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A RANDOM FIXED POINT ITERATION FOR THREE RANDOM OPERATORS ON UNIFORMLY CONVEX BANACH SPACES

Binayak S. Choudhury

(Deemend University, India)

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Abstract

In the present paper we introduce a random iteration scheme for three random operators defined on a closed and convex subset of a uniformly convex Banach space and prove its convergence to a common fixed point of three random operators. The result is also an extension of a known theorem in the corresponding non-random case.

Key Words *random iteration, fixed point, uniformly convex Banach space*

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1 Introduction

There have been efforts to construct general fixed point iterations for different types of nonlinear operators in linear spaces. Ishikawa iteration[9] is one such general iteration which and variations of which have successfully been applied to a number of cases of non-linear operators [1],[2],[3] and [10].

Random iteration schemes have been defined and their convergences to random fixed points or to solutions of random operator equations have been considered in [4],[5] and [6]. In particular random Ishikawa iteration scheme has been defined in [4].

In this paper we introduce a random iteration scheme involving three random operators and consider its convergence to a common random fixed point of the three operators. These operators are defined on closed and convex subsets of a uniformly convex separable

Banach space. This iteration is based on the same idea as that of Random Ishikawa Iteration[4]. The corresponding result in the non-random case is a generalization of a theorem in [10].

The following concepts are essential for our discussion in this paper.

Unless otherwise stated, throughout this paper (Ω, Σ) stands for the measurable space, B denotes a uniformly convex separable Banach space and C denotes a non-empty, closed and convex subset of B.

Any function $f:\Omega\to C$ is said to be measurable if $f^{-1}(C\cap E)\in\Sigma$ for all Borel subset E of B.

$T:\Omega\times C\to C$ is said to be a random operator if $T(\cdot, x):\Omega\to C$ is measurable for all $x \in C$.

A random operator $T:\Omega\times C\to C$ is said to be continuous if $T(t,\cdot):C\to C$ is continuous for all $t\in\Omega$.

Definition 1.1. Random Iteration:

Let $R,S,T:\ \Omega\times C\to C$ where C is a non-empty convex subset of a separable uniformly convex Banach space B. Starting with an arbitrary measurable function

$$g_0:\ \Omega\to C, \tag{1.1}$$

we define a sequence of functions $\{g_n\}$ as in the following:

$$g_{n+1}(t) = \alpha_n R(t,g_n(t)) + \beta_n S(t,g_n(t)) + \gamma_n T(t,h_n(t)), \tag{1.2}$$

where

$$h_n(t) = \alpha'_n R(t,g_n(t)) + \beta'_n S(t,g_n(t)) + \gamma'_n T(t,g_n(t)), \tag{1.3}$$

$$0 < a \leqslant \alpha_n,\ \beta_n,\gamma_n \leqslant b < 1 \text{ (a and } b \text{ are given)}, \tag{1.4}$$

$$0 \leqslant \alpha'_n,\ \beta'_n,\ \gamma'_n \leqslant 1, \tag{1.5}$$

$$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1 \tag{1.6}$$

and

$$0 \leqslant \gamma'_n \leqslant M_1 < 1 \qquad (M_1 \text{ is given}). \tag{1.7}$$

The following result was proved in [7]. We state the result in the following lemma.

Lemma 1[7]. *If $\{x_n\}$ and $\{y_n\}$ are sequences in the closed unit ball of a uniformly convex Banach space B and if $z_n=(1-\alpha_n)x_n+\alpha_n y_n$ satisfies*

$$\lim_{n\to\infty} \| z_n \| = 1, \tag{1.8}$$

where

$$0 < a \leqslant \alpha_n \leqslant b < 1, \tag{1.9}$$

then

$$\lim_{n\to\infty} \| x_n - y_n \| = 0. \tag{1.10}$$

A quasi non-expansive mapping[10] is a mapping $A:C\to C$ which has at least one fixed point and further

$$Ap = p \text{ implies } \| Ax - p \| \leqslant \| x - p \| \text{ for all } x \in C. \tag{1.11}$$

We denote by $F(A)$ the set of all fixed points of a mapping A. The following result is quoted as a lemma.

