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ASYMPTOTIC APPROXIMATION OF FUNCTIONS AND THEIR DERIVATIVES BY GENERALIZED BASKAKOV-SZÁZS-DURRMEYER OPERATORS

Ulrich Abel
(*University of Applied Sciences Giessen-Friedberg, Germany*)
Vijay Gupta
(*Netaji Subhas Institute of Technology, New Delhi, India*)
Mircea Ivan
(*Technical University of Cluj-Napoca, Romania*)

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Abstract

We present the complete asymptotic expansion for a generalization of the Baskakov-Szász-Durrmeyer operators and their derivatives.

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1 Introduction

For each $\gamma > 0$, let $W_\gamma(0,\infty)$ denote the space of all locally integrable functions f on $(0,\infty)$ satisfying $f(t) = O(e^{\gamma t})$ as $t \to +\infty$.

For parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}$ we define the summation-integral operators $\hat{V}_{n,\alpha,\beta}$ by

$$\hat{V}_{n,\alpha,\beta}(f;x) = n \sum_{\nu=\max\{0,-\beta\}}^{\infty} b_{n+\alpha,\nu}(x) \int_0^\infty s_{n,\nu+\beta}(t)\ f(t)\ \mathrm{d}t, \qquad x \in [0,\infty), \tag{1}$$

where

$$b_{n,\nu}(x) = \binom{n+\nu-1}{\nu} \frac{x^\nu}{(1+x)^{n+\nu}}$$

and

$$s_{n,\nu}(t) = e^{-nt} \frac{(nt)^\nu}{\Gamma(\nu+1)}$$

are the Baskakov and the Szász-Mirakjan basis functions, respectively.

Note that, for $f \in W_\gamma(0,\infty)$, the expressions $\hat{V}_{n,\alpha,\beta}(f;x)$ are defined for each integer $n > \gamma$, provided $n + \alpha > 0$. It is easily verified that the operators (1) are positive linear operators on the spaces $W_\gamma(0,\infty)$, for all n sufficiently large. We call these hybrid operators generalized Baskakov-Szász-Durrmeyer operators (abbreviated generalized BSD operators).

The special case $\alpha = 0$, $\beta = 0$ which should be regarded as ordinary BSD operators was studied by Gupta and Srivastava[13]. They proved, for $f \in W_\gamma(0,\infty)$ bounded on every finite subinterval of $(0,\infty)$ and admitting a derivative of order $r+2$, the Voronovskaja-type result

$$\lim_{n\to\infty} n\left(\hat{V}^{(r)}_{n,0,0}(f;x) - f^{(r)}(x)\right) = \frac{x(2+x)}{2} f^{(r+2)}(x) + (1 + r(x+1)) f^{(r+1)}(x) + \frac{r(r-1)}{2} f^{(r)}(x). \quad (2)$$

See also [15], where a Voronovskaja-type result for simultaneous approximation is derived for linear combinations of the operators $\hat{V}_{n,0,0}$.

The Voronovskaja-type result in the case $\alpha = 1$, $\beta = 0$

$$\lim_{n\to\infty} n\left(\hat{V}^{(r)}_{n,1,0}(f;x) - f^{(r)}(x)\right) = \frac{x(2+x)}{2} f^{(r+2)}(x) + (r+1)(1+x) f^{(r+1)}(x) + \frac{r(r+1)}{2} f^{(r)}(x). \quad (3)$$

was derived by Gupta and Srivastava[14, Theorem 3.1]. An inverse result for the operators $\hat{V}_{n,1,0}$ is presented in [9].

Furthermore, we mention that L_p-approximation by linear combinations of the operators $\hat{V}_{n,0,-1}$ (case $\alpha = 0$, $\beta = -1$) was studied by Agrawal and Mohammad[11]. To be precise, their definition[11, Eq. (1.1)] contains an additional term $(1+x)^{-n} f(0)$ assuring that the operators preserve constant functions.

Note that our definition (1) also includes the supposed more general setting

$$\hat{V}_{n,\alpha_1,\alpha_2,\beta_1,\beta_2}(f;x) = (n+\beta_1) \sum_{\nu=\max\{0,-\alpha_2,-\beta_2\}}^{\infty} b_{n+\alpha_1,\nu+\alpha_2}(x) \int_0^\infty s_{n+\beta_1,\nu+\beta_2}(t)\, f(t)\, dt$$

$(x \in (0,\infty))$ by a suited shifting of the parameters.

