LEBESGUE CONSTANT FOR LAGRANGE INTERPOLATION ON EQUIDISTANT NODES

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Abstract

Properties of Lebesgue function for Lagrange interpolation on equidistant nodes are investigated. It is proved that Lebesgue function can be formulated both in terms of a hypergeometric function $_2F_1$ *and Jacobi polynomials. Moreover, an integral expression of Lebesgue function is also obtained and the asymptotic behavior of Lebesgue constant is studied.*

Key words *Lagrange interpolation, Lebesgue function, Lebesgue constant*

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1 Introduction

The importance of the study of Lebesgue functions for polynomial interpolation was demonstrated in [1] through the investigation of their local maxima. In the last four decades, such an analysis has received constant attention from researchers for some sets of nodes which are of special importance in interpolation theory, such as equidistant nodes[2]-[5], Chebyshev roots[6]-[7] and extrema or other ones.

Let $C[-1,1]$ be the Banach space of continuous functions on $[-1,1]$ equipped with uniform norm

$$||f|| = \max_{x\in[-1,1]} |f(x)|$$

and let $L_n(f, X, x)$ the Lagrange interpolating polynomial of degree at most $n-1$ coinciding with $f(x)$ at the nodes

$$X = \{x_{k,n}\}, \qquad k = 1, 2, \cdots, n,$$

where

$$-1 \le x_{1,n} < x_{2,n} < \cdots < x_{n,n} \le 1, \qquad n \in \mathbf{N}.$$

The Lagrange interpolating polynomial is

$$L_n(f, X, x) = \sum_{k=1}^{n} f(x_{k,n}) l_{k,n}(X, x), \qquad n \in \mathbf{N}, \tag{1}$$

where

$$l_{k,n}(X, x) = \frac{\omega_n(X, x)}{(x - x_{k,n})\omega_n'(X, x_{k,n})}, \qquad k = 1, 2, \cdots, n \tag{2}$$

are the so-called fundamental polynomials of degree exactly $n-1$[8] and

$$\omega_n(X, x) = \prod_{k=1}^{n} (x - x_{k,n}).$$

The Lebesgue function is defined as

$$\lambda_n(X, x) = \sum_{k=1}^{n} |l_{k,n}(X, x)| \tag{3}$$

and the Lebesgue constant

$$\Lambda_n(X) = \max_{x \in [-1,1]} \lambda_n(X, x) \tag{4}$$

is closely connected with convergence and divergence of the Lagrange interpolation polynomials (e.g. see [9]).

Although Runge[10] proved that the set of equally spaced points represents a "bad" choice for Lagrange interpolation, there still exist considerable literatures concerning the behavior of Lebesgue function corresponding to equidistant nodes. Such an interest is perhaps due to the fact that this choice frequently occurs in many applications[2].

Let

$$E = \left\{ x_{k,n} = \frac{2k - n - 1}{n - 1} \right\}, \qquad k = 1, 2, \cdots, n \tag{5}$$

be the set of equidistant nodes on the interval $I = [-1, 1]$.

A first result, due to Turetskii[3], proposes an asymptotic expression for the largest maximum

$$\Lambda_n(E) = \frac{2^{n+1}}{en\log(n)}, \quad n \to \infty. \tag{6}$$

Schönhage[4] derived an asymptotic expression for $\Lambda_n(E)$ that is a little bit more precise than (6), namely

$$\Lambda_n(E) = \frac{2^{n+1}}{en\,[\log n + \gamma]}, \qquad n \to \infty. \tag{7}$$

In [5], Mills and Smith improved the expression (7) by finding an asymptotic expansion of $\log\Lambda_n(E)$

$$\log \Lambda_n(E) = (n+1)\log 2 - \log n - \log\log n - 1 + \sum_{k=1}^{m} \frac{A_k}{[\log n]^k}, \qquad n \to \infty, \tag{8}$$

where

$$A_1 = -\gamma,\ A_2 = \gamma^2/2 - \pi^2/12,\ A_3 = -\gamma^3/3 + \gamma\pi^2/6 - \zeta(3)/3, \cdots$$

and $\zeta(3) = \sum_{r=1}^{\infty} r^{-3}$.

2 Main Results

In the sequel, without loss of generality, we shall consider the following nodes set

$$\overline{E} = \left\{ z_{k,n} = \frac{k-1}{n-1} \right\}, \qquad k = 1, 2, \cdots, n. \tag{9}$$

Since through the change of variable

$$x = 2z - 1,$$

one can map the interval I into $\overline{I} = [0,1]$.