Lemma 1. 2[7]. *Let C be a non-empty closed convex subset of a uniformly convex Banach space and $T:C\to C$ be a quasi-nonexpansive mapping. Then*

$$F(T) = \{x : x \in C \text{ and } Tx = x\}$$

is non-empty closed and convex on which T is continuous.

The lemma was actually proved for strictly convex spaces. Uniformly convex spaces being strictly convex, the lemma also holds for the former. We have stated it for uniformly convex Banach spaces.

For three mappings $A_1, A_2, A_3 : C\to C$, we write $F = F(A_1)\cap F(A_2)\cap F(A_3)$, that is, F is precisely the set of all common fixed points of A_1, A_2 and A_3.

Condition-* Three mappings A_1, A_2, A_3: $C\to C$ satisfies 'condition-*' if $F=F(A_1)\cap F(A_2)\cap F(A_3)$ is non-empty and there exists a non-decreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for $r\in(0,\infty)$ such that

$$\| (1-\lambda)A_1x + \lambda A_2x - A_3y \| \geqslant f(d(x,F)), \tag{1.12}$$

where

$$x \in C,\ y = \alpha A_1x + \beta A_2x + \gamma A_3x, \tag{1.13}$$

$$d(x,F) = \inf\{ \| x - z \| : z \in F\}, \tag{1.14}$$

$$0 \leqslant \gamma \leqslant M_2 < 1 (M_2 \text{ is given}), \tag{1.15}$$

and

$$\alpha + \beta + \gamma = 1. \tag{1.16}$$

The following condition was introduced in [10].

Condition-A[10]. A mapping $T:C\to C\subset B$ with $F(T)\neq\varnothing$ is said to satisfy 'condition-A' if there exists $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(0)>0$ for all $r\in(0,\infty)$ such that

$$\| x - Ty \| \geqslant f(d(,F)) \text{ for all } x \in C, \tag{1.17}$$

where

$$y = (1-t)x + tTx, \tag{1.18}$$

and

$$0 \leqslant t \leqslant M_3 < 1 \ (M_3 \text{ is given}). \tag{1.19}$$

It may be noted that if we put $A_1=A_2=I$ (identity mapping) then 'condition-*' reduces to 'condition-A'. It may be noted that 'condition-A' in [10] is a further generalization of 'condition-I' in [11].

2 Main Results

In this section we prove that under certain conditions the random iteration in definition 1.1 converges to a common random fixed point of the three random operators R, S and T.

Theorem 2.1. *Let $R,S,T:\Omega\times C\to C$ be three continuous random operators defined on a non-empty closed convex subset C of a separable uniformly convex Banach space B such that following conditions are satisfied*

(a) *for all $t\in\Omega$, $R(t,\cdot)$, $S(t,\cdot)$, $T(t,\cdot)$: $C\to C$ are quasi non-expansive.*

(b) *for all $t\in\Omega$, the set $F(t)$ of common fixed points for $R(t,\cdot)$, $S(t,\cdot)$ and $T(t,\cdot)$ is non-empty.*

(c) *for all $t\in\Omega$, $R(t,\cdot)$, $S(t,\cdot)$ and $T(t,\cdot)$ satisfy 'condition-*'.*

Then the random iteration in definition 1.1 converges to a common random fixed point of R, S and T.

proof. The construction (1.1)—(1.6) along with the fact that C is nonempty convex shows that $g_n(t)\in\Omega$ for all $t\in\Omega$ and $n=0,1,2$.

Since B is separable, for any continuous random operator $A:\Omega\times C\to C$ and any measurable function $f:\Omega\to C$, the function $h(t)=A(t,f(t))$ is a measurable function[8]. Since g_0 is measurable and C is convex, it follows that $\{g_n\}$ constructed in the random iteration (definition 1.1) is a sequence of measurable functions.

For fixed $t\in\Omega$, let $k(t)\in F(t)$. This is possible since by condition (b) of the theorem, $F(t)$ is non-empty.

