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Review and Evaluation of Different Methods for Force and Torque Calculations in the Strip Rolling Process

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Abstract. This paper presents a review and an evaluation of slip line methods, upper bound methods, slab methods, and finite element methods for force and torque calculations in the hot and cold strip rolling processes. Based on the study, it is recommended that a specific method that suits the analysis should be selected. The following factors should be considered: (a) application of the results, (b) accuracy of the results, and (c) requirements of computer and personnel time. In general, an enhanced slab method for a preliminary analysis and a finite element method for a detailed analysis are recommended.

Nomenclature

f_i = surface traction
k = shear flow stress
k_{ijkl} = elasticity constants
n_j = normal unit vector
q_i = body forces
t = time
T_i = surface forces
$u_{i,j}$ = displacement
$du_{i,j}$ = displacement increments
α = thermal expansion coefficient
$\bar{\varepsilon}$ = effective strain
$\dot{\bar{\varepsilon}}$ = effective strain rate
$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ = generalized state of strain
μ = coefficient of friction
$\bar{\sigma}$ = equivalent or effective stress
$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ = generalized state of stress
σ_f = flow stress in a uniaxial test
σ_m = average stress
σ_y = yield stress
σ_o = constant yield stress
τ = friction shear stress or tangential shear stress
V = volume of the body

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Introduction

Increasing competition in the metal industry and advances in technology lead to additional demands for accurate analysis so that better strip quality in terms of gauge and shape can be obtained with minimum energy consumption. In strip rolling, the accurate determination of rolling force, torque, and slip, as well as the values of stress, strain, strain rate, and temperature distributions, are extremely important to the proper design and control of the process. In order to select an appropriate method for obtaining this information, it is necessary to review the different methods. In this paper, slip line methods, upper bound methods, slab methods, and finite element methods for force and torque calculations in the hot and cold strip rolling processes are reviewed and evaluated.

Strip Rolling Process

The reduction of material by rolling it between rolls is simple to visualize. Figure 1 illustrates diagram-

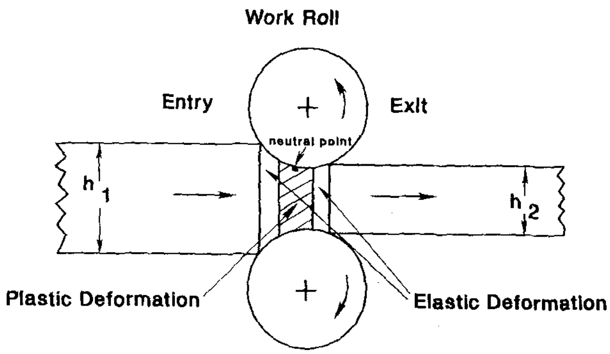

Fig. 1. Plane strip rolling.

matically the basic deformation that takes place. The strip material deforms elastically before entry and after exit of the roll gap. During deformation within the roll gap, the material behavior is elastic-plastic. Friction plays a major role in transferring the energy from the rolls to deform the strip. The speed of the strip changes while passing through the roll gap. At the entrance, the speed of the strip is less than the roll speed; at the exit, the speed of the strip is more than the roll speed. Therefore, a neutral point exists within the roll gap. High rolling forces make the rolls deflect quite considerably due to imposed bending, so that the rolled strip may have a bad profile, for example, be thicker in the middle than at the edges. Apart from the above elastic deformation of the rolls, they also deform elastically while compressing the strip. The act of continuous loading and unloading of the circumference of the rolls also leads to fatigue. Material properties change as material passes through the roll gap. Heat transfer from the strip to the rolls and to the atmosphere, and heat generation due to plastic deformation affect the whole process considerably.

In order to properly roll the strip, the following problems need to be addressed:

1. Force on the rolls during the rolling operation.
2. Roll torque required to deform the strip.
3. The entrance and exit speeds of the strip.
4. The stress, strain, strain rate, and temperature distributions within the strip during the deformation process.
5. Heat generated due to deformation and transferred to the rolls.
6. Effect of one or more of the parameters (e.g., roll diameter, draft, entry and exit tensions, etc.) on the rolling process.
7. Residual stresses in the strip after rolling (in case of cold rolling).

In order to solve these problems, it is essential to use the theory of plasticity. Although at first sight it might seem possible to predict the macroscopic behavior in terms of stress and strain from the knowledge of interatomic forces and the movement of dislocations, in practice, the difficulties associated with such an approach are insurmountable. At the microscopic level, the phenomena of plastic deformation is very complicated. There are usually several possible slip systems in any crystal, and practical metals are polycrystalline so the movement and interaction of dislocations can be exceedingly complex. As such, one must employ the "macroscopic" theory of plasticity [2–5]. This theory has for its starting point the assumption of an idealized metal whose properties (such as when it will yield and how it will behave under applied stresses) are considered to be completely definable in terms of basic stress–strain curves. By using such a theory, a limited amount of information may be obtained about the way the metal will behave.