The purpose of this paper is the study of the local rate of convergence of the operators (1). We investigate their asymptotic behaviour also for simultaneous approximation. As main result we derive the complete asymptotic expansion for the operators $\hat{V}_{n,\alpha,\beta}(f;x)$ and their derivatives in the form

$$\hat{V}_{n,\alpha,\beta}(f;x) = f(x) + \sum_{k=1}^{\infty} c_k(f;x)\, n^{-k}, \qquad n \to \infty\,, \tag{4}$$

provided that $f \in W_\gamma(0,\infty)$ and f possesses derivatives of sufficiently high order at x ($x \in (0,\infty)$). Throughout the paper $n^{\overline{k}}$ resp. $n^{\underline{k}}$ denotes the rising factorial $n^{\overline{k}} = n(n+1)\cdots(n+k-1)$, $n^{\overline{0}} = 1$ resp. falling factorial $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$, $n^{\underline{0}} = 1$. Formula (4) means that, for all $q \in \mathbb{N}$,

$$\hat{V}_{n,\alpha,\beta}(f;x) = f(x) + \sum_{k=1}^{q} c_k(f;x)\, n^{-k} + o(n^{-q})$$

as $n \to \infty$.

We remark that in [1,3,2,5,8,7] the first author gave analogous results for the Meyer-König and Zeller operators, for the operators of Bleimann, Butzer and Hahn, the Bernstein-Kantorovich operators, the Bernstein-Durrmeyer operators, and the operators of K. Balázs and Szabados, respectively. Asymptotic expansions of multivariate operators can be found in [6,10].

2 Main Results

For $q \in \mathbb{N}$ and $x \in (0,\infty)$, let $K[q;x]$ be the class of all functions $f \in W_\gamma(0,\infty)$ which are q times differentiable at x. Throughout the paper let the numbers $Z_{\alpha,\beta}(s,k,j)$, for $0 \le j \le k \le s$, be given by

$$Z_{\alpha,\beta}(s,k,j) = \binom{s}{j} \sum_{m=k}^{s} (-1)^{s-m} \binom{s-j}{m-j} (\beta+m)^{\underline{j}} \sum_{i=0}^{k-j} \binom{m-j-i}{m-k} \left[{m-j \atop m-j-i} \right] \alpha^{k-j-i}. \tag{5}$$

The quantities $\left[{j \atop i} \right]$ in Eq. (5) denote the (signless) Stirling numbers of the first kind defined by

$$x^{\overline{j}} = \sum_{i=0}^{j} \left[{j \atop i} \right] x^i \qquad (j = 0,1,\cdots)\,. \tag{6}$$

(see, e.g., [12, Eq. (6.11)]). We mention that in the traditional notation of the Stirling numbers of the first kind we have $S_j^i = (-1)^{j-i} \left[{j \atop i} \right]$.

The following theorem is our main result.

Theorem 1. *Let $q \in \mathbb{N}$ and $x \in (0,\infty)$. For each function $f \in K[2q;x]$, the operators*

$\hat{V}_{n,\alpha,\beta}$ possess the asymptotic expansion

$$\hat{V}_{n,\alpha,\beta}(f;x) = f(x) + \sum_{k=1}^{q} c_{k,\alpha,\beta}(f;x)\, n^{-k} + o\left(n^{-q}\right), \qquad n \to \infty, \tag{7}$$

where the coefficients $c_{k,\alpha,\beta}(f;x)$ $(k = 1, 2, \cdots)$ are given by

$$c_{k,\alpha,\beta}(f;x) = \sum_{s=k}^{2k} \frac{f^{(s)}(x)}{s!} \sum_{j=0}^{k} x^{s-j} Z_{\alpha,\beta}(s,k,j) \tag{8}$$

and the $Z_{\alpha,\beta}(s,k,j)$ are as defined by Eq. (5). Moreover, if $f \in K[2(q+r);x]$, for $r \in \mathbb{N}_0$, the differentiated operators $\hat{V}^{(r)}_{n,\alpha,\beta}$ possess the asymptotic expansion

$$\hat{V}^{(r)}_{n,\alpha,\beta}(f;x) = f^{(r)}(x) + \sum_{k=1}^{q} c^{(r)}_{k,\alpha,\beta}(f;x)\, n^{-k} + o\left(n^{-q}\right), \qquad n \to \infty. \tag{9}$$

Remark 1. *It turns out that the coefficients in the asymptotic expansion of $\hat{V}^{(r)}_{n,\alpha,\beta}(f;x)$ are the r-th derivatives of the coefficients $c_{k,\alpha,\beta}(f;x)$ in the asymptotic expansion of $\hat{V}_{n,\alpha,\beta}(f;x)$, i.e., in Eq. (7) one can differentiate term by term.*

Remark 2. *For functions $f \in \bigcap_{q=1}^{\infty} K[q;x]$ and $r \in \mathbb{N}_0$, we have the complete asymptotic expansion*

$$\hat{V}^{(r)}_{n,\alpha,\beta}(f;x) \sim f^{(r)}(x) + \sum_{k=1}^{\infty} c^{(r)}_{k,\alpha,\beta}(f;x)\, n^{-k}, \qquad n \to \infty.$$

