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FRACTAL DIMENSION ESTIMATES FOR INVARIANT SETS OF NON-INJECTIVE MAPS

Qu Chengqin
(*Naval Institute of Computer Technology, China*)
Zhou Zuoling
(*Zhongshan University, China*)

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Abstract

For a special class of non-injective maps on Riemannian manifolds an upper bound for the fractal dimension of invariant set in terms of singular values of the tangent map and degree of non-injectivity is given.

Key Words *Riemannian manifold, degree of non-injectivity, fractal dimension*

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1 Introduction

In [1] Hausdorff dimension estimate for compact set $K\subset\mathbb{R}^n$ that is invariant under C^1-map φ is given, and this estimate is generalized in [6] to maps on Riemannian manifolds. Analogously fractal dimension estimate is derived in [2] for C^1-maps on Riemannian manifolds.

In practice it is well-known that the maps describing concrete physical or technical systems are often non-injective[1]. For such non-injective maps, the Douady-Oesterlé-type Hausdorff dimension estimates using the "degree of non-injectivity" are considered in [5] for the first time.

Let (M,g) be a smooth n-dimensional Riemannian manifold, let $U\subset M$ be an open subset and let $\varphi:U\to M$ be a map. One possibility to handle non-differentiable maps φ on an n-dimensional Riemannian manifold is to assume that

(C1) There are a finite number of compact subsets $K \subset M$, $K_1 \subset M, \cdots, K_k \subset M$ with $K = \bigcup_{i=1}^{k} K_i$, $\varphi(K_i) = K$ for all $i = 1, \cdots, k$.

(C2) Every partial map φ_i can be extended to a C^1-diffeomorphism on an open neighbourhood $U_i \subset M$ of $K_i (i = 1, \cdots, k)$.

In the following we will refer the map $\varphi_i := \varphi|_{K_i}$ in (C1) as partial maps.

To estimate the Hausdorff and fractal dimensions, the singular value function of this map is used. Consider a liear operator $L: E \to E'$ between two n-dimensional Euclidean spaces E and E' with the scalar products $\langle \cdot , \cdot \rangle_E$ and $\langle \cdot , \cdot \rangle_{E'}$, respectively, and L^* denote the adjoint operator of L, i. e. the unique linear operator $L^* : E' \to E$ satisfying $\langle Lx, y\rangle_{E'} = \langle x, L^* y\rangle_E$ for all $x \in E$ and $y \in E'$. The singular value of L are defined as the eigenvalues of the positive semi-definite operator $(L^* L)^{\frac{1}{2}}$. They are denoted by $\alpha_1(L) \geqslant \cdots \geqslant \alpha_n(L)$. For any $j \in \{0, 1, \cdots, n\}$ we introduce for L the notation

$$\omega_j(L) = \begin{cases} \alpha_1(L) \cdots \alpha_j(L), & \text{for } j > 0, \\ 1, & \text{for } j = 0. \end{cases}$$

For an arbitrary number $d \in (0, n]$, we define

$$\omega_d(L) = \omega_{[d]}(L)^{1-s} \omega_{[d]+1}(L)^s$$

the singular value function of order d, where $[d]$ is the integer part of $d, s = d - [d]$.

Let (M, g) be an n-dimensional Riemannian manifold, $U \subset M$ be an open subset and $\varphi: U \to M$ be a C^1 map, then the tangential map $d_p\varphi: T_pM \to T_{\varphi(p)}M$ is a linear operator. Let $\alpha_1(d_p\varphi) \geqslant \cdots \geqslant \alpha_n(d_p\varphi)$ denote the singular values of $d_p\varphi$. Put

$$\omega_{d,K}(\varphi) := \sup_{u \in K} \omega_d(d_u\varphi), \quad \rho_*(\varphi) := \inf_{u \in K} \alpha_n(d_u\varphi), \quad \rho^*(\varphi) := \sup_{u \in K} \alpha_n(d_u\varphi).$$

Theorem A[5]. *Suppose that (M, g) is a smooth n-dimensional Riemannian manifold, let $U \subset M$ be an open subset and $\varphi: U \to M$ be a map satisfying (C1) and (C2). If there exists a number d such that*

$$\max_{i=1,\cdots,k} \omega_{d,K}(\varphi_i^{-1}) < \frac{1}{k},$$

then the Hausdorff dimension of K *can be estimated by*

$$\dim_H(K) \leqslant d.$$

In this paper we give the upper bound of the fractal dimension using the technique in [2].

