CONSTRUCTION OF SOME KIESSWETTER-LIKE FUNCTIONS –THE CONTINUOUS BUT NON-DIFFERENTIABLE FUNCTION DEFINED BY QUINARY DECIMAL

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Abstract

In this paper, we construct some continuous but non-differentiable functions defined by quinary decimal, that are Kiesswetter-like functions. We discuss their properties, then investigate the Hausdorff dimensions of graphs of these functions and give a detailed proof.

Key words *Kiesswetter-like functions, continuous but non-differentiable, quinary decimal, iterated function system, inequality; Hausdorff dimension*

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1 Introduction

In 1966, Karl Kiesswetter[1] gave a simple example of everywhere continuous and nowhere differentiable function, namely, so-called Kiesswetter's function.

Let $x \in [0,1)$, its quartenary expansion:

$$x = \sum_{n=1}^{\infty} \frac{x_k}{4^k}, \qquad x_k \in \{0,1,2,3\}.$$

Then, Kiesswetter's function is defined as follows:

$$K(x) = \sum_{n=1}^{\infty} \frac{(-1)^{N_n} U(x_n)}{2^n},$$

where

$$U(x_n) = \begin{cases} x_n - 2, & x_n = 1,2,3, \\ 0, & x_n = 0, \end{cases}$$

and N_n is the number of $x_k = 0$ only if $k < n$.

This is a very noticeable fractal function (see [2],[3]), G.A.Edgar regarded it as one of the nineteen typical fractal articles that included Weierstrass's function, Von Koch curve and Cantor set etc, and introduced them in the collected papers: " Classics on Fractal"[4].

Since the continuous but non-differentiable function was proposed for the first time by Weierstrass in 1872, some other examples have been obtained. A question which naturally arises is how to construct the class of functions, and what is the relationship between classical fractals and modern fractals?

Recently, a general method by using the combining b-adic expansion with iterated function system to define the everywhere continuous and nowhere differentiable functions is found in papers [5] and [6]. However, their discussion is merely on the situation: $f(x) : [0,1] \to [0,1]$, while Kiesswetter's function showed that $K(x) : [0,1] \to [-1,1]$. On the base of [5] and [6], we extend the Kiesswetter's function on the situation of quinary decimal and analyse their properties in this paper. Finally, we have proved that the Hausdorff dimensions of graphs of these functions are equal to $2 - \dfrac{\log 3}{\log 5}$.

2 Construction of some Kiesswetter-like functions

Now, we construct some continuous but non-differentiable function defined by quinary decimal, which are similar to Kiesswetter's functions.

With the same idea of [5] and [6], consider the following five affine-mapping expressions $W_j (j = 0,1,2,3,4)$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = W_j = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & (-1)^{\alpha_j}\frac{1}{3} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} \frac{j}{5} \\ \frac{U(j)}{3} \end{pmatrix}, \qquad j = 0,1,2,3,4, \tag{1}$$

where $U(j)$ are some constants depending on j and $\alpha_j = 0$ or 1. Let $\eta = f(\xi)$ and $y = f(x)$, then (1) can be written as

$$\begin{cases} x = \dfrac{j}{5} + \dfrac{\xi}{5}, \\ f(x) = (-1)^{\alpha_j}\dfrac{1}{3}f(\xi) + \dfrac{U(j)}{3}, \end{cases} \qquad j = 0,1,2,3,4. \tag{2}$$

Suppose that $x \in [0,1]$, its quinary decimal expansion is

$$x = \sum_{k=1}^{\infty} \frac{x_k}{5^k}, \qquad x_k \in \{0,1,2,3,4\}.$$

We can obtain

$$j = x_1, \qquad \xi = \frac{x_2}{5} + \sum_{k=2}^{\infty} \frac{x_{k+1}}{5^k} = \frac{x_2}{5} + \frac{\lambda}{5}, \qquad x_2 \in \{0,1,2,3,4\}, \quad 0 \le \lambda \le 1, \tag{3}$$

and

$$f(x) = (-1)^{\alpha_{x_1}} \frac{1}{3} f(\xi) + \frac{U(x_1)}{3}. \tag{4}$$

From (4), using ξ instead of x, we have

$$f(\xi) = (-1)^{\alpha_{x_2}} \frac{1}{3} f(\lambda) + \frac{U(x_2)}{3}.$$

Hence,

$$f(x) = \frac{U(x_1)}{3} + \frac{(-1)^{\alpha_{x_1}} U(x_2)}{3^2} + \frac{(-1)^{\alpha_{x_1}+\alpha_{x_2}}}{3^2} f(\lambda).$$