Remark 3. *For the convenience of the reader, we list the initial coefficients explicitly:*

$$c_{1,\alpha,\beta}(f;x) = (\alpha x + 1 + \beta)\, f'(x) + \frac{1}{2} x\,(2+x)\, f''(x),$$

$$\begin{aligned} c_{2,\alpha,\beta}(f;x) = {} & \frac{1}{2}\left(\alpha(1+\alpha)x^2 + 2\,\alpha\,(2+\beta)x + (1+\beta)(2+\beta)\right) f^{(2)}(x) \\ & + \frac{1}{6}\left((2+3\alpha)x^3 + 3(3+2\alpha+\beta)x^2 + 6(2+\beta)x\right) f^{(3)}(x) \\ & + \frac{1}{8}x^2\,(2+x)^2\, f^{(4)}(x), \end{aligned}$$

$$\begin{aligned} c_{3,\alpha,\beta}(f;x) = {} & \frac{1}{6}\left(\alpha^{\bar{3}} x^3 + 3\alpha^{\bar{2}}(3+\beta)x^2 + 3\alpha(3+\beta)^{\underline{2}}x + (3+\beta)^{\underline{3}}\right) f^{(3)}(x) \\ & + \frac{1}{12}\Big(\left(3+7\alpha+3\alpha^2\right)x^4 + 2\left(8+15\alpha+3\alpha^2+2\beta+3\alpha\beta\right)x^3 \\ & + 3(3+\beta)(4+4\alpha+\beta)x^2 + 6\,(3+\beta)^{\underline{2}}x\Big) f^{(4)}(x) \\ & + \frac{1}{24}\left((4+3\alpha)x^5 + (23+12\alpha+3\beta)x^4 + 12(4+\alpha+\beta)x^3 + 12(3+\beta)x^2\right) f^{(5)}(x) \\ & + \frac{1}{48}x^3\,(2+x)^3\, f^{(6)}(x). \end{aligned}$$

An immediate consequence of Theorem 1 is the following Voronovskaja-type formula.

Corollary 2. *Let $x \in (0,\infty)$. For each function $f \in K[2;x]$, the operators $\hat{V}_{n,\alpha,\beta}$ satisfy the asymptotic relation*

$$\lim_{n\to\infty} n\left(\hat{V}_{n,\alpha,\beta}(f;x) - f(x)\right) = (1+\beta+\alpha x)\, f'(x) + \frac{1}{2}x\,(2+x)\, f''(x). \qquad (10)$$

Moreover, if $f \in K[2r+2;x]$, for $r \in \mathbb{N}_0$, we have

$$\begin{aligned}\lim_{n\to\infty} n\left(\hat{V}^{(r)}_{n,\alpha,\beta}(f;x) - f^{(r)}(x)\right) &= \frac{1}{2}x\,(2+x)\, f^{(r+2)}(x) + ((r+\alpha)x + 1 + \beta + r)\, f^{(r+1)}(x) \\ &\quad + \frac{1}{2}r\,(r-1+2\alpha)\, f^{(r)}(x).\end{aligned}$$

In the special case $\alpha = \beta = 0$, we get back the result (2) by Gupta and Srivastava. Formula (2) is the special case $\alpha = 1$, $\beta = 0$.

In the special case $\alpha = 0$, the expression (5) obviously simplifies to

$$Z_{0,\beta}(s,k,j) = \binom{s}{j} \sum_{m=k}^{s} (-1)^{s-m} \binom{s-j}{m-j} (\beta+m)^{\underline{j}} \begin{bmatrix} m-j \\ m-k \end{bmatrix}.$$

In the special case $\alpha = 1$, the inner sum of Eq. (5) becomes

$$\begin{aligned}\sum_{i=0}^{k-j} \binom{m-j-i}{m-k} \begin{bmatrix} m-j \\ m-j-i \end{bmatrix} \alpha^{k-j-i} &= \sum_{i=0}^{k-j} \binom{m-j-i}{k-j-i} \begin{bmatrix} m-j \\ m-j-i \end{bmatrix} \\ &= \sum_{i=m-k}^{m-j} \binom{i}{m-k} \begin{bmatrix} m-j \\ i \end{bmatrix} = \begin{bmatrix} m-j+1 \\ m-k+1 \end{bmatrix},\end{aligned}$$

where we used [12, Eq. (6.16)]. Hence, Theorem 1 implies the following generalization of the result (2) for the operators $\hat{V}_{n,1,0}$.

Corollary 3. *In the case $\alpha = 1$ and $\beta = 0$ the assertion of Theorem 1 holds with*

$$Z_{1,0}(s,k,j) = \binom{s}{j} \sum_{m=k}^{s} (-1)^{s-m} \binom{s-j}{m-j} m^{\underline{j}} \begin{bmatrix} m+1-j \\ m+1-k \end{bmatrix}.$$

Remark 4. *Note that in the special case $\alpha = 0$ and $\beta = -1$ the right-hand side of Eq. (10) does not contain $f'(x)$. In this case the initial coefficients are as follows:*

$$\begin{aligned}c_{1,0,-1}(f;x) &= \frac{1}{2}x\,(2+x)\, f''(x), \\ c_{2,0,-1}(f;x) &= \frac{1}{3}\left(3x+3x^2+x^3\right) f^{(3)}(x) + \frac{1}{8}x^2\,(2+x)^2\, f^{(4)}(x), \\ c_{3,0,-1}(f;x) &= \frac{1}{4}x\,(2+x)\left(2+2x+x^2\right) f^{(4)}(x) \\ &\quad + \frac{1}{6}x^2\,(2+x)\left(3+3x+x^2\right) f^{(5)}(x) + \frac{1}{48}x^3\,(2+x)^3\, f^{(6)}(x).\end{aligned}$$

3 Auxiliary Results

In order to prove our main result we shall need the following lemmas. As usual, we denote the monomials by $e_m(x) = x^m$ $(m = 0, 1, 2, \cdots)$. Furthermore, for each real x, put $\psi_x(t) = t - x$.

