MULTIPLICITY OF POSITIVE SOLUTIONS FOR AN ELLIPTIC SYSTEM

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Abstract

By considering the properties of $\frac{f(t,u,v)}{u+v}$, $\frac{g(t,u,v)}{u+v}$, *we show the multiplicity of at least two positive solutions of the elliptic system.*

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1 Introduction

In this paper we consider the multiplicity of positive solutions for elliptic system of the form

$$\begin{aligned} &u''(t) + f(t,u(t),v(t)) = 0, \qquad t \in (0,1),\\ &v''(t) + g(t,u(t),v(t)) = 0, \qquad t \in (0,1),\\ &\alpha_1 u(0) - \beta_1 u'(0) = 0, \qquad \alpha_2 v(0) - \beta_2 v'(0) = 0,\\ &\gamma_1 u(1) + \delta_1 u'(1) = 0, \qquad \gamma_2 v(1) + \delta_2 v'(1) = 0, \end{aligned} \tag{1}$$

where $(u,v)\in C^2[0,1]\times C^2[0,1]$, $\alpha_i,\beta_i,\gamma_i,\delta_i\geqslant 0$, and $\rho_i=\gamma_i\beta_i+\alpha_i\gamma_i+\alpha_i\delta_i>0$, $i=1,2$.

The following condition will be assumed throughout:

$$f,g:[0,1]\times[0,\infty)\times[0,\infty)\to[0,\infty) \text{ are continuous.}$$

By a positive solution of (1) we understand a solution $(u,v)\in C^2[0,1]\times C^2[0,1]$ with $u\geqslant 0$, $v\geqslant 0$ and either $u\not\equiv 0$ or $v\not\equiv 0$. By the maximum principle and above conditions, each notrivial component of (u,v) is thus positive for $t\in(0,1)$.

In recent years it has been proved that for a single equation, superlinearity at one end and sublinearity at the other end (zero and infinity) can guarantee the existence of a positive solution on an annulus. See [4,5], for instance. On the other hand, as was shown in [3], superlinearity or sublinearity of the nonlinearity at both ends can imply the existence of at least two positive solutions.

Seeing such a fact, we can not but ask "whether or not we can obtain a similar conclusion in elliptic systems". Inspired by the above mentioned results, we attempt to establish a simple criterion for the multiplicity of positive solutions of (1).

We introduce the following notation

$$f_0 \equiv \lim_{(u,v)\to 0} \frac{f(t,u,v)}{u+v}, \qquad g_0 \equiv \lim_{(u,v)\to 0} \frac{g(t,u,v)}{u+v},$$

$$f_\infty \equiv \lim_{(u,v)\to \infty} \frac{f(t,u,v)}{u+v}, \qquad g_\infty \equiv \lim_{(u,v)\to \infty} \frac{g(t,u,v)}{u+v}.$$

By Fixed-Point Theorem of cone expansion/compression type, we prove the following:

Theorem 1.1. *Assume that the following cases hold:*

(H1) *either* $f_0=\infty$, *or* $g_0=\infty$,

(H2) *either* $f_\infty=\infty$, *or* $g_\infty=\infty$,

and

(H3) *there is a* $p>0$, *such that* $0\leqslant u$, $v\leqslant p$, $t\in[0,1]$, *imply*

$$f(t,u(t),v(t))\leqslant \lambda_1 p, \qquad g(t,u(t),v(t))\leqslant \lambda_2 p,$$

where

$$\lambda_i = \left(2\int_0^1 G_i(s,s)\mathrm{d}s\right)^{-1}, \qquad i=1,2.$$

Here $G_i(t,s)$, $i=1,2$, *is the Green's function*,

$$G_i(t,s) = \frac{1}{\rho_i}\begin{cases}(\gamma_i+\delta_i-\gamma_i t)(\beta_i+\alpha_i s), & 0\leqslant s\leqslant t\leqslant 1,\\ (\beta_i+\alpha_i t)(\gamma_i+\delta_i-\gamma_i s), & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

The problem (1) *has at least two positive solutions* (u_1,v_1), (u_2,v_2) *such that*

$$0<\|(u_1,v_1)\|<p<\|(u_2,v_2)\|.$$

where $\|(u,v)\|=\|u\|+\|v\|$, *and* $\|u\|=\max_{0\leqslant t\leqslant 1}|u(t)|$.