Repeating the same iteration and so forth

$$f(x) = \sum_{n=1}^{m} \frac{(-1)^{\alpha_{x_1}+\cdots+\alpha_{x_{n-1}}} U(x_n)}{3^n} + \frac{(-1)^{\alpha_{x_1}+\cdots+\alpha_{x_m}}}{3^m} f(\tau), \qquad 0 \le \tau \le 1.$$

Let $m \to \infty$, suppose that $f(x)$ is bounded, then

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{\alpha_{x_1}+\cdots+\alpha_{x_{n-1}}} U(x_n)}{3^n}. \tag{5}$$

We note that there are the following two different quinary decimal representations for each rational point $x = \frac{m}{5^l}, m = 1, 2, \cdots, 5^l - 1, l = 1, 2, \cdots$

$$\sum_{n=1}^{l-1} \frac{x_n}{5^n} + \frac{x_l}{5^l} + \sum_{n=l+1}^{\infty} \frac{0}{5^n} \quad \text{and} \quad \sum_{n=1}^{l-1} \frac{x_n}{5^n} + \frac{x_l - 1}{5^l} + \sum_{n=l+1}^{\infty} \frac{4}{5^n}, \qquad x_1, \cdots, x_n \in \{0,1,2,3,4\}. \tag{6}$$

Therefore, to insure the function (5) is well defined, for the above two quinary decimal expansions(6), the corresponding value $f(\frac{m}{5^l})$ must be uniquely determined.

Theorem 1. *If $U(j)$ and α_j are satisfying one of following conditions:*

$$\begin{cases} U(j) = U_1(j) = \begin{cases} 2 - j, & j = 0,1,2,3, \\ 0, & j = 4, \end{cases} \\ \alpha(j) = \alpha_1(j) = \begin{cases} 1, & j = 4, \\ 0, & j \ne 4, \end{cases} \end{cases} \tag{7}$$

$$\begin{cases} U(j) = U_2(j) = \begin{cases} 0, & j = 0, \\ j - 2, & j = 1,2,3,4, \end{cases} \\ \alpha(j) = \alpha_2(j) = \begin{cases} 1, & j = 0, \\ 0, & j \ne 0, \end{cases} \end{cases} \tag{8}$$

$$\begin{cases} U(j)=U_3(j)=\begin{cases} 0, & j=0, \\ 2-j, & j=1,2,3,4, \end{cases} \\ \alpha(j)=\alpha_3(j)=\begin{cases} 1, & j=0, \\ 0, & j\neq 0, \end{cases} \end{cases} \tag{9}$$

$$\begin{cases} U(j)=U_4(j)=\begin{cases} j-2, & j=0,1,2,3, \\ 0, & j=4, \end{cases} \\ \alpha(j)=\alpha_4(j)=\begin{cases} 1, & j=4, \\ 0, & j\neq 4, \end{cases} \end{cases} \tag{10}$$

then values of function (5) *are independent of the representations* (6) *chosen of* $x=\dfrac{m}{5^l}$.

Proof. Corresponding to conditions (7)-(10), we obtain four functions:

$$f_1(x)=\sum_{n=1}^{\infty}\frac{(-1)^{\alpha_{1n}}U_1(x_n)}{3^n}, \tag{11}$$

$$f_2(x)=\sum_{n=1}^{\infty}\frac{(-1)^{\alpha_{2n}}U_2(x_n)}{3^n}, \tag{12}$$

$$f_3(x)=\sum_{n=1}^{\infty}\frac{(-1)^{\alpha_{3n}}U_3(x_n)}{3^n}, \tag{13}$$

$$f_4(x)=\sum_{n=1}^{\infty}\frac{(-1)^{\alpha_{4n}}U_4(x_n)}{3^n}, \tag{14}$$

where

$$\alpha_{1n}=\alpha_1(x_1)+\cdots+\alpha_1(x_{n-1}),$$

$$\alpha_{2n}=\alpha_2(x_1)+\cdots+\alpha_2(x_{n-1}),$$

$$\alpha_{3n}=\alpha_3(x_1)+\cdots+\alpha_3(x_{n-1}),$$

$$\alpha_{4n}=\alpha_4(x_1)+\cdots+\alpha_4(x_{n-1}).$$

We only verify that the value of function $f_1(x)$ is uniquely determined, for other functions $f_i(x)(i=2,3,4)$ are similar.