For fixed $t\in\Omega$, $n=0,1,2,\cdots$.

$$\begin{aligned}
&\| g_{n+1}(t)-k(t)\| \\
&= \|\alpha_nR(t,g_n(t))+\beta_nS(t,g_n(t))+\gamma_nT(t,h_n(t))-k(t)\| \quad \text{(by (1.2))}\\
&\leqslant \alpha_n\|R(t,g_n(t))-k(t)\|+\beta_n\|S(t,g_n(t))-k(t)\|+\gamma_n\|T(t,h_n(t))-k(t)\|\\
&\leqslant \alpha_n\|g_n(t)-k(t)\|+\beta_n\|g_n(t)-k(t)\|+\\
&\qquad \gamma_n\|h_n(t)-k(t)\| \quad \text{(by condition (a))}\\
&= \alpha_n\|g_n(t)-k(t)\|+\beta_n\|g_n(t)-k(t)\|+\gamma_n\|\alpha'_nR(t,g_n(t))+\\
&\qquad \beta'_nS(t,g_n(t))+\gamma'_nT(t,g_n(t))-k(t)\| \quad \text{(by (1.3))}
\end{aligned}$$

$$\leqslant (\alpha_n + \beta_n) \| g_n(t) - k(t) \| + \gamma_n \{ \alpha'_n \| R(t, g_n(t)) - k(t) \| + \beta'_n \| S(t, g_n(t)) - k(t) \| + \gamma'_n \| T(t, g_n(t)) - k(t) \| \}$$

$$\leqslant (\alpha_n + \beta_n) \| g_n(t) - k(t) \| + \gamma_n \{ \alpha'_n + \beta'_n + \gamma_n) \| g_n(t) - k(t) \| \} \quad \text{(by condition (a))}$$

$$= \| g_n(t) - k(t) \| \quad \text{(by (1.6)} \tag{2.1}$$

If follows that for $t \in \Omega, n = 0, 1, 2, \cdots$

$$d(g_{n+1}(t), F(t)) \leqslant d(g_n(t), F(t))$$

or

$$\{ \| g_n(t) - k(t) \| \} \text{ and } \{ d(g_n(t), F(t)) \}$$

are decreasing sequences and hence are convergent.

Let for $t \in \Omega$,

$$\lim_{n \to \infty} d(g_n(t), F(t)) = a(t).$$

Then for all $t \in \Omega$,

$$\lim_{n \to \infty} \| g_n(t) - k(t) \| \geqslant \lim_{n \to \infty} d(g_n(t), F(t)) = a(t). \tag{2.2}$$

We next show that $a(t) = 0$ for all $t \in \Omega$.

If it is not true, let $a(t') \neq 0$ for some $t' \in \Omega$.

Then

$$\lim_{n \to \infty} \| g_n(t') - k(t') \| = b(t') \geqslant a(t') > 0. \tag{2.3}$$

For $t \in \Omega$, $n = 1, 2, \cdots$, let

$$x_n(t) = \frac{\alpha_n R(t, g_n(t)) + \beta_n S(t, g_n(t)) - (1 - \gamma_n) k(t)}{(1 - \gamma_n) \| g_n(t) - k(t) \|} \tag{2.4}$$

and

$$y_n(t) = \frac{T(t, h_n(t)) - k(t)}{\| g_n(t) - k(t) \|}. \tag{2.5}$$

Then for all $t \in \Omega$, $n = 1, 2, \cdots$,

$$\| x_n(t) \| = \frac{\| \alpha_n (R(t, g_n(t)) - k(t)) + \beta_n (S, (t, g_n(t)) - k(t)) \|}{(1 - \gamma_n) \| g_n(t) - k(t) \|} \quad \text{(by (1.6))}$$

$$\leqslant \frac{(\alpha_n + \beta_n) \| g_n(t) - k(t) \|}{(\alpha_n + \beta_n) \| g_n(t) - k(t) \|} \leqslant 1 \quad \text{(by condition (a))}. \tag{2.6}$$

