < \ln^2[n(1+\|S_{n,m,r+1}(X)\|)]/n$, then by Lemma F, (3.16) we have

$$d_k^2\|A_{r+1,k}(x)\|_{d\mu,p} \leq c\left\{\frac{\ln^2[n(1+\|S_{n,m,r+1}(X)\|)]}{n}\right\}^4 n^{\Gamma^*/(1+\gamma^*)}\|(x-x_k)A_{r+1,k}(x)\|_{d\mu,p}$$
$$\leq c\|S_{n,m,r+1}(X)\|_{d\mu,p}n^{\Gamma^*/(1+\gamma^*)-4}\ln^8[n(1+\|S_{n,m,r+1}(X)\|)].$$

Therefore by (3.15) and (3.17) we have

$$S \leq c\|S_{n,m,r+1}\|_{d\mu,p}n^{\max\{\Gamma^*-2,\Gamma^*/(1+\gamma^*)-4\}}\ln^{10}\left[n(1+n^{\Gamma^*/p}\|S_{n,m,r+1}\|_{d\mu,p})\right]. \tag{3.19}$$

Now by applying (3.8), (3.9), (3.10), (3.18), (3.19), (2.12) and Lemma D we finish the proof of this lemma.

Theorem 3. *Let $m_k \equiv m \in \mathbf{N}_2$, $r \in \mathbf{N}_2$, $\Gamma^* < p < +\infty$ and $d\mu$ be a measure in $[-1,1]$ satisfying* (3.1). *If* (2.1) *holds for the polynomials $f = f_{r+1}$ and $f = f_{r+2}$, then* (2.1) *holds for every polynomial f; more* (2.1) *holds for every $f \in C^r[-1,1]$ if and only if* (2.2) *is true and* (2.1) *holds for the polynomials $f = f_{r+1}$ and $f = f_{r+2}$.*

Proof. This is a direct application of Lemma 5.

Developing and properly modifying P. Vértesi[3], Y. G.Shi[7], we can give the rate of convergence in term of the modulus $\omega_\varphi(f,t)$ of Ditzian and Totik in [11] for truncated Hermite interpolation on an arbitrary system of nodes.

First we assume that $\varphi(x) = (1-x^2)^{1/2}$. The weighted modulus of continuity defined by Ditzian and Totik is given by

$$\omega_\varphi(f;t) = \sup_{0<h\le t} \|f(\cdot + h\varphi(\cdot)/2) - f(\cdot - h\varphi(\cdot)/2)\|,$$

where the expression inside $\|\cdot\|$ is taken to be zero if $x \pm h\varphi(x)/2 \notin (-1,1)$.

As usual we set

$$E_n(f) = \min_{P\in P_n} \|f-P\|.$$

We need the following known result.

Lemma H[3, Lemma 3.3]. *For every $f \in C^r[-1,1] (r \ge 0)$ there exists a polynomial $P_n \in \boldsymbol{P}_n$ such that for all $x \in [-1,1]$*

$$\left|f^{(j)}(x) - P_n^{(j)}(x)\right| \le c\Delta_n(x)^{r-j} E_{n-r}(f^{(r)}), \quad 0 \le j \le r, \tag{3.20}$$

$$|P_n^{(j)}(x)| \le c\Delta_n(x)^{r-j}\omega_\varphi(f^{(r)};\frac{1}{n}), \qquad j > r, \tag{3.21}$$

where $\Delta_n(x) = \dfrac{(1-x^2)^{1/2}}{n} + \dfrac{1}{n^2}$.

Theorem 4. *Let $d\mu$ be a measure in $[-1,1]$ satisfying $\dfrac{d\mu}{dx} \ge cw^{(\alpha,\beta)}$, a.e. , $0 < p < \infty$, $m_k \equiv m \in \mathbf{N}_2$, and let*

$$\mu_{n1}(X) = \left\|\sum_{k=1}^{n} |A_{0k}(x)|\right\|_{d\mu,p}, \tag{3.22}$$

$$\mu_{n2}(X) = \left\|\sum_{k=1}^{n} |A_{1k}(x)(x-x_k)|\right\|_{d\mu,p}. \tag{3.23}$$

Let $\ln[n(1+n^{\Gamma^*/p}\mu_{n2}(X))] = \mu(\Gamma^*,p)$, *then for* $f \in C^r[-1,1]$

$$\|H_{nmr}(X,f)-f\|_{d\mu,p} \le \begin{cases} cn^{-r}\omega_\varphi(f^{(r)};\frac{1}{n})\left\{\mu_{n1}(X)+n\mu_{n2}(X)[\mu(\Gamma^*,p)]^{2(m-2)}\right\}, & 0\le r\le m-2, \\ cn^{1-m}E_n(f^{(m-1)})\left\{\mu_{n1}(X)+n\mu_{n2}(X)[\mu(\Gamma^*,p)]^{2(m-2)}\right\}, & r=m-1. \end{cases} \tag{3.24}$$