By the definition

$$f_1\Big(\sum_{n=1}^{l-1}\frac{x_n}{5^n}+\frac{x_l}{5^l}+\sum_{n=l+1}^{\infty}\frac{0}{5^n}\Big)=\sum_{n=1}^{l-1}\frac{(-1)^{\alpha_{1n}}U_1(x_n)}{3^n}+\frac{(-1)^{\alpha_{1l}}U_1(x_l)}{3^l}+\sum_{n=l+1}^{\infty}\frac{(-1)^{\alpha_{1(l+1)}}\cdot 2}{3^n},$$

$$f_1\Big(\sum_{n=1}^{l-1}\frac{x_n}{5^n}+\frac{x_l-1}{5^l}+\sum_{n=l+1}^{\infty}\frac{4}{5^n}\Big)=\sum_{n=1}^{l-1}\frac{(-1)^{\alpha_{1n}}U_1(x_n)}{3^n}+\frac{(-1)^{\alpha_{1l}}U_1(x_l-1)}{3^l}.$$

For the following four cases

$$(1)\ x_l=1,\quad U_1(x_l)=1,\quad U_1(x_l-1)=2,\quad \alpha_{1(l+1)}=\alpha_{1l},$$

$$(2)\ x_l=2,\quad U_1(x_l)=0,\quad U_1(x_l-1)=1,\quad \alpha_{1(l+1)}=\alpha_{1l},$$

$$(3)\ x_l=3,\quad U_1(x_l)=-1,\quad U_1(x_l-1)=0,\quad \alpha_{1(l+1)}=\alpha_{1l},$$

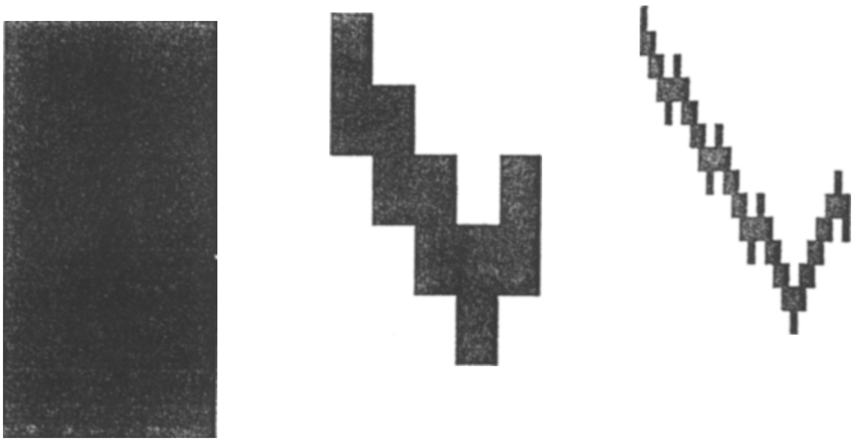
$$(4)\ x_l=4,\quad U_1(x_l)=0,\quad U_1(x_l-1)=-1,\quad \alpha_{1(l+1)}=\alpha_{1l}+1,$$

it is easy to see

$$f_1\Big(\sum_{n=1}^{l-1}\frac{x_n}{5^n}+\frac{x_l}{5^l}+\sum_{n=l+1}^{\infty}\frac{0}{5^n}\Big)=f_1\Big(\sum_{n=1}^{l-1}\frac{x_n}{5^n}+\frac{x_l-1}{5^l}+\sum_{n=l+1}^{\infty}\frac{4}{5^n}\Big),$$