Theorem 1. 1. *Suppose that (M, g) is a smooth n-dimensional Riemannian manifold, let $U \subset M$ be an open subset and $\varphi: U \to M$ be a map satisfying (C1) and (C2). If there exists a number d such that*

$$\max_{i=1,\cdots,k} \omega_{n,K}(\varphi_i^{-1}) \frac{(\rho^*(\varphi_i^{-1}))^d}{(\rho_*(\varphi_i^{-1}))^n} < \frac{1}{k},$$

then the fractal dimension of K can be estimated by

$$\dim_F(K) \leqslant d.$$

2 Some Lemmas

Lemma 2. 1[2]. *For two liear operators $L: E \to E'$ and $L': E' \to E''$ between n-dimensional Euclidean spaces there holds the relation*

$$\omega_d(LL') \leqslant \omega_d(L)\omega_d(L') \quad \text{for} \quad d \in [0,n].$$

Let $\mathscr{E}$ be an ellipsoid with semiaxes $\alpha_1(\mathscr{E}) \geqslant \cdots \geqslant \alpha_1(\mathscr{E}) > 0$, and for any $j \in \{0,1,\cdots,n\}$ define

$$\omega_j(\mathscr{E}) = \begin{cases} \alpha_1(\mathscr{E})\cdots\alpha_j(\mathscr{E}), & \text{for } j > 0, \\ 1, & \text{for } j = 0. \end{cases}$$

For an arbitrary number $d \in (0,n]$, we define

$$\omega_d(\mathscr{E}) = \omega_{[d]}(\mathscr{E})^{1-s}\omega_{[d]+1}(\mathscr{E})^s.$$

Lemma 2. 2[2]. *Let $(E, \langle , \rangle_E)$ be an n-dimensional Euclidean space, $u_1, \cdots, u_n$ an orthonormal basis and*

$$\mathscr{E} = \{a_1u_1 + \cdots + a_nu_n \in E : (a_1, \cdots, a_n) \in \mathrm{R}^n, (\frac{a_1}{\alpha_1(\mathscr{E})})^2 + \cdots + (\frac{a_n}{\alpha_n(\mathscr{E})})^2 \leqslant 1\}$$

an ellipsoid. Then for any $\eta > 0$, the set $\mathscr{E} + B_\eta(0)$, is contained in the ellipsoid $\mathscr{E}' = (1 + \frac{\eta}{\alpha_n(\mathscr{E})})\mathscr{E}$, where $B_\eta(0)$ denotes the ball with radius η centered at the origin.

Let $N_\delta(A)$ be the smallest number of balls of radius δ needed to cover A, then we have.

Lemma 2. 3[2]. *Let $(E, \langle , \rangle_E)$ be an n-dimensional Euclidean space, $u_1, \cdots, u_n$ an orthonormal basis and*

$$\mathscr{E} = \{a_1u_1 + \cdots + a_nu_n \in E : (a_1, \cdots, a_n) \in \mathrm{R}^n, (\frac{a_1}{\alpha_1(\mathscr{E})})^2 + \cdots + (\frac{a_n}{\alpha_n(\mathscr{E})})^2 \leqslant 1\}$$

an ellipsoid and $0 < r < \alpha_n(\mathscr{E})$. Then the relation $N_{\sqrt{n}r}(\mathscr{E}) \leqslant \frac{2^n\omega_n(\varepsilon)}{r^n}$ holds.