Mathematical Problem Formulation

The strip rolling problem is a boundary value problem, in which a volume, V, of material is bound by a surface, S, as in Figure 2. Compatibility equations and equilibrium equations apply within the volume. In order to simplify the written arrangement of the equations, a tensor notation is used. The compatibility equations mean that for any force producing an infinitesimal strain increment field $d\varepsilon_{ij}$, it must be possible to derive such a field from a displacement increment field du_i through:

$$d\varepsilon_{ij} = \frac{1}{2}(du_{i,j} + du_{j,i}) \tag{1}$$

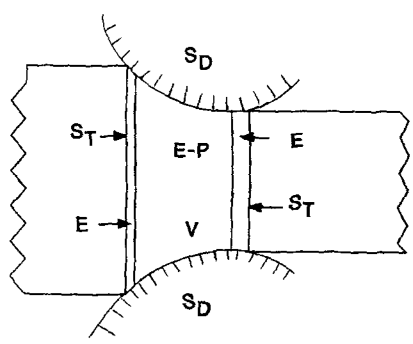

Fig. 2. Solid mechanics boundary value problem.

where $du_{i,j} = du/dj$.

Furthermore, the plastic part of strain increment satisfies its incompressibility:

$$d\varepsilon_{ii}^{p} = 0 \tag{2}$$

The equilibrium equations mean that for any force producing a stress field σ_{ij} in which body forces, q_i, are present, the following applies:

$$\sigma_{ij,j} + q_i = 0 \tag{3}$$

where a summation convention is implied. Furthermore, a part of the surface on which force/unit area, T_i, is applied and on which force boundary conditions have to be satisfied:

$$\sigma_{ij} n_j = T_i \tag{4}$$

where n_j is a unit vector normal to the surface.

The stresses and strain increments produced by a certain force are related by constitutive equations. The body may be divided into two parts:one E, in which yielding does not take place, and another part E-P where yielding has occurred. In the E part, where elasticity equations apply:

$$d\varepsilon_{ij} - \alpha \Delta T \delta_{ij} = K_{ijkl} d\sigma_{ij} \tag{5}$$

where K_{ijkl} are the elasticity constants, α is the thermal expansion coefficient, and δ is the Kronecker delta representation. In the other part, E-P, where yielding has occurred, the yield criterion applies:

$$\bar{\sigma} = \sigma_f \tag{6}$$

where $\bar{\sigma}$ is effective stress and σ_f is flow stress. Elasticity and plasticity stress–strain relations:

$$d\varepsilon_{ij} - d\varepsilon_{ij}^{p} - \alpha \Delta T \delta_{ij} = K_{ijkl} d\sigma_{ij} \tag{7}$$

and all the Levy-Mises flow rules are applicable in this region.

During rolling, there is a certain amount of heat generated by deformation as well as heat lost to the environment. This produces two effects. One, if material properties are very sensitive to temperature, deformation patterns can be modified substantially. Two, if the thermal gradients are large, heat transferred to the rolls can reduce the roll strength substantially. It is important in many cases to perform an analysis of the thermal aspect of a problem parallel with a mechanics analysis. They are coupled in the sense that temperatures influence material properties and deformation generates heat.

Methods of Solution

Obtaining a closed form solution that satisfies all the above requirements is very difficult. Several approximate analytical and numerical solutions have been developed through the years even before the theory was completely defined.

All the well-known methods of solving the strip rolling problems can be divided into four groups, namely (a) slip line field methods, (b) upper bound methods, (c) slab methods, and (d) finite element methods. The detailed description of these methods can be found in [1–5] and elsewhere.

The major simplification for ease of solution is to consider the strip rolling problem as plane strain. This means that there is no increase in the width of the strip due to the rolling operation. This simplification allows us to consider this problem as a two-dimensional problem. However, for spread analysis, the problem can be solved with consideration of three-dimensional deformation using upper bound or finite element methods.

All the above methods are briefly discussed here to see their applicability in solving the strip rolling problems.

Slip Line Methods

The first approach to the analysis of metalworking processes that did not assume homogeneous deformation is the slip line method. Slip line models postulate a geometrical arrangement of two sets (α and β) of slip planes, in the roll bite, which cut each other orthogonally. Figure 3 shows a slip line field in the roll bite. The orthogonal network of lines of maximum shear is commonly called the slip line field. The upper bound to the true deforming force is obtained simply by finding the configuration of shear planes. Usually, a number of assumptions are taken to construct these fields.

Having constructed a slip line network, the vertical stresses exerted on the base of each triangle coincident with the center line of the workpiece may be computed. The sum of these values corresponds to the specific rolling force. Similarly, the resultant of the shearing stresses along the bases of the triangles coincident with the roll surface can be used to compute the specific roll torque.

This method permits a point-by-point calculation of stresses and velocity distributions. But the method