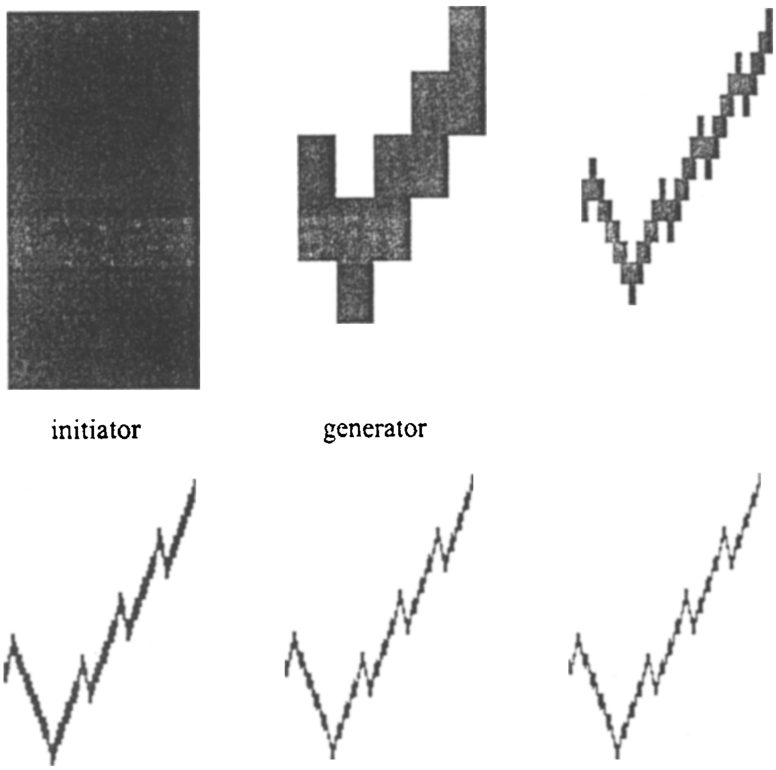
and this completes the proof of Theorem 1.