As a consequence the formula (3) becomes

$$\lambda_n(\overline{E}, z) = \sum_{k=1}^{n} |l_{k,n}(\overline{E}, z)|. \tag{10}$$

Remark 1. Note that the Lebesgue constant is invariant under affine transformations[2]

$$\Lambda_n(E) = \max_{z \in [0,1]} \lambda_n(\overline{E}, z). \tag{11}$$

The first proposition gives an alternative formulation of the Lagrange fundamental polynomials at equidistant nodes in terms of binomial coefficients which is simpler than that known in literature.

Proposition 1.

$$l_{k,n}(\overline{E}, z) = (-1)^{n+k} \binom{nz - z}{k-1} \binom{nz - z - k}{n-k}, \qquad k = 1, 2, \cdots, n, \tag{12}$$

where $\binom{p}{q}$ *is the binomial coefficient*[11]

$$\binom{p}{q} = \frac{p(p-1)\dots(p-q+1)}{q(q-1)\dots 1}. \tag{13}$$

Proof. The equidistant Lagrange fundamental polynomials are defined by means of the condition

$$l_{k,n}(\overline{E}, z_{q,n}) = \delta_{k,q}, \qquad k, q = 1, 2, \cdots, n, \tag{14}$$

where $\delta_{r,s}$ is the Kronecker function.

By using the identity[11]

$$\binom{-r}{s} = (-1)^s \binom{r+s-1}{s}, \qquad s \text{ integer}$$

one derives that

$$l_{k,n}(\overline{E}, z_{q,n}) = \binom{q-1}{k-1}\binom{n-q}{n-k}, \qquad k, q = 1, 2, \cdots, n \tag{15}$$

and by inspection arguments the expression (14) follows.

2.1 Lebesgue Function

As is pointed out in [2], $\lambda_n(\overline{E}, z)$ takes its maximum when $z \in \left[0, \frac{1}{n-1}\right]$, therefore our attention will be focused on the behavior of the Lebesgue function in the first subinterval, namely $\left[0, \frac{1}{n-1}\right]$. Following the same approach as in [7] we derive

$$\lambda_n(\overline{E}, z) = l_{1,n}(\overline{E}, z) + \sum_{k=2}^{n} (-1)^k l_{k,n}(\overline{E}, z), \qquad z \in \left[0, \frac{1}{n-1}\right]. \tag{16}$$

The next result proposes two alternative expressions of the Lebesgue function restricted to such a subinterval.

Proposition 2.

$$\lambda_n(\overline{E}, z) = (-1)^{n+1}\binom{nz-z-1}{n-1}\left[2 - {}_2F_1(1-n, z-nz; 1+z-nz; -1)\right] \tag{17}$$

$$\lambda_n(\overline{E}, z) = (-1)^{n+1} 2\binom{nz-z-1}{n-1} - P_{n-1}^{(-nz+z,-n)}(3). \tag{18}$$

The proof will apply the following lemma.

Lemma 1.

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{nz-z}{nz-z-k+1} = {}_2F_1(1-n, z-nz; 1+z-nz; -1), \tag{19}$$

where

$$ {}_2F_1(a,b;c;w) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{w^k}{k!} \tag{20} $$

and $(\alpha)_q$ *is the Pochhammer function defined as*

$$ \begin{cases} (\alpha)_q = \prod_{j=0}^{q-1} (\alpha + j), \\ (\alpha)_0 = 1. \end{cases} $$

Note that the formula (19) is the analytical extension of the hypergeometric function ${}_2F_1(a,b;c;w)$ defined for $|w| < 1$ [9,12].

Proof of Lemma 1. Using the formula (20) and the definition of Pochhammer function in terms of Gamma functions[13]

$$ (\alpha)_q = \frac{\Gamma(\alpha+q)}{\Gamma(\alpha)}, \qquad q = 0, 1, \cdots, $$

the right hand side of (19) becomes

$$ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(1-n+k)\Gamma(z-nz+k)\Gamma(1+z-nz)}{\Gamma(1-n)\Gamma(z-nz)\Gamma(1+z-nz+k)}. \tag{21} $$

Since it can easily proved that the indeterminate term of (21), $\dfrac{\Gamma(1-n+k)}{\Gamma(1-n)}$, is equal to

$$ (-1)^k \frac{(n-1)!}{(n-k-1)!} \tag{22} $$

and using the recursive properties of the Gamma function[13]

$$ \begin{cases} \Gamma(1+z-nz) = (z-nz)\Gamma(z-nz), \\ \Gamma(1+z-nz+k) = (z-nz+k)\Gamma(z-nz+k), \end{cases} \tag{23} $$

from (21) one obtains

$$ {}_2F_1(1-n, z-nz; 1+z-nz; -1) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{nz-z}{nz-z-k+1}. \tag{24} $$