For $t \in \Omega$,

$$y_n(t) = \frac{\| T(t, h_n(t)) - k(t) \|}{\| g_n(t) - k(t) \|}$$

$$\leqslant \frac{\| h_n(t) - k(t) \|}{\| g_n(t) - k(t) \|} \quad \text{(by condition (a))}$$

$$= \frac{\| \alpha'_n R(t, g_n(t)) + \beta'_n S(t, g_n(t)) + \gamma'_n T(t, g_n(t)) - k(t) \|}{\| g_n(t) - k(t) \|} \quad \text{(by (1.3))}$$

$$\leqslant \frac{(\alpha_n' + \beta_n' + \gamma_n) \| g_n(t) - k(t) \|}{\| g_n(t) - k(t) \|} \leqslant 1. \tag{2.7}$$

For $t=t'$,

$$\| (1-\gamma_n)x_n(t') + \gamma_n y_n(t') \|$$

$$= \frac{\| \alpha_n(R(t',g_n(t')) + \beta_n(S(t',g_n(t')) + \gamma_n T(t',h_n(t')) - k(t') \|}{\| g_n(t') - k(t') \|}$$

$$= \frac{\| g_{n+1}(t') - k(t') \|}{\| g_n(t') - k(t') \|} \to \frac{b(t')}{b(t')} = 1 \text{ as } n \to \infty \qquad (\text{by } (2.3)). \tag{2.8}$$

(2.6), (2.7) and (2.8) imply by virtue of lemma 1.1 that

$$\lim_{n\to\infty} \| x_n(t') - y_n(t') \| = 0. \tag{2.9}$$

But

$$\| x_n(t') - y_n(t') \|$$

$$= \frac{\| \frac{\alpha_n}{(\alpha_n + \beta_n)} R(t',g_n(t')) + \frac{\beta_n}{(\alpha_n + \beta_n)} S(t',g_n(t')) - T(t,h_n(t')) \|}{\| g_n(t') - k(t') \|} \quad (\text{by } (1.6))$$

$$\geqslant \frac{f(d(g_n(t'),F(t')))}{\| g_n(t') - k(t') \|} \quad (\text{by } (1.5)-(1.7)) \text{ and condition (c) of the theorem})$$

$$\geqslant \frac{f(b(t'))}{b(t')} > 0 \quad (\text{since } f(r) > 0 \text{ for } r > 0).$$

This contradicts (2.9). Hence we have proved that for all $t \in \Omega$,

$$\lim_{n\to\infty} d(g_n(t), F(t)) = 0. \tag{2.10}$$

From (2.1) and (2.10) it follows that for $t \in \Omega$, given $\varepsilon > 0$, there exists $N(t,\varepsilon) > 0$ and $y(t,\varepsilon) \in F(t)$ such that

$$\| g_n(t) - y(t,\varepsilon) \| < \varepsilon \quad \text{for all } n > N(t,\varepsilon). \tag{2.11}$$

Let $\varepsilon_p = \frac{1}{2^p}$ where p is any positive integer. Then for all $t \in \Omega$, there exists a positive integer $N(t,p)$ and $y(t,p) \in F(t)$ such that

$$\| g_n(t) - y(t,p) \| \leqslant \frac{\varepsilon_p}{4} \text{ for all } n > N(t,p). \tag{2.12}$$

The constuction in (2.12) shows that for all $t \in \Omega$,

$$N(t,p) \leqslant N(t,p+1). \tag{2.13}$$

Again for all $t \in \Omega$, and any positive integer p, and $n > N(t,p+1)$,

$$\| y(t,p) - y(t,p+1) \|$$

$$= \| y(t,p) - g_n(t) + g_n(t) - y(t,p+1) \|$$

$$\leqslant \| y(t,p) - g_n(t) \| + \| g_n(t) - y(t,p+1) \|$$

$$< \frac{\varepsilon_p}{4} + \frac{\varepsilon_{p+1}}{4} + 1 \leqslant \frac{1}{4}\left(\frac{1}{2^p} + \frac{1}{2^{p+1}}\right) \leqslant \frac{3}{4}\,\frac{1}{2^{p+1}} = \frac{3\varepsilon_{p+1}}{4}. \tag{2.14}$$

$$S(y,\varepsilon) = \{x \in B: \| x - y \| = \varepsilon\}$$

is the closed sphere of radius ε and centre y. Let $z \in S(y(t,p+1),\varepsilon_{p+1})$, then

$$\| y(t,p) - z \| \leqslant \| y(t,p) - y(t,p+1) \| + \| y(t,p+1) - z \|$$

$$\leqslant \frac{3\varepsilon_{p+1}}{4} + \frac{\varepsilon_{p+1}}{4} < \varepsilon_p. \quad \text{(by (2.14))}$$