The graphs of $f_i(x)(i=1,2,3,4)$ are as follows

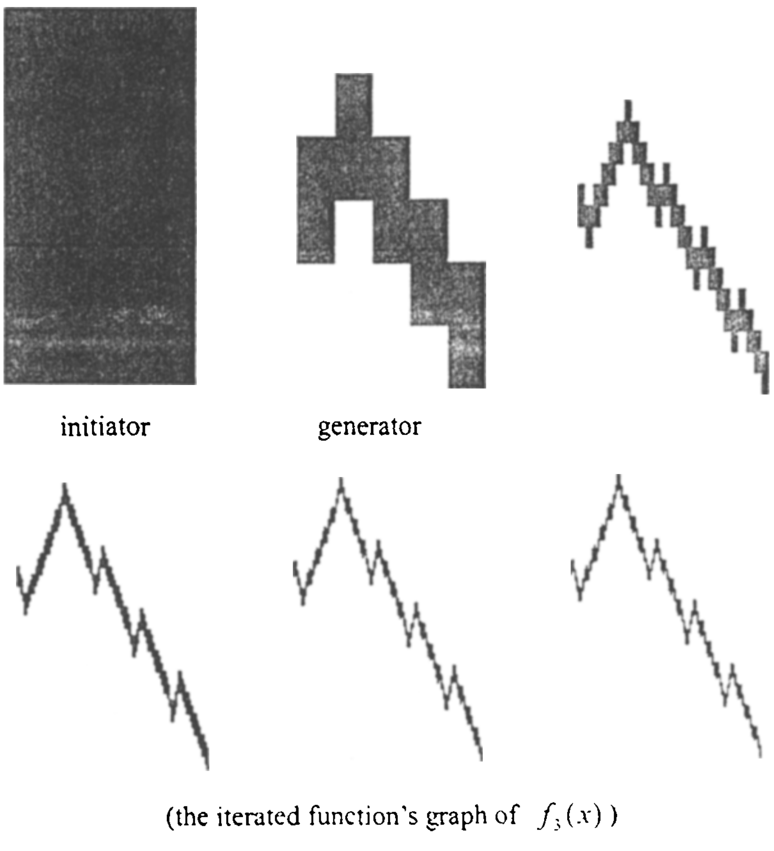


initiator generator

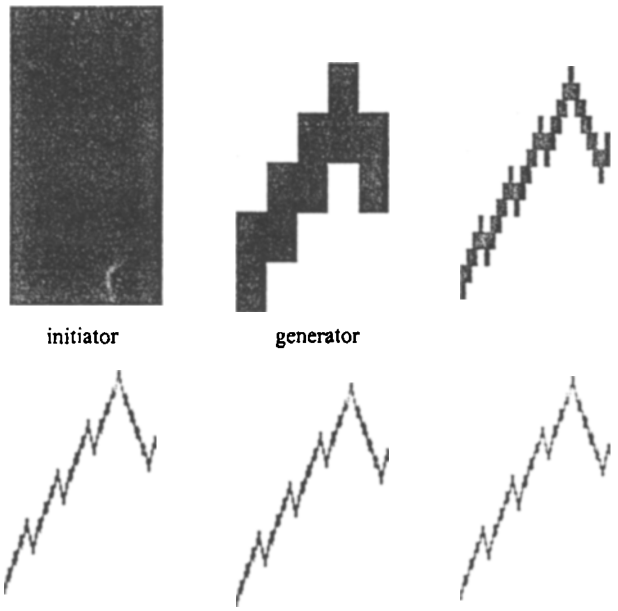
(the iterated function's graph of $f_1(x)$)



(the iterated function's graph of $f_2(x)$)



(the iterated function's graph of $f_3(x)$)



(the iterated function's graph of $f_4(x)$)

3 Smoothness Analysis

We investigate the Hölder continuity of functions $f_i(x), i = 1, 2, 3, 4$; this is an important characteristic of fractal function.

Theorem 2. *The function $f_i(x), i = 1, 2, 3, 4$, satisfy the Hölder condition with index $\alpha = \frac{\log 3}{\log 5}$, speaking exactly, there exist constants $c_1, c > 0$, such that*

$$c_1|x-y|^\alpha \le \max_{0\le x\le\xi\le\eta\le y} |f_i(\xi) - f_i(\eta)| \le c|x-y|^\alpha. \tag{15}$$

Proof. We only give the proof of $f_1(x)$. The proofs of $f_i(x), i = 2, 3, 4$, are similar.

Firstly, we prove that $f_1(x)$ satisfies the Hölder inequality:

$$|f_1(x) - f_1(y)| \le c|x-y|^\alpha, \tag{16}$$

where

$$x, y \in [0, 1), \qquad \alpha = \frac{\log 3}{\log 5},$$

where $c > 0$ is constant.

In fact, for any $0 \le x < y < 1$, there exists a positive integer l, such that

$$\frac{1}{5^{l+1}} < |x-y| < \frac{1}{5^l}.$$

There are two cases.

1. For one positive integer $m \in \{0,1,2,\cdots,5^l-1\}$, $\frac{m}{5^l} \le x < y < \frac{m+1}{5^l}$. That is

$$x = \frac{m}{5^l} + \sum_{n=l+1}^{\infty} \frac{x_n}{5^n}, \qquad y = \frac{m}{5^l} + \sum_{n=l+1}^{\infty} \frac{y_n}{5^n}, \qquad x_n, y_n \in \{0,1,2,3,4\}.$$

Thus, we have

$$\begin{aligned} |f_1(x) - f_1(y)| &\le \sum_{n=l+1}^{\infty} \left| \frac{(-1)^{\alpha_{1n}} U_1(x_n)}{3^n} - \frac{(-1)^{\beta_{1n}} U_1(y_n)}{3^n} \right| \\ &\le 3 \cdot \sum_{n=l+1}^{\infty} \frac{1}{3^n} = \frac{9}{2} \cdot 5^{-(l+1)\frac{\log 3}{\log 5}} \le \frac{9}{2} |x-y|^{\frac{\log 3}{\log 5}}. \end{aligned}$$

2. For the certain positive integer $m \in \{0,1,2,\cdots,5^l-1\}$,

$$\frac{m-1}{5^l} \le x < \frac{m}{5^l} < y < \frac{m+1}{5^l}.$$

That is

$$x = \frac{m-1}{5^l} + \sum_{n=l+1}^{\infty} \frac{x_n}{5^n}, \qquad y = \frac{m}{5^l} + \sum_{n=l+1}^{\infty} \frac{y_n}{5^n}.$$