Proof. Let $P_n \in \boldsymbol{P}_n$ be given as in Lemma H . Then (3.20) and (3.21) hold. Clearly $P_n(x) = H^*_{nm}(P_n,x)$, thus

$$\begin{aligned} &\|H_{nmr}(f)-f\|_{d\mu,p} = \|P_n - f + H_{nmr}(f) - H^*_{nm}(P_n)\|_{d\mu,p} \\ &\le c\left\{\|P_n - f\|_{d\mu,p} + \left\|\sum_{k=1}^n [f(x_k)-P_n(x_k)]\,A_{0k}\right\|_{d\mu,p}\right\} \\ &+c\left\{\left\|\sum_{j=1}^r\sum_{k=1}^n \left[f^{(j)}(x_k)-P_n^{(j)}(x_k)\right]A_{jk}(x)\right\|_{d\mu,p} + \sum_{j=r+1}^{m-1}\left\|\sum_{k=1}^n P_n^{(j)}(x_k)A_{jk}(x)\right\|_{d\mu,p}\right\} \\ &:= c\{\delta_0+\delta_1+\delta_2+\delta_3\}. \end{aligned} \tag{3.25}$$

If $r = m-1$ then $\delta_3 = 0$.

Since

$$E_{n-r}(f^{(r)}) \le cE_n(f^{(r)}). \tag{3.26}$$

it follows from by Lemma H and (3.26) that

$$\delta_0 \le c\|\Delta_n(x)\|^r E_{n-r}(f^{(r)}) \le cn^{-r}E_n(f^{(r)}).$$

By this estimation, (3.26) and Lemma H, similarly we have

$$\delta_1 \le cn^{-r}E_n(f^{(r)})\mu_{n1}(x).$$

To estimate δ_2 and δ_3 we need to estimate d_k. By (3.23) and Lemma A we have

$$\mu_{n2} \ge c\sum_{k=1}^n d_k^{2-m}|(x-x_k)l_k(x)|^m$$

which gives $|(x-x_k)l_k(x)| \le c[\mu_{n2}]^{1/m}$. Similarly as (3.16), using [5, Theorem 4.1(Claim 2)] we have

Case 1. If $\sin\theta_k \le \dfrac{\ln^2[n(1+\mu_{n2})]}{n}$, then

$$d_k \le c\frac{\ln^4[n(1+\mu_{n2})]}{n^2}. \tag{3.27}$$

Case 2. If $\sin\theta_k > \dfrac{\ln^2[n(1+\mu_{n2})]}{n}$, then

$$d_k \le c\frac{\ln^2[n(1+\mu_{n2})]}{n}. \tag{3.28}$$

Let us estimate d_k^{j-1} in the first case. If j is odd then

$$d_k^{j-1} = d_k^{2\frac{j-1}{2}} \le c\frac{\ln^{2(j-1)}[n(1+\mu_{n2})]}{n^{j-1}}, \quad 1 \le j \le m-1; \tag{3.29}$$

and if j is even, then

$$d_k^{j-1} = d_k^{\frac{1}{2}+\frac{j-2}{2}} \le c\frac{\ln^{2j}[n(1+\mu_{n2})]}{n^j}, \quad 2 \le j \le m-2. \tag{3.30}$$

Then by Lemma H, (3.28), (3.29), (3.30) and Lemma A we obtain

$$\begin{aligned}\delta_2 &\le c\sum_{j=1}^{r}\left\|\sum_{k=1}^{n}\Delta_n(x_k)^{r-j}E_{n-r}(f^{(r)})d_k^{j-1}|A_{1k}|\right\|_{d\mu,p} \\ &\le cn^{-r+1}\mu_{n2}(X)E_n(f^{(r)})\left[\ln\left[n(1+n^{1+\Gamma^*/p}\mu_{n2}(X))\right]\right]^{2(r-2)}\end{aligned}$$

and

$$\begin{aligned}\delta_3 &\le c\sum_{j=r+1}^{m-1}\left\|\sum_{k=1}^{n}\Delta_n(x_k)^{r-j}\omega_\varphi(f^{(r)};\frac{1}{n})d_k^{j-1}|A_{1k}|\right\|_{d\mu,p} \\ &\le cn^{-m+1}\mu_{n2}(X)\omega_\varphi(f^{(r)};\frac{1}{n})\left[\ln\left[n(1+n^{\Gamma^*/p}\mu_{n2}(X))\right]\right]^{2(m-2)},\end{aligned}$$

respectively. Substituting the estimates of $\delta_i, i=0,1,2,3$ into (3.25) and using the relation in [10, Theorem 7.1.1]

$$E_n(f^{(r)}) \le c\omega_\varphi(f^{(r)};\frac{1}{n})$$

we obtain (3.24).

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