Proof of Proposition 2. The expression (16) through (12) becomes

$$ \lambda_n(\overline{E}, z) = (-1)^{n+1} \binom{nz-z-1}{n-1} + \sum_{k=2}^{n} (-1)^n \binom{nz-z}{k-1} \binom{nz-z-k}{n-k}, \tag{25} $$

and by standard algebraic manipulations it assumes the following form:

$$ \lambda_n(\overline{E}, z) = (-1)^{n+1} \binom{nz-z-1}{n-1} \left[2 - \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{nz-z}{nz-z-k+1} \right]. \tag{26} $$

Hence by Lemma 1 (17) follows.

In view of the identity[12]

$$P_n^{(-nz,-n-1)}(3) = \frac{(1-nz)_n}{n!}\,{}_2F_1(-n,-nz;1-nz;-1),$$

where $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial and by using the equivalence

$$(-1)^{(n+1)}\binom{nz-z-1}{n-1}\frac{(n-1)!}{(1-nz+z)_{n-1}} = 1,$$

we can write (17) as (18).

Remark 2. A further integral expression of the Lebesgue function can be retrieved by recalling that[12]

$${}_2F_1(-n,-nz;1-nz;-1) = \frac{2^n}{\pi}\frac{(2n\pi z)}{\sin[2n\pi z]}\int_0^\pi (\cos\phi)^n \cos[(2nz-n)\phi]\mathrm{d}\phi,$$

consequently (17) can be rewritten as

$$\begin{aligned}&\lambda_n(\overline{E},z)\\ &= (-1)^{n+1}\binom{nz-z-1}{n-1}\left[2-\frac{2^{n-1}}{\pi}\frac{2(n-1)\pi z}{\sin[2(n-1)\pi z]}\int_0^\pi (\cos\phi)^{n-1}\cos[(n-1)(2z-1)\phi]\mathrm{d}\phi\right].\end{aligned} \tag{27}$$

2.2 The Lebesgue Constant

In this section we prove a formula for the asymptotic behavior of the Lebesgue constant, by showing a relationship among Λ_n, combinatoric theory and Jacobi polynomials.

Proposition 3.

$$\Lambda_n = (-1)^{(n+1)}\binom{\frac{1}{\log(n-1)+\gamma}-1}{n-1} - P_{n-1}^{\left(-\frac{1}{\log(n-1)+\gamma},-n\right)}(3),\ \ n\to\infty, \tag{28}$$

where the asymptotic value of z corresponding to the maximum of the Lebesgue function in the first subinterval is given by

$$z^* = \frac{1}{(n-1)\left[\log(n-1)+\gamma\right]},\ \ n\to\infty, \tag{29}$$

with $\gamma \approx 0.577215665$ *the Eulero-Mascheroni constant.*

The following identities will be used for the proof of Proposition 3.

$$\lim_{n\to\infty}\frac{\displaystyle\sum_{k=1}^{n}\frac{1}{k}}{\displaystyle\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}} = 0, \tag{30}$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k^2}}{\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}}=0. \tag{31}$$

The proof of (30) and (31) is obtained by means of simple algebraic arguments.

Proof. First, let us derive the expression for z^*. The derivative of $\lambda_n(\overline{E},z)$ can be expressed as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\lambda_n(\overline{E},z)=2\sum_{k=1}^{n-1}\frac{1}{nz-z-k}-\sum_{k=1}^{n-1}\sum_{q=1}^{n}\binom{n-1}{q-1}\frac{nz-z}{(nz-z-k)(nz-z-q+1)}\\+\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{k-1}{(z-nz+k-1)^2}\end{aligned} \tag{32}$$

due to the fact that

$$\frac{\mathrm{d}}{\mathrm{d}z}\binom{nz-z-1}{n-1}=(n-1)\binom{nz-z-1}{n-1}\sum_{k=1}^{n-1}\frac{1}{nz-z-k}. \tag{33}$$

By some algebraic manipulations, it can be rewritten as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\binom{nz-z-1}{n-1}=\sum_{k=1}^{n-1}\frac{1}{nz-z-k}-\sum_{k=1}^{n-1}\sum_{q=2}^{n}\binom{n-1}{q-1}\frac{nz-z}{(nz-z-k)(nz-z-q+1)}\\+\sum_{k=2}^{n}\binom{n-1}{k-1}\frac{k-1}{(z-nz+k-1)^2}.\end{aligned} \tag{34}$$