Lemma 2. 4[5]. *Suppose that (M, g) is a smooth n-dimensional Riemannian manifold, let $U \subset M$ be an open subset and $\varphi: U \to M$ be a map satisfying (C1) and (C2), then for arbitrary natural number p, the map φ^p also satisfies (C1) and (C2).*

3 Proof of Theorem

Proof of Theorem 1. 1. Since every partial map φ_i can be extended to a C^1-diffeomorphism on an open neighbourhood $U_i \subset M$ of $K_i (i = 1, \cdots, k)$, by Lemma 2. 4 for every map

$\varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1}$ there is an open set $U_{i_1,\cdots,i_p}$ containing K such that $\varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1}:U_{i_1,\cdots,i_p}\to M$ is a C^1-map. Since $K\subset U_{i_1,\cdots,i_p}$ is compact, there exists an open set $V_{i_1,\cdots,i_p}\subset M$ containing K which itself lies inside a compact subset of $U_{i_1,\cdots,i_p}$. For convenience, put

$$\varphi:=\varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1},\qquad V:=V_{i_1,\cdots,i_p}.$$

Let $\eta\in(0,\rho_*(\varphi))$ be an arbitrary number and $r_1(\varphi)>0$ be small enough such that

$$\|\tau_{\varphi(v)}^{\varphi(u)}d_v\varphi\tau_u^v-d_u\varphi\|\leqslant\eta \tag{1}$$

for any $u,v\in V$ with $\rho(u,v)\leqslant r_1(\varphi)$ is satisfied, where $\|\cdot\|$ denotes the operator norm. By $\rho(\cdot)$ we mean the geodesic distance between the points of M and by τ_u^v we denote the isometry between the tangent spaces T_uM and T_vM defined by parallel transport. Let $\exp_u$: $T_uM\to M$ denote the exponential map at an arbitrary point $u\in M$. Since $\exp_u$ is a smooth map satisfying $\|d_{O_u}\exp_u\|=1$ for any point $u\in M$ we find a number $r_u>0$ such that $\|d_v\exp_u\|\leqslant 2$ for any $v\in B_{r_u}(O_u)$, where O_u denotes the origin of the tangent space T_uM. Since V is contained in a compact set there is a number $r_2>0$ such that

$$\|d_v\exp_u\|\leqslant 2 \tag{2}$$

for any $u\in V$ and any $v\in B_{r_2}(O_u)$.

Since K is compact, there is $\alpha(\varphi)>0$ such that $\sup\limits_{u\in V}\alpha_1(d_u\varphi)<\alpha(\varphi)$. Now we can find a number $r_0(\varphi)\leqslant\min\{r_1(\varphi),\frac{r_2}{2+\alpha(\varphi)+\eta}\}$ such that any ball $B_{r_0(\varphi)}(u)$ containing points of K is entirely contained in V. Let $r\in(0,r_0(\varphi))$ be fixed. Since K is compact there is a finite number of points $u_j\in V$ $(j=1,\cdots,N_r(K))$ such that $K=\bigcup\limits_{j=1}^{N_r(K)}B_r(u_j)\cap K$ and therefore

$$\varphi(K)=\bigcup_{j=1}^{N_r(K)}\varphi(B_r(u_j)\cap K).$$

The Taylor formula for the differentiable map φ guartantees the relation

$$\|\exp_{\varphi(u_j)}^{-1}\varphi(v)-d_{u_j}\varphi(\exp_{u_j}^{-1}v)\|=\sup_{w\in B_r(u_j)}\|\tau_{\varphi(w)}^{\varphi(u_j)}d_w\varphi\tau_{u_j}^w-d_{u_j}\varphi\|\cdot\|\exp_{u_j}^{-1}(w)\|$$

for every $v\in B_r(u_j)$. Combing with (1) we obtain

$$\varphi(B_r(u_j))\subset\exp_{\varphi(u_j)}(d_{u_j}\varphi(B_r(O_{u_j})))+B_{\eta r}(O_{\varphi(u_j)}).$$

Since $\mathscr{E}_j=d_{u_j}\varphi(B_1(O_{u_j}))$ is an ellipsoid in $E=T_{\varphi(u_j)}M$, by Lemma 2.2 we have

$$\varphi(B_r(u_j))\subset\exp_{\varphi(u_j)}(r(\mathscr{E}_j+B_\eta(O_{\varphi(u_j)})))\subset\exp_{\varphi(u_j)}(r\mathscr{E}'_j),$$

where $\mathscr{E}'_j=(1+\frac{\eta}{(\alpha_n(\mathscr{E})})\mathscr{E}_j$. Put

$$\sigma(\varphi):=\sqrt{n}\,\rho_*(\varphi),$$

we have

$$N_{\sigma(\varphi)r}(\varphi(K))\leqslant N_r(K)\max_{j=1,\cdots,N_r(K)}N_{\sigma(\varphi)r}(\exp_{\varphi(u_j)}(r\mathscr{E}'_j)).$$