Lemma 4. *For all $m = 0, 1, 2, \cdots$, and $x \in [0, \infty)$, we have the representation*

$$\begin{aligned}\hat{V}_{n,\alpha,\beta}(e_m; x) &= \frac{m!}{n^m}\sum_{j=0}^{m}\binom{n+\alpha+j-1}{j}\binom{m+\beta}{m-j}x^j\\ &\quad -n^{-m}\sum_{\nu=0}^{-\beta-1}\binom{n+\alpha+\nu-1}{\nu}(\nu+\beta+m)^{\underline{m}}\frac{x^\nu}{(1+x)^{n+\alpha+\nu}},\end{aligned}$$

where the second sum is to be read as zero if $\beta \geq 0$.

Lemma 5. *For all $m = 0, 1, 2, \cdots$, and $x \in (0, \infty)$, we have the complete asymptotic expansion*

$$\begin{aligned}\hat{V}_{n,\alpha,\beta}(e_m; x) &= \sum_{k=0}^{m} n^{-k}\sum_{j=0}^{k}\binom{m}{j}(\beta+m)^{\underline{j}}x^{m-j}\sum_{i=0}^{k-j}\left[\begin{matrix}m-j\\ m-j-i\end{matrix}\right]\binom{m-j-i}{m-k}\alpha^{k-j-i}\\ &\quad +O\left(\frac{n^{-\beta-1}}{(1+x)^{n+\alpha}}\right),\end{aligned}$$

as $n \to \infty$. The O-term can be replaced by zero if $\beta \geq 0$.

Lemma 6. *For each integer $q > 0$ and each $x \in (0, \infty)$, the central moments $\hat{V}_{n,\alpha,\beta}(\psi_x^s; x)$ $(s = 0, 1, 2, \cdots)$ satisfy the asymptotic relation*

$$\hat{V}_{n,\alpha,\beta}(\psi_x^s; x) = \sum_{k=\lfloor (s+1)/2 \rfloor}^{q} n^{-k}\sum_{s=k}^{2k}\frac{f^{(s)}(x)}{s!}\sum_{j=0}^{k} Z_{\alpha,\beta}(s,k,j)\,x^{s-j} + o\left(n^{-q}\right), \qquad n \to \infty.$$

Remark 5. *An immediate consequence of Lemma 6 is*

$$\hat{V}_{n,\alpha,\beta}(\psi_x^s; x) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right), \qquad n \to \infty.$$

In order to derive as our main result the complete asymptotic expansion of the operators $\hat{V}_{n,\alpha,\beta}$ we use the following general approximation theorem for positive linear operators due to Sikkema [17, Theorem 3] (cf. [18, Theorems 1 and 2]).

Lemma 7. *Let I be an interval. For $q \in \mathbb{N}$ and fixed $x \in I$, let $A_n : L_\infty(I) \to C(I)$ be a sequence of positive linear operators with the property*

$$A_n(\psi_x^s; x) = O(n^{-\lfloor (s+1)/2 \rfloor}) \qquad (n \to \infty) \qquad (s = 0, 1, \cdots, 2q+2).$$

Then, we have for each $f \in L_\infty(I)$ which is $2q$ times differentiable at x the asymptotic relation

$$A_n(f; x) = \sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}A_n(\psi_x^s; x) + o(n^{-q}), \qquad n \to \infty. \tag{11}$$

If, in addition, $f^{(2q+2)}(x)$ exists, the term $o(n^{-q})$ in (11) can be replaced by $O(n^{-(q+1)})$.

Lemma 8(Localization result). *Let $x > 0$ be given. Assume that $f \in W_\gamma(0,\infty)$ vanishes in a neighborhood of x. Then for each positive constant q there holds $\hat{V}_{n,\alpha,\beta}(f;x) = O(n^{-q})$ as $n \to \infty$.*

Lemma 9. *The derivatives of the operators $\hat{V}_{n,\alpha,\beta}$ can be expressed by*

$$\left(\frac{d}{dx}\right)^r \hat{V}_{n,\alpha,\beta}(f;x) = (n+\alpha)^{\overline{r}} \Delta_\beta^r \hat{V}_{n,\alpha+r,\beta}(f;x), \qquad r = 0,1,2,\cdots,$$

where Δ_β denotes the forward difference operator of step 1 with respect to β, defined by

$$\Delta_\beta \hat{V}_{n,\alpha,\beta} = \hat{V}_{n,\alpha,\beta+1} - \hat{V}_{n,\alpha,\beta}.$$

4 Proofs

Proof of Lemma 4. For $m = 0,1,2,\cdots$, there holds

$$n\int_0^\infty s_{n,\nu+\beta}(t)\; t^m \,\mathrm{d}t = \int_0^\infty e^{-t}\frac{t^{\nu+\beta}}{(\nu+\beta)!}\left(\frac{t}{n}\right)^m \mathrm{d}t = n^{-m}(\nu+\beta+m)^{\underline{m}}.$$

Therefore, we have

$$\hat{V}_{n,\alpha,\beta}(e_m;x) = n^{-m}\sum_{\nu=\max\{0,-\beta\}}^{\infty} b_{n+\alpha,\nu}(x)\;(\nu+\beta+m)^{\underline{m}} = S_1 - S_2,$$

say, where

$$S_2 = n^{-m}\sum_{\nu=0}^{-\beta-1}\binom{n+\alpha+\nu-1}{\nu}(\nu+\beta+m)^{\underline{m}}\frac{x^\nu}{(1+x)^{n+\alpha+\nu}}$$

and

$$\begin{aligned} S_1 &= n^{-m}\sum_{\nu=0}^{\infty}\binom{n+\alpha+\nu-1}{\nu}\frac{x^\nu}{(1+x)^{n+\alpha+\nu}}\;(\nu+\beta+m)^{\underline{m}} \\ &= \frac{(1+x)^{-(n+\alpha)}}{n^m y^\beta}\sum_{\nu=0}^{\infty}\binom{n+\alpha+\nu-1}{\nu}\left(\frac{\partial}{\partial y}\right)^m y^{\nu+\beta+m}\bigg|_{y=x/(1+x)} \\ &= \frac{(1+x)^{-(n+\alpha)}}{n^m}\,y^{-\beta}\left(\frac{\partial}{\partial y}\right)^m \frac{y^{\beta+m}}{(1-y)^{n+\alpha}}\bigg|_{y=x/(1+x)}. \end{aligned}$$

Using the Leibniz formula we obtain

$$\begin{aligned} S_1 &= \frac{(1+x)^{-(n+\alpha)}y^{-\beta}}{n^m}\sum_{j=0}^{m}\binom{m}{j}(\beta+m)^{\underline{m-j}}y^{\beta+j}\frac{(-1)^j(-n-\alpha)^{\underline{j}}}{(1-y)^{n+\alpha+j}}\bigg|_{y=x/(1+x)} \\ &= \frac{1}{n^m}\sum_{j=0}^{m}\binom{m}{j}(\beta+m)^{\underline{m-j}}(n+\alpha)^{\overline{j}}x^j. \end{aligned}$$

This proves Lemma 4.

Proof of Lemma 5 By Lemma 4, we have

$$\hat{V}_{n,\alpha,\beta}(e_m;x) = S_1 - S_2,$$

say. Since, for $\beta \le -1$, there holds $S_2 = O\left(\frac{n^{|\beta|-1-m}}{(1+x)^n}\right)$ $(n \to \infty)$, resp. $S_2 = 0$ $(\beta \in \mathbb{N}_0)$, we have only to consider the first term

$$S_1 = \frac{1}{n^m}\sum_{j=0}^{m}\binom{m}{j}(\beta+m)^{\underline{m-j}}(n+\alpha)^{\overline{j}}x^j.$$

Using

$$(n+\alpha)^{\overline{j}} = \sum_{i=0}^{j}\left[\begin{matrix} j \\ i \end{matrix}\right](n+\alpha)^i$$

and the binomial formula we obtain

$$\begin{aligned} S_1 &= \frac{1}{n^m}\sum_{j=0}^{m}\binom{m}{j}(\beta+m)^{\underline{m-j}}x^j\sum_{i=0}^{j}\left[\begin{matrix} j \\ i \end{matrix}\right]\sum_{k=0}^{i}\binom{i}{k}n^k\alpha^{i-k} \\ &= \sum_{k=0}^{m}\frac{1}{n^{m-k}}\sum_{j=k}^{m}\binom{m}{j}(\beta+m)^{\underline{m-j}}x^j\sum_{i=k}^{j}\left[\begin{matrix} j \\ i \end{matrix}\right]\binom{i}{k}\alpha^{i-k} \\ &= \sum_{k=0}^{m}n^{-k}\sum_{j=0}^{k}\binom{m}{j}(\beta+m)^{\underline{j}}x^{m-j}\sum_{i=0}^{k-j}\left[\begin{matrix} m-j \\ m-j-i \end{matrix}\right]\binom{m-j-i}{m-k}\alpha^{k-j-i} \end{aligned}$$

This proves Lemma 5.

Proof of Lemma 6. Since

$$\hat{V}_{n,\alpha,\beta}(\psi_x^s;x) = \sum_{m=0}^{s}\binom{s}{m}(-x)^{s-m}\hat{V}_{n,\alpha,\beta}(e_m;x).$$

Lemma 5 implies

$$\begin{aligned} &\hat{V}_{n,\alpha,\beta}(\psi_x^s;x) \\ =& \sum_{k=0}^{s}n^{-k}\sum_{m=k}^{s}(-1)^{s-m}\binom{s}{m}\sum_{j=0}^{k}\binom{m}{j}(\beta+m)^{\underline{j}}x^{s-j}\sum_{i=0}^{k-j}\left[\begin{matrix} m-j \\ m-j-i \end{matrix}\right]\binom{m-j-i}{m-k}\alpha^{k-j-i} \\ =& \sum_{k=0}^{s}n^{-k}\sum_{j=0}^{k}Z_{\alpha,\beta}(s,k,j)\,x^{s-j}, \end{aligned}$$

where the numbers $Z_{\alpha,\beta}(s,k,j)$ are as defined in Eq. (5).

In order to prove $\hat{V}_{n,\alpha,\beta}(\psi_x^s;x) = 0$, for $k < \lfloor (s+1)/2 \rfloor$, we show that $Z_{\alpha,\beta}(s,k,j) = 0$, for

$2k < s$. We have

$$\begin{aligned} Z_{\alpha,\beta}(s,k,j) &= \binom{s}{j} \sum_{m=k-j}^{s-j} (-1)^{s-j-m} \binom{s-j}{m} (\beta+m+j)^{\underline{j}} \sum_{i=0}^{k-j} \left[\begin{matrix} m \\ m-i \end{matrix} \right] \binom{m-i}{k-j-i} \alpha^{k-j-i} \\ &= \binom{s}{j} \sum_{i=0}^{k-j} \alpha^{k-j-i} \sum_{m=k-j}^{s-j} (-1)^{s-j-m} \binom{s-j}{m} (\beta+m+j)^{\underline{j}} \binom{m-i}{k-j-i} \left[\begin{matrix} m \\ m-i \end{matrix} \right] \end{aligned}$$

and it is sufficient to show that the inner sum vanishes when $2k < s$.

Recall that the Stirling numbers of first kind possess the representation

$$(-1)^i \left[\begin{matrix} m \\ m-i \end{matrix} \right] = \sum_{\mu=0}^{i} C_{i,i-\mu} \binom{m}{i+\mu}, \qquad i = 0, \cdots, m, \tag{12}$$

(see [16, p.151, Eq. (5)]). The coefficients $C_{i,k}$ are independent of m and satisfy certain partial difference equations ([16, p. 150]). Some closed expressions for $C_{i,k}$ can be found in [3, p. 113]. Taking advantage of representation (12) we obtain

$$\begin{aligned} &\sum_{m=k-j}^{s-j} (-1)^{s-j-m} \binom{s-j}{m} (\beta+m+j)^{\underline{j}} \binom{m-i}{k-j-i} \left[\begin{matrix} m \\ m-i \end{matrix} \right] \\ &= \sum_{\mu=0}^{i} \frac{(-1)^i C_{i,i-\mu} (s-j)^{\underline{k-j}}}{(i+\mu)!(k-j-i)!} \sum_{m=0}^{s-k} (-1)^{s-k-m} \binom{s-k}{m} (\beta+m+k)^{\underline{j}} (m+k-j-i)^{\underline{\mu}}. \end{aligned}$$

Inserting this into Eq. (5) yields

$$Z(s,k,j) = \binom{s}{j} \sum_{i=0}^{k-j} \alpha^{k-j-i} \sum_{m=0}^{s-k} (-1)^{s-k-m} \binom{s-k}{m} P_{\alpha,\beta}(k,j,i;r),$$

where

$$P_{\alpha,\beta}(k,j,i;r) = \sum_{\mu=0}^{i} \frac{(-1)^i C_{i,i-\mu} (s-j)^{\underline{k-j}}}{(i+\mu)!(k-j-i)!} (\beta+m+k)^{\underline{j}} (m+k-j-i)^{\underline{\mu}}$$

is a polynomial in the variable m of degree at most $j+i \le k$. Thus, the inner sum vanishes if $k < s-k$, i.e., if $2k < s$. This completes the proof of Lemma 6.

Proof of Lemma 8. Let $|f(t)| \le Me^{\gamma t}$ and assume that $f(t) = 0$ if $|t-x| \le \delta$, for some $\delta > 0$. Putting

$$K_{n,\alpha,\beta}(x,t) = n \sum_{\nu=\max\{0,-\beta\}}^{\infty} b_{n+\alpha,\nu}(x)\, s_{n,\nu+\beta}(t),$$

we obtain

$$\left|\hat{V}_{n,\alpha,\beta}(f;x)\right| \le M \int_{|t-x|\ge\delta} K_{n,\alpha,\beta}(x,t)\ e^{\gamma t}\ dt \le M\delta^{-2m} \int_0^{\infty} K_{n,\alpha,\beta}(x,t)\, \psi_x^{2m}(t)\ e^{\gamma t}\ dt,$$

for each integer $m > q$. Application of the Schwarz inequality yields

$$\left|\hat{V}_{n,\alpha,\beta}(f;x)\right| \le M\delta^{-2m} \left(\int_0^{\infty} K_{n,\alpha,\beta}(x,t)\, \psi_x^{4m}(t)\ dt \right)^{1/2} \left(\int_0^{\infty} K_{n,\alpha,\beta}(x,t)\ e^{2\gamma t}\ dt \right)^{1/2}.$$