Hence, using the Mclaurin series expansion, the right hand side of (34) becomes

$$\begin{aligned}&\sum_{k=2}^{n}\binom{n-1}{k-1}\frac{1}{k-1}-\sum_{k=1}^{n-1}\frac{1}{k}-\sum_{k=1}^{n-1}\sum_{s=1}^{\infty}\frac{(n-1)^s z^s}{k^{s+1}}\\&-\sum_{k=1}^{n-1}\sum_{q=2}^{n}\sum_{s=1}^{\infty}\sum_{r=1}^{s}\binom{n-1}{q-1}\frac{(n-1)^s z^s}{k^r(q-1)^{s-r+1}}+\sum_{k=2}^{n}\sum_{s=1}^{\infty}\binom{n-1}{k-1}\frac{(s+1)(n-1)^s z^s}{(k-1)^{s+1}},\end{aligned} \tag{35}$$

from which, through identities (30), (31) and for large values of n, one obtains

$$1-\frac{\sum_{s=1}^{\infty}\sum_{r=1}^{s}\left[\sum_{k=1}^{n-1}\frac{1}{k^r}\right]\left[\sum_{k=2}^{n}\binom{n-1}{k-1}\frac{1}{(k-1)^{s-r+1}}\right]}{\sum_{k=2}^{n}\binom{n-1}{k-1}\frac{1}{k-1}}(n-1)^s z^s. \tag{36}$$

Then, noting that (36) has only sense for $s=r$, one has

$$1-\sum_{s=1}^{\infty}(n-1)^s z^s\left[\sum_{k=1}^{n-1}\frac{1}{k^s}\right], \tag{37}$$

or in a different form

$$1-\sum_{k=1}^{n-1}\frac{1}{k}(n-1)z-\sum_{s=2}^{\infty}(n-1)^s z^s\left[\sum_{k=1}^{n-1}\frac{1}{k^s}\right]. \tag{38}$$

Moreover, by identity (31), it results that (38) is asymptotically equal to

$$1-(n-1)z\sum_{k=1}^{n-1}\frac{1}{k} \tag{39}$$

and, finally, since[11]

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{k}=\log(n-1)+\gamma,$$

it is trivial to conclude that

$$z^*=\frac{1}{(n-1)\left[\log(n-1)+\gamma\right]}.$$

The asymptotic expansion of the Lebesgue constant readily follows by substitution of (29) into (18).

In order to show the accuracy of the expression (28) of the Lebesgue constant with respect to the other formulas proposed in literature and here cited, an extensive numerical comparisons of the relative errors are performed. In Fig. 1 the numerical results are shown up to $n = 200$. As clearly it can be noted, the proposed estimation (Formula (28)) always allows one to obtain a better degree of accuracy than the other ones, except that in a brief interval $n \in [25, 32]$.

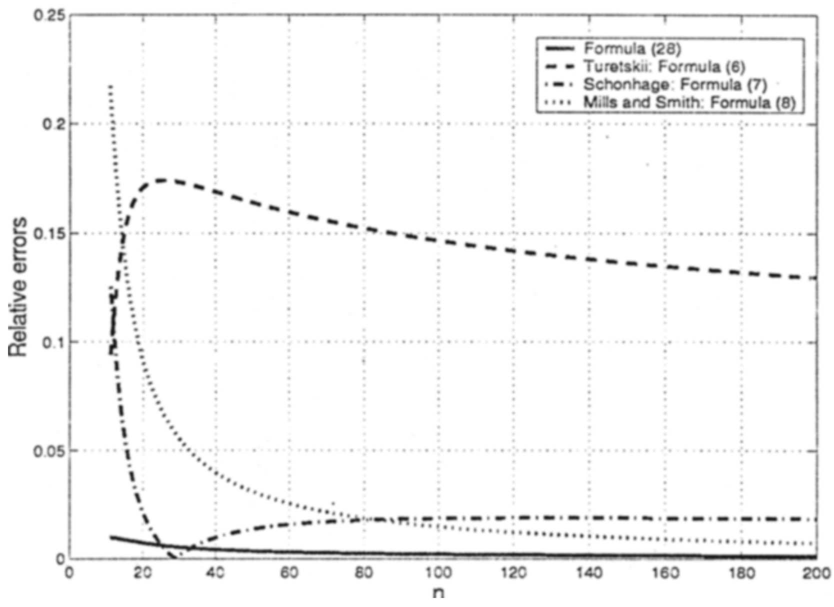

Figure 1 Comparison of Lebesgue Constant Estimations

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