For any $v\in M$, if $B_{\sigma r}(v)\cap\exp_{\varphi(u_j)}(r\mathscr{E}'_j)\neq\varnothing$, then

$$B_{\sigma r}(v) \subset B_{(2+\epsilon_1(\mathscr{E}'_j))r}(\varphi(u_j)) \subset B_{r_2}(\varphi(u_j)).$$

Using (2) we have

$$B_{\sigma(\varphi)r}(v) \supset \exp_{\varphi(u_j)}(B_{\frac{1}{2}\sigma(\varphi)r}(\exp^{-1}_{\varphi(u_j)}v)),$$

which means

$$N_{\sigma(\varphi)r}(\exp_{\varphi(u_j)}(r\mathscr{E}'_j)) \leqslant N_{\frac{1}{2}\sigma(\varphi)r}(r\mathscr{E}'_j).$$

Since $\rho_*(\varphi) \leqslant \alpha_n(d_{u_j}\varphi) = \alpha_n(\mathscr{E}_j) \leqslant \alpha_n(\mathscr{E}'_j)$, by Lemma 2.2 we have

$$N_{\frac{1}{2}\sigma(\varphi)r}(r\mathscr{E}'_j) \leqslant \frac{2^n\omega_n(r\mathscr{E}'_j)}{(\frac{1}{2}r\rho_*(\varphi))^n} \leqslant \frac{4^n(1+\frac{\eta}{\rho_*(\varphi)})^n\omega_n(\mathscr{E}_j)}{\rho^n_*(\varphi)} \leqslant \frac{8^n\omega_n(d_{u_j}\varphi)}{\rho^n_*(\varphi)}.$$

Therefore

$$N_{\sigma(\varphi)r}(\varphi(K)) \leqslant N_r(K)\,\frac{8^n\omega_n(d_{u_j}\varphi)}{\rho^n_*(\varphi)}. \tag{3}$$

Now denote $\max\limits_{i=1,\cdots,k}\omega_{n,K}(\varphi_i^{-1})\frac{(\rho^*(\varphi_i^{-1}))^d}{(\rho_*(\varphi_i^{-1}))^n}$ by ν, then $\nu \leqslant \frac{1}{k}$, this implies that there is a number p_0 such that $(k\nu)^p < (8^n n^{\frac{d}{2}})^{-1}$ for any $p > p_0$, and by Lemma 2.1 for any $i_1,\cdots,i_p \in \{1,\cdots,k\}$ we have

$$\omega_{n,K}(\varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1})\frac{(\max\limits_{i_1,\cdots,i_p}\rho^*(\varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1}))^d}{(\rho_*(\varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1}))^n} \leqslant \nu^p < \frac{(8^n n^{\frac{d}{2}})^{-1}}{k^p}. \tag{4}$$

Set $K_{i_1\cdots i_p} := \varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1}(K)$, by Lemma 2.4, we have

$$K = \bigcup_{i_1,\cdots,i_p} K_{i_1\cdots i_p}. \tag{5}$$

On the other hand, take $\varphi = \varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1}$, from (3) and (4), we have

$$N_{\sigma(\varphi)r}(\varphi(K)) \leqslant N_r(K)n^{-\frac{d}{2}}k^{-p}(\max_{i_1,\cdots,i_p}(\rho^*(\varphi_{i_1}^{-1}\circ\cdots\circ_{i_p}^{-1}))^{-d}. \tag{6}$$

Take

$$r_0 = \min_{i_1,\cdots,i_p}\{r_0(\varphi) : \varphi = \varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1}\},$$

$$\sigma = \max_{i_1,\cdots,i_p}\{\sigma(\varphi) : \varphi = \varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1}\}.$$