Therefore, $z \in S(y(t,p),\varepsilon_p)$.

This establishes the fact that

$$S(y(t,p+1),\varepsilon_{p+1}) \subseteq S(y(t,p),\varepsilon_p). \tag{2.15}$$

But $\varepsilon_p = \frac{1}{2^k} \to 0$ as $k \to \infty$, then by Cantor's intersection theorem for $t \in \Omega$,

$$\bigcap_{p=1}^{\infty} S(y(t,p),\varepsilon_p)$$

contains exactly one point $q(t)$. Then for $t \in \Omega$,

$$\lim_{p \to \infty} y(t,p) = q(t). \tag{2.16}$$

Again by lemma 1.2, $F(t)$ is closed. It follows by virtue of (2.16) that for all $t \in \Omega$,

$$q(t) \in F(t). \tag{2.17}$$

(2.12) and (2.16) jointly imply that for all $t \in \Omega$,

$$\lim_{n \to \infty} g_n(t) = q(t). \tag{2.18}$$

Again $\{g_n\}$ is a sequence of measurable functions and hence $q(t)$ being the limit of a sequence of measurable function is also measurable. This fact along with (2.17) and (2.18) establishes that $\{g_n\}$ actually converges to a common random fixed point of R, S and T.

Taking Ω to be a singleton set, we obtain the following corollary.

Corollary 2.1. *Let $R, S, T: C \to C$ be three continuous mappings defined on a nonempty closed convex subset of a uniformly convex Banach space B such that the following conditions are satisfied:*

(a) *R, S and T are quasi-non-expansive,*

(b) $F(R) \cap F(S) \cap F(T) \neq \varnothing$,

(c) *R, S, T satisfy 'condition-$*$'.*

Then the sequence $\{x_n\}$ defined as

$$x_0 \in C \text{ is arbitrary} \tag{2.19}$$

$$x_{n+1} = \alpha_n R x_n + \beta_n S x_n + \gamma_n T y_n, \tag{2.20}$$

where

$$y_n = \alpha'_n R x_n + \beta'_n S x_n + \gamma'_n T x_n, \tag{2.21}$$

$$0 < a \leqslant \alpha_n,\ \beta_n, \gamma_n \leqslant b < 1 \quad (a, b \text{ are given}), \tag{2.22}$$

$$0 \leqslant \alpha'_n, \beta'_n, \gamma'_n \leqslant 1, \tag{2.23}$$

$$\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n, \tag{2.24}$$

and

$$0 \leqslant \gamma'_n \leqslant M_4 < 1 \quad (M_4 \text{ is given}) \tag{2.25}$$

converges to a common fixed point of R, S and T.

Choosing $R=S=I$ (identity mapping) we obtain the following result of [10].

Corollary 2.2[10]. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space B and T is a quasi-nonexpansive mapping of C into itself. If T satisfies condition—A then for arbitrary $x_0 \in C$ the sequence $\{x_n\}$ construct like the following.*

For all $n=0,1,2,\cdots$

$$X_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \tag{2.26}$$

where

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \tag{2.27}$$

$$0 < a \leqslant \alpha_n \leqslant b < 1 \ (a, b \text{ are given}), \tag{2.28}$$

$$0 \leqslant \beta_n \leqslant M_5 < 1 \ (M_5 \text{ is given}), \tag{2.29}$$

actually converges to a fixed point of T.

Remark. It may be noted that the Banach spaces in the corollaries need not be separable.

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Department of Mathematics
B. E. College
Deemed University
Howrah-711103, West Bengal
India