By Lemma 6, the first term on the right-hand side is of order $O(n^{-m})$ as $n \to \infty$. By direct calculation, the second term appears to be equal to

$$\left(\sum_{\nu=\max\{0,-\beta\}}^{\infty} b_{n+\alpha,\nu}(x)\left(\frac{n}{n-2\gamma}\right)^{\nu+\beta+1}\right)^{1/2} \le \left(\frac{n}{n-2\gamma}\right)^{(\beta+1)/2}\left(1-\frac{2\gamma x}{n-2\gamma}\right)^{-(n+\alpha)/2}$$

and the observation that the last term tends to $e^{\gamma x}$ as $n \to \infty$ completes the proof.

Proof of Lemma 9. The identity

$$b'_{n+\alpha,k}(x) = (n+\alpha)\left(b_{n+\alpha+1,\nu-1}(x) - b_{n+\alpha+1,\nu}(x)\right), \qquad \nu = 0,1,2,\cdots,$$

with the convention $b_{n+\alpha+1,-1}(x) \equiv 0$ implies

$$\frac{\mathrm{d}}{\mathrm{d}x}\hat{V}_{n,\alpha,\beta}(f;x) = (n+\alpha)\left(\hat{V}_{n,\alpha+1,\beta+1}(f;x) - \hat{V}_{n,\alpha+1,\beta}(f;x)\right)$$

and Lemma 9 follows by mathematical induction.

Proof of Theorem 1. In view of Lemma 8, we can assume without loss of generality, that the function $f \in K[2q;x]$ is bounded on $(0,\infty)$. By Lemma 6 and application of Lemma 7, we have

$$\begin{aligned}\hat{V}_{n,\alpha,\beta}(f;x) &= \sum_{s=0}^{2q}\frac{f^{(s)}(x)}{s!}\hat{V}_{n,\alpha,\beta}(\psi_x^s;x) + o(n^{-q}) \\ &= \sum_{k=0}^{q} n^{-k}\sum_{s=k}^{2k}\frac{f^{(s)}(x)}{s!}\sum_{j=0}^{k} Z_{\alpha,\beta}(s,k,j)\,x^{s-j} + o(n^{-q})\end{aligned}$$

as $n \to \infty$, which proves Eq. (7).

Now we are going to prove Eq. (9). By Lemma 9, we have, for $f \in K[2(q+r);x]$,

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r}\hat{V}_{n,\alpha,\beta}(f;x) &= (n+\alpha)^{\overline{r}}\,\Delta_\beta^r\hat{V}_{n,\alpha+r,\beta}(f;x) \\ &= (n+\alpha)^{\overline{r}}\sum_{k=0}^{q+r} n^{-k}\sum_{s=k}^{2k} f^{(s)}(x)\sum_{j=0}^{k}\Delta_\beta^r Z_{\alpha+r,\beta}(s,k,j)\,x^{s-j} + o(n^{-q}).\end{aligned}$$

The obvious identity

$$\Delta_\beta^r(\beta+m)^{\underline{j}} = j^{\underline{r}}(\beta+m)^{\underline{j-r}}$$

reveals that

$$\Delta_\beta^r Z_{\alpha+r,\beta}(s,k,j) = 0 \qquad (k = 0,\cdots,r-1).$$

Thus, we conclude that the differentiated operators possess an asymptotic expansion of the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r}\hat{V}_{n,\alpha,\beta}(f;x) = \sum_{k=0}^{q} d_{k,\alpha,\beta}^{[r]}(f;x)\,n^{-k} + o(n^{-q}).$$

The terms $d_{k,\alpha,\beta}^{[r]}(f;x)$ are linear expressions of $f^{(s)}(x)$, for $s = k+r, \cdots, 2(k+r)$, where the coefficient of $f^{(s)}(x)$ is a polynomial in x of degree at most s. In order to prove Eq. (9) it is sufficient to show that $d_{k,\alpha,\beta}^{[r]}(f;x) = c_{k,\alpha,\beta}^{(r)}(f;x)$ is valid for all polynomial functions f.

Let $m \in \mathbb{N}$. By Lemma 4, there exists a positive constant c, such that

$$\hat{V}_{n,\alpha,\beta}(e_m;x) = \frac{1}{n^m}\sum_{j=0}^{m}\binom{m}{j}(\beta+m)^{\underline{j}}(n+\alpha)^{\overline{m-j}}x^{m-j} + O\left(e^{-cn}\right)$$

as $n \to \infty$. Applying Lemma 9, we obtain

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^r \hat{V}_{n,\alpha,\beta}(e_m;x) &= (n+\alpha)^{\overline{r}}\Delta_\beta^r \hat{V}_{n,\alpha+r,\beta}(e_m;x)\\ &= \frac{1}{n^m}\sum_{j=r}^{m}\binom{m}{j}j^{\underline{r}}(\beta+m)^{\underline{j-r}}(n+\alpha)^{\overline{r+m-j}}x^{m-j} + O\left(e^{-cn}\right)\\ &= \frac{1}{n^m}\sum_{j=0}^{m-r}\binom{m}{j}(\beta+m)^{\underline{j}}(n+\alpha)^{\overline{m-j}}(m-j)^{\underline{r}}x^{m-j-r} + O\left(e^{-cn}\right)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^r \frac{1}{n^m}\sum_{j=0}^{m-r}\binom{m}{j}(\beta+m)^{\underline{j}}(n+\alpha)^{\overline{m-j}}x^{m-j} + O\left(e^{-cn}\right)\end{aligned}$$

as $n \to \infty$. This implies $d_{k,\alpha,\beta}^{[r]}(e_m;x) = c_{k,\alpha,\beta}^{(r)}(e_m;x)$ which completes the proof of Theorem 1.

References

[1] Abel, U., The Moments for the Meyer-König and Zeller Operators, J. Approx. Theory, 82(1995), 352-361.

[2] Abel, U., On the Asymptotic Approximation with Operators of Bleimann, Butzer and Hahn, Indag. Math., New Ser., 7(1996), 1-9.

[3] Abel, U., The Complete Asymptotic Expansion for Meyer-König and Zeller operators, J. Math. Anal. Appl., 208(1997), 109-119.

[4] Abel, U., Asymptotic Approximation with Stancu Beta Operators, Rev. Anal. Numér. Theor. Approx., 27:1(1998), 5-13.

[5] Abel, U., Asymptotic Approximation with Kantorovich Polynomials, Approx. Theory Appl., (N.S.) 14:3(1998), 106-116.

[6] Abel, U., On the Asymptotic Approximation with Bivariate Operators of Bleimann, Butzer and Hahn, J. Approx. Theory, 97(1999), 181-198.

[7] Abel, U. and Della Vecchia, B., Asymptotic Approximation by the Operators of K. Balázs and Szabados, Acta Sci. Math. (Szeged), 66(2000), 137-145.

[8] Abel, U., Asymptotic Approximation by Bernstein-Durrmeyer Operators and Their Derivatives, Approx. Theory Appl. (N.S.), 16:2(2000), 1-12.

[9] Abel, U., Gupta, V. and Ivan, M., Inverse Theorem and Rate of Convergence for Baskakov-Szász-Durrmeyer Operators, Preprint.

[10] Abel, U. and Ivan, M., Asymptotic Expansion of the Multivariate Bernstein Polynomials on a Simplex, Approx. Theory Appl. (N.S.), 16:3(2000), 85-93.

[11] Agrawal, P. N. and Mohammad, A. J., On L_p-Approximation by a Linear Combination of a New Sequence of Linear Positive Operators, Turk. J. Math., 27(2003), 389-405.

[12] Graham, R. L., Knuth, D. E. and Patashnik, O., Concrete Mathematics", Addison-Wesley, 1994.

[13] Gupta, V. and Srivastava, G. S., Simultaneous Approximation by Baskakov-Szasz Type Operators, Bull. Math. Soc. Sci. Math. de Roumanie, 37:85(1993), 73-85.

[14] Gupta, V. and Srivastava, G. S., Convergence of Derivatives by Summation-Integral Type Operators, Revista Colombiana de Mate., 29:1(1995), 1-11.

[15] Gupta, V. and Srivastava, G. S., On Simultaneous Approximation by Combinations of Baskakov-Szasz Type Operators, Fasc. Math., 27(1997), 29-42.

[16] Jordan, C., "Calculus of Finite Differences", Chelsea, New York, 1965.

[17] Sikkema, P. C., On Some Linear Positive Operators, Indag. Math., 32(1970), 327-337.

[18] Sikkema, P. C., On the Asymptotic Approximation with Operators of Meyer-König and Zeller, Indag. Math., 32(1970), 428-440.

Ulrich Abel

Fachbereich MND

Fachhochschule Giessen–Friedberg

University of Applied Sciences

Wilhelm–Leuschner–Straße 13, 61169 Friedberg, Germany

e-mail: Ulrich.Abel@mnd.fh-friedberg.de

Vijay Gupta

School of Applied Sciences

Netaji Subhas Institute of Technology

Azad Hind Fauj Marg

Sector-3, Dwarka,

New Delhi-110045, India

e-mail: vijay@nsit.ac.in

Mircea Ivan

Department of Mathematics

Technical University of Cluj-Napoca

400020 Cluj-Napoca, Romania

e-mail: Mircea.Ivan@math.utcluj.ro