For any $r \in (0, r_0)$, by (5) and (6) we have

$$\mu(K,d,\sigma r) \leqslant N_r(K)\left(\frac{\max\limits_{i_1,\cdots,i_p}\rho_*(\varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1})}{\max\limits_{i_1,\cdots,i_p}\rho^*(\varphi_{i_1}^{-1}\circ\cdots\circ\varphi_{i_p}^{-1})}\right)^d r^d \leqslant \mu(K,d,r). \tag{7}$$

From (4) we have $\sigma < 1$. Therefore, for any $\varepsilon \in (0, r_0)$ we can find a number $l \in \mathrm{N}$ such that $\sigma^{l+1}r_0 \leqslant \varepsilon < \sigma^l r_0$, combining with (7) we have

$$\mu(K,d,\varepsilon) \leqslant \mu(K,d,\sigma^{-l}\varepsilon) \leqslant N_{\sigma r_0}(K)r_0^d.$$

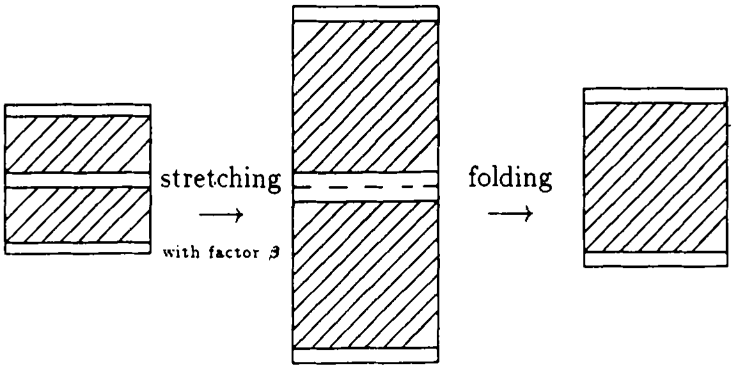
Therefore $\dim_F(K) \leqslant d$, which completes the proof of Theorem 1.1.

4 An Application

Example 4.1. Let $\varphi:(0-\varepsilon,1+\varepsilon)\times(0-\varepsilon,1+\varepsilon)\to \mathrm{R}^2$ be defined by

$$\varphi(u,v)=\begin{cases}(u,\beta v-\dfrac{1-\beta}{2}), & v\leqslant\dfrac{1}{2},\\[2ex] (u,\dfrac{1+\beta}{2}-\beta v), & v>\dfrac{1}{2},\end{cases}$$

with parameters $\varepsilon>0$ and $\beta>2$. This map when applied to the unit squere $[0,1]\times[0,1]$ first stretches the square in v-direction with factor β and then folds it.



The set $K=\bigcap_{i=1}^{\infty}\varphi^{-i}([0,1]\times[0,1])$ is a compact invariant set. Let

$$K_1=\{(u,v)\in K:u\leqslant\frac{1}{2}\}$$

and

$$K_2=\{(u,v)\in K:u\geqslant\frac{1}{2}\}.$$

Since the partial maps are affine-linear, the properties (C1) and (C2) are satisfied, and we have $\alpha_1(d_u\varphi_i^{-1})=1$, $\alpha_2(d_u\varphi_i^{-1})=\frac{1}{\beta}$, thus

$$\max_{i=1,2}\omega_{2,K}(\varphi_i^{-1})\frac{(\rho^*(\varphi_i^{-1}))^d}{(\rho_*(\varphi_i^{-1}))^2}=\begin{cases}1, & \text{for } 0\leqslant d\leqslant 1,\\ \beta^{1-d}, & \text{for } 1<d\leqslant 2.\end{cases}$$

If $d>1+\frac{\log 2}{\log\beta}$, we have $\beta^{1-d}<\frac{1}{2}$, by Theorem 1.1 we have $\dim_F(K)\leqslant d$, thus $\dim_F(K)\leqslant 1+\frac{\log 2}{\log\beta}$. On the other hand, by [5] we have $\dim_H(K)=1+\frac{\log 2}{\log\beta}$. Thus $\dim_F(K)=1+\frac{\log 2}{\log\beta}$, which means that the dimension estimate of Theorem 1.1 is sharp.

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Qu Chengqin
Naval Institute of Computer Technology
Beijing 100841
P. R. China

Zhou Zuoling
Lingnan College
Zhongshan University
Guangzhou 510275
P. R. China