We have

$$\left| f_1(\frac{m}{5^l}) - f_1(y) \right| \le \left| f_1(\frac{m}{5^l}) - \left[f_1(\frac{m}{5^l}) + \sum_{n=l+1}^{\infty} \frac{(-1)^{\beta_{1n}} U_1(y_n)}{3^n} \right] \right| \le \sum_{n=l+1}^{\infty} \frac{2}{3^n},$$

$$\left| f_1(\frac{m-1}{5^l}) - f_1(\frac{m}{5^l}) \right| \le \left| f_1(\frac{m-1}{5^l}) - \left[f_1(\frac{m-1}{5^l}) + \sum_{n=l+1}^{\infty} \frac{(-1)^{\alpha_{1n}} U_1(x_n)}{3^n} \right] \right| \le \sum_{n=l+1}^{\infty} \frac{2}{3^n},$$

$$\left| f_1(x) - f_1(\frac{m-1}{5^l}) \right| \le \left| \left[f_1(\frac{m-1}{5^l}) + \sum_{n=l+1}^{\infty} \frac{(-1)^{\alpha_{1n}} U_1(x_n)}{3^n} \right] - f_1(\frac{m-1}{5^l}) \right| \le \sum_{n=l+1}^{\infty} \frac{2}{3^n}.$$

Hence,

$$\begin{aligned} |f_1(x) - f_1(y)| &\le \left| f_1(x) - f_1(\frac{m-1}{5^l}) \right| + \left| f_1(\frac{m-1}{5^l}) - f_1(\frac{m}{5^l}) \right| + \left| f_1(\frac{m}{5^l}) - f_1(y) \right| \\ &\le 3 \cdot \sum_{n=l+1}^{\infty} \frac{2}{3^n} = 9 \cdot 5^{-(l+1)\frac{\log 3}{\log 5}} \le 9|x-y|^{\frac{\log 3}{\log 5}}. \end{aligned}$$

Consequently, the inequality (16) is valid.

Now, for any $x \in [0,1)$, we can get $y (x \neq y)$, such that

$$|x-y| \le \frac{1}{5^l}, \qquad |f_1(x) - f_1(y)| \ge c_1 |x-y|^{\frac{\log 3}{\log 5}}. \tag{17}$$

That is, the Hölder index $\alpha = \frac{\log 3}{\log 5}$ in (16) is exact.

In fact, there exists one positive integer $m \in \{0,1,2,\cdots,5^l-1\}$, such that

$$\frac{m}{5^l} \le x < \frac{m+1}{5^l}.$$

If $x = \frac{m}{5^l}$, we shall select $y = \frac{m+1}{5^l}$ (it satisfies the condition), because

$$|f_1(x) - f_1(y)| = \left|f_1(\frac{m}{5^l}) - f_1(\frac{m+1}{5^l})\right| \geq \sum_{n=l+1}^{\infty} \frac{1}{3^n} \geq \frac{1}{2}|x-y|^{\frac{\log 3}{\log 5}}.$$

Consequently, the inequality (17) is valid.

Now, we only need to consider the case of $\frac{m}{5^l} < x < \frac{m+1}{5^l}$. Since one of the two following inequalities:

$$\left|f_1(\frac{m}{5^l}) - f_1(x)\right| \geq \frac{1}{2}\left|f_1(\frac{m}{5^l}) - f_1(\frac{m+1}{5^l})\right| = \frac{1}{4 \cdot 3^l}$$

and

$$\left|f_1(x) - f_1(\frac{m+1}{5^l})\right| \geq \frac{1}{2}\left|f_1(\frac{m+1}{5^l}) - f_1(\frac{m}{5^l})\right| = \frac{1}{4 \cdot 3^l}$$

must be hold. We can select $y = \frac{m}{5^l}$ or $y = \frac{m+1}{5^l}$, such that $|x-y| \leq \frac{1}{5^l}$, and we have

$$|f_1(x) - f_1(y)| \geq \sum_{n=l+1}^{\infty} \frac{1}{3^n} \geq \frac{1}{2}|x-y|^{\frac{\log 3}{\log 5}}.$$

Consequently, the inequality (17) is still valid.

By inequalities (16) and (17), we infer that the inequality (15) is valid.

Corollary. *The $f_i(x)(i = 1,2,3,4)$ are continuous but non-differentiable.*

4 Hausdorff Dimension

It is known that it is difficult to obtain the Hausdorff dimension of general self-affine fractal[7], but as a particular, G.A.Edgar[8] has shown that the graph of Kiesswetter's function has Hausdorff dimension 3/2. However, the argument is complicated. The proof of next theorem will proceed following the technique in [5].