Theorem 1.2. *Assume that the following cases hold:*

(H4) $f_0=g_0=f_\infty=g_\infty=0$,

and

(H5) *there is a* $p>0$, *such that* $\sigma p\leqslant u+v\leqslant p$, $t\in[\frac{1}{4},\frac{3}{4}]$, *imply*

$$f(t,u(t),v(t)) \geqslant \eta_1 p, \qquad g(t,u(t),v(t)) \geqslant \eta_2 p,$$

where

$$\eta_i = \left[\max_{0\leqslant t\leqslant 1}\int_{\frac{1}{4}}^{\frac{3}{4}} G_i(t,s)\mathrm{d}s\right]^{-1}, \qquad i = 1,2$$

and

$$\sigma = \min\left\{\frac{\alpha_1 + 4\beta_1}{4(\alpha_1 + \beta_1)}, \frac{\alpha_2 + 4\beta_2}{4(\alpha_2 + \beta_2)}, \frac{\gamma_1 + 4\delta_1}{4(\gamma_1 + \delta_1)}, \frac{\gamma_2 + \delta_2}{4(\gamma_2 + \delta_2)}\right\}.$$

Then the problem (1) *has at least two positive solutions* (u_1,v_1), (u_2,v_2) *such that*

$$0 < \|(u_1,v_1)\| < p < \|(u_2,v_2)\|.$$

2 Preliminaries

It is easy to check (see also [5]) that for $i=1,2$,

$$G_i(t,s) \leqslant G_i(s,s), \quad (t,s) \in [0,1] \times [0,1],$$

$$G_i(t,s) \geqslant \sigma G_i(s,s), \quad (t,s) \in [\frac{1}{4},\frac{3}{4}] \times [0,1]. \tag{2}$$

On the other hand, (1) is equivalent to the system of integral equations

$$u(t) = \int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s, \qquad t \in (0,1),$$

$$v(t) = \int_0^1 G_2(t,s)g(s,u(s),v(s))\mathrm{d}s, \qquad t \in (0,1). \tag{3}$$

From now on we concentrate on (3). Indeed, any positive solution of (3) is a positive solution of (1).

Let

$$A(u,v)(t) = \int_0^1 G_1(t,s)f(s,u(s),v(s))\mathrm{d}s,$$

$$B(u,v)(t) = \int_0^1 G_2(t,s)g(s,u(s),v(s))\mathrm{d}s,$$

$$T(u,v)(t) = (A(u,v)(t),B(u,v)(t)).$$

Then (3) is equivalent to the fixed point equation $T(u,v)=(u,v)$ in the usual Banach space $X=C[0,1]\times C[0,1]$ with the norm $\|(u,v)\| = \|u\| + \|v\|$, where

$$\|u\| = \max_{0\leqslant t\leqslant 1}|u(t)|.$$

Let K be the cone in X defined by

$$K = \{(u,v) \in X: u,v \geqslant 0, \min_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4}}(u(t) + v(t)) \geqslant \sigma(\|u\| + \|v\|)\}.$$

Lemma 2.1. *$T: K \to K$, and T is completely continuous.*

Proof. To proof $T(K) \subset K$, choose $(u,v) \in K$. Then for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} A(u,v)(t) = \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s$$

$$\geqslant \sigma \int_0^1 G_1(s,s) f(s,u(s),v(s)) \mathrm{d}s$$

$$\geqslant \sigma \max_{0 \leqslant t \leqslant 1} \int_0^1 G_1(t,s) f(s,u(s),v(s)) \mathrm{d}s$$

$$= \sigma \| A(u,v) \| .$$

Similarly,

$$\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} B(u,v)(t) \geqslant \sigma \| B(u,v) \| .$$

Hence, $T(u,v)(t) \in K$. The complete continuity of T is obvious. See also [1].

Lemma 2.2[2]. *Let X be a Banach space and K a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and*

$$A: K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that either

(1) $Ax \geqslant x,\ \forall\ x \in K \cap \partial\Omega_1,\ Ax \leqslant x,\ \forall\ x \in K \cap \partial\Omega_2$;

or

(2) $Ax \geqslant x,\ \forall\ x \in K \cap \partial\Omega_2,\ Ax \leqslant x,\ \forall\ x \in K \cap \partial\Omega_1$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Proofs of Theorems

In this section we give the proofs of the theorems 1.1 and 1.2.

Proof of Theorem 1.1. Since (H1), $f_0 = \infty$, we may choose $0 < 2H_1 < p$ so that $f(t,u,v) \geqslant \eta(u+v)$, for $0 \leqslant u,v \leqslant H_1$, where the constant $\eta > 0$ satisfies

$$\sigma\eta \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},s) \mathrm{d}s \geqslant 1, \qquad \sigma\eta \int_{\frac{1}{4}}^{\frac{3}{4}} G_2(\frac{1}{2},s) \mathrm{d}s \geqslant 1.$$

Set $\Omega_1 = \{(u,v) \in K : \| (u,v) \| < H_1\}$. If $(u,v) \in \partial\Omega_1$, we have

$$A(u,v)(\frac{1}{2}) \geqslant \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},s) f(s,u(s),v(s)) \mathrm{d}s$$

$$\geqslant \eta \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},s)(u(s) + v(s)) \mathrm{d}s$$

$$\geqslant \sigma\eta(\|u\| + \|v\|)\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},s)\mathrm{d}s$$

$$\geqslant \|(u,v)\|.$$

Hence

$$\|T(u,v)\| \geqslant A(u,v)(\frac{1}{2}) \geqslant \|(u,v)\| \text{ for } (u,v) \in \partial\Omega_1.$$

An analogous estimate holds if $g_0=\infty$.

Since (H2), $f_\infty=\infty$, we can choose $H_3>2p$, so that $f(t,u,v)\geqslant\eta(u+v)$ for $u+v\geqslant H_3$, where the constant $\eta>0$ satisfies

$$\sigma\eta\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},s)\mathrm{d}s \geqslant 1, \qquad \sigma\eta\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(\frac{1}{2},s)\mathrm{d}s \geqslant 1.$$

Set $\Omega_3=\{(u,v)\in K: \|(u,v)\| < H_3\}$. If $(u,v)\in\partial\Omega_3$, we have

$$\min_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4}} (u(t) + v(t)) \geqslant \sigma(\|(u,v)\|).$$

Hence

$$A(u,v)(\frac{1}{2})\geqslant \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},s)f(s,u(s),v(s))\mathrm{d}s$$

$$\geqslant \eta\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},s)(u(s) + v(s))\mathrm{d}s$$

$$\geqslant \sigma\eta(\|u\| + \|v\|)\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(\frac{1}{2},s)\mathrm{d}s$$

$$\geqslant \|(u,v)\|.$$

Then

$$\|T(u,v)\| \geqslant A(u,v)(\frac{1}{2}) \geqslant \|(u,v)\|.$$

An analogous estimate holds if $g_\infty=\infty$.

If we further assume (H3) holds, we set $\Omega_2=\{(u,v)\in K: \|(u,v)\| < p\}$. If $(u,v)\in\partial\Omega_2$, we have

$$A(u,v)(t)\leqslant \int_0^1 G_1(s,s)f(s,u(s),v(s))\mathrm{d}s$$

$$\leqslant \lambda_1 p\int_0^1 G_1(s,s)\mathrm{d}s$$

$$\leqslant \frac{\|(u,v)\|}{2}.$$

Similarly, we have

$$B(u,v)(t) \leqslant \frac{\|(u,v)\|}{2},$$

Therefore

$$\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \leqslant \|(u,v)\|.$$

Now by Lemma 2.2, T has least one positive solutions (u_1,v_1) in $\bar{\Omega}_2\backslash\Omega_1$, and another positive solution (u_2,v_2) in $\bar{\Omega}_3\backslash\Omega_2$ with $0<\|(u_1,v_1)\|<p<\|(u_2,v_2)\|$.

Proof of Theorem 1.2. Since (H4), $f_0=g_0=0$, we may choose $0<2H_1<p$ so that

$$f(t,u,v) \leqslant \varepsilon(u+v), \qquad g(t,u,v) \leqslant \varepsilon(u+v)$$

for $0\leqslant u,v\leqslant H_1$, where the constant $\varepsilon>0$ satisfies

$$2\varepsilon\int_0^1 G_1(s,s)h(s)\mathrm{d}s \leqslant 1, \qquad 2\varepsilon\int_0^1 G_2(s,s)\mathrm{d}s \leqslant 1.$$

Set

$$\Omega_1 = \{(u,v) \in K: \|(u,v)\| < H_1\}.$$

If $(u,v)\in\partial\Omega_1$, we have

$$\begin{aligned} A(u,v)(t) &\leqslant \int_0^1 G_1(s,s)f(s,u(s),v(s))\mathrm{d}s \\ &\leqslant \varepsilon\int_0^1 G_1(s,s)(u(s)+v(s))\mathrm{d}s \\ &\leqslant \varepsilon(\|u\| + \|v\|)\int_0^1 G_1(s,s)\mathrm{d}s \\ &\leqslant \frac{\|(u,v)\|}{2}. \end{aligned}$$

Similarly,

$$B(u,v)(t) \leqslant \frac{\|(u,v)\|}{2}.$$

Hence,

$$\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \leqslant \|(u,v)\|$$

for $(u,v)\in\partial\Omega_1$.

Let us define two new functions $f^*(l)=\max\limits_{0\leqslant u+v\leqslant l} f(t,u,v)$, $g^*(l)=\max\limits_{0\leqslant u+v\leqslant l} g(t,u,v)$. Note that $f^*(l)$, $g^*(l)$ are nondecreasing in their respective arguments. Moreover, from $f_\infty=g_\infty=0$, it follows that

$$\lim_{l\to\infty}\frac{f^*(l)}{l} = 0, \qquad \lim_{l\to\infty}\frac{g^*(l)}{l} = 0.$$

Therefore, there is an $H_3>\max\{2p,\frac{H_1}{\sigma}\}$ such that $f^*(l)\leqslant\varepsilon l$, $g^*(l)\leqslant\varepsilon l$ for $l\geqslant H_3$, where the constant $\varepsilon>0$ satisfies

$$2\varepsilon\int_0^1 G_1(s,s)h(s)\mathrm{d}s \leqslant 1, \qquad 2\varepsilon\int_0^1 G_2(s,s)\mathrm{d}s \leqslant 1.$$

Set $\Omega_3=\{(u,v)\in K: \|(u,v)\|<H_3\}$. If $(u,v)\in\partial\Omega_3$, we have

$$\begin{aligned} A(u,v)(t) &\leqslant \int_0^1 G_1(s,s)f(s,u(s),v(s))\mathrm{d}s \\ &\leqslant \int_0^1 G_1(s,s)f^{*}(H_3)\mathrm{d}s \\ &\leqslant \varepsilon\int_0^1 G_1(s,s)(u(s)+v(s))\mathrm{d}s \\ &\leqslant \varepsilon H_3\int_0^1 G_1(s,s)\mathrm{d}s \\ &\leqslant \frac{H_3}{2}, \end{aligned}$$

hence $A(u,v)(t)\leqslant \|(u,v)\|/2$.

Similarly,

$$B(u,v)(t)\leqslant\frac{\|(u,v)\|}{2}.$$

Then

$$\|T(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \leqslant \|(u,v)\| \text{ for } (u,v)\in\partial\Omega_3.$$

Since (H5), we set $\Omega_2=\{(u,v)\in K: \|(u,v)\|<p\}$. If $(u,v)\in\partial\Omega_2$, $\min\limits_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4}}(u+v)\geqslant \sigma\|(u,v)\| = \sigma p$, we have

$$\begin{aligned} \|A(u,v)\| &\geqslant \max_{0\leqslant t\leqslant 1}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(t,s)f(s,u(s),v(s))\mathrm{d}s \\ &\geqslant \eta_1 p\max_{0\leqslant t\leqslant 1}\int_{\frac{1}{4}}^{\frac{3}{4}} G_2(t,s)\mathrm{d}s \\ &= p = \|(u,v)\|. \end{aligned}$$

Then

$$\|T(u,v)\| \geqslant \|A(u,v)(t)\| \geqslant \|(u,v)\|.$$

Similarly,

$$\|T(u,v)\| \geqslant \|B(u,v)(t)\| \geqslant \|(u,v)\|$$

for $(u,v)\in\partial\Omega_2$, $\min\limits_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4}}(u+v)\geqslant\sigma\|(u,v)\| = \sigma p$.

Applying Lemma 2.2, we obtain the multiplicity of positive solution (u,v) for (1).

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