Theorem 3. *The Hausdorff dimensions of graphs of $f_i(x)(i = 1,2,3,4)$ are*

$$s = \dim_H(\text{graph} f_i) = 2 - \frac{\log 3}{\log 5}, \qquad i = 1,2,3,4.$$

Proof. We define that $E_0 = [0,1] \times [-1,1]$ as an initiator, then generator E_1 are five rectangles E_j with width 5^{-1} and height 3^{-1}, and each rectangle is contained in E_0. Through k-iteration of affine-mapping $W_j, j = 1,2,3,4,5, 5^k$ rectangles E_{jk} are obtained with width 5^{-k} and height 3^{-k}, (see the above iterated graphs), such that

$$\text{graph} f_i = (x, f_i(x)) = \bigcap_k E_{jk}, \qquad j = 0,1,2,3,4 \quad \text{and} \quad k \quad \text{is a positive integer.}$$

These rectangular collections $\{E_{jk}\}$ are the covering of the graphs of function $f_i(x), i = 1,2,3,4$. Every E_{jk} can be covered by $[(\frac{5}{3})^k]+1$ small squares S_k with length of edge 5^{-k}, ($[(\frac{5}{3})^k]$ indicates the integer part of $(\frac{5}{3})^k$).

Suppose that $\sqrt{2}\cdot 5^{-k} < \delta$, we have

$$\begin{aligned} H^s_\delta(\mathrm{graph} f_i) &= \inf_\delta\{\sum_i |\overline{U}_i|^s,\ \mathrm{diam}|\overline{U}_i| \le \delta\} \\ &\le \sum_{i=1}^{5^k}([(\frac{5}{3})^k]+1)(\sqrt{2}\cdot 5^{-k})^s \le 2^{\frac{s}{2}}\cdot 5^k(1+3^{-k}\cdot 5^k)5^{-ks} \le 2^{1+\frac{s}{2}}. \end{aligned}$$

Consequently,

$$\dim_H(\mathrm{graph} f_i) \le s. \tag{18}$$

Applying the mass distribution principle[7] to estimate its lower bound of the dimension, we define a measure μ on the rectangle E_0, such that

$$\begin{aligned} &\mu(E_0) = 1, \qquad \mu(E_{jk}) = 5^{-k}, \qquad && j = 0,1,2,3,4; \quad k = 1,2,3,\cdots, \\ &\mu(E_0 - \mathrm{graph} f_i) = 0, && i = 1,2,3,4. \end{aligned}$$

Then, μ limited in the graph $f_i, i = 1,2,3,4$, is a mass distribution.

Let S_k be squares with length 5^{-k}, if $S_k \cap E_{jk} = \emptyset$, then $\mu(S_k) = 0$; if $S_k \cap E_{jk} \ne \emptyset$, then

$$\frac{\mu(S_k)}{\mu(E_{jk})} = \frac{\mu(S_k\cap E_{jk})}{\mu(E_{jk})} \le \frac{\mathrm{area} S_k}{\mathrm{area} E_{jk}} = (\frac{3}{5})^k.$$

Consequently,

$$\mu(S_k) \le \frac{3^k}{5^{2k}} = 5^{-ks}.$$

For any given Borel set $\overline{U} \subset E_0$, let k be a positive integer, such that

$$\sqrt{2}\cdot 5^{-(k+1)} \le \mathrm{diam}\overline{U} \le \sqrt{2}\cdot 5^{-k}.$$

Then, there are at most 9 small squares S_k intersect $\overline{U}$, so we have

$$\mu(\overline{U}) \le 9\cdot\mu(S_k) \le 9\cdot 5^{-ks} = 9\cdot 5^s\cdot 5^{-s(k+1)} \le 2^{-\frac{s}{2}}\cdot 9\cdot 5^s\cdot(\mathrm{diam}\overline{U})^s.$$

Hence,

$$H^s_\delta(\mathrm{graph} f_i) \ge \frac{1}{9}\cdot 2^{\frac{s}{2}}\cdot 5^{-s}\mu(\mathrm{graph} f_i) > 0.$$

This implies

$$\dim_H(\mathrm{graph} f_i) \ge s. \tag{19}$$

By (18) and (19), we infer

$$\dim_H(\mathrm{graph} f_i) = s = 2 - \frac{\log 3}{\log 5}, \qquad i = 1,2,3,4.$$

The theorem 3 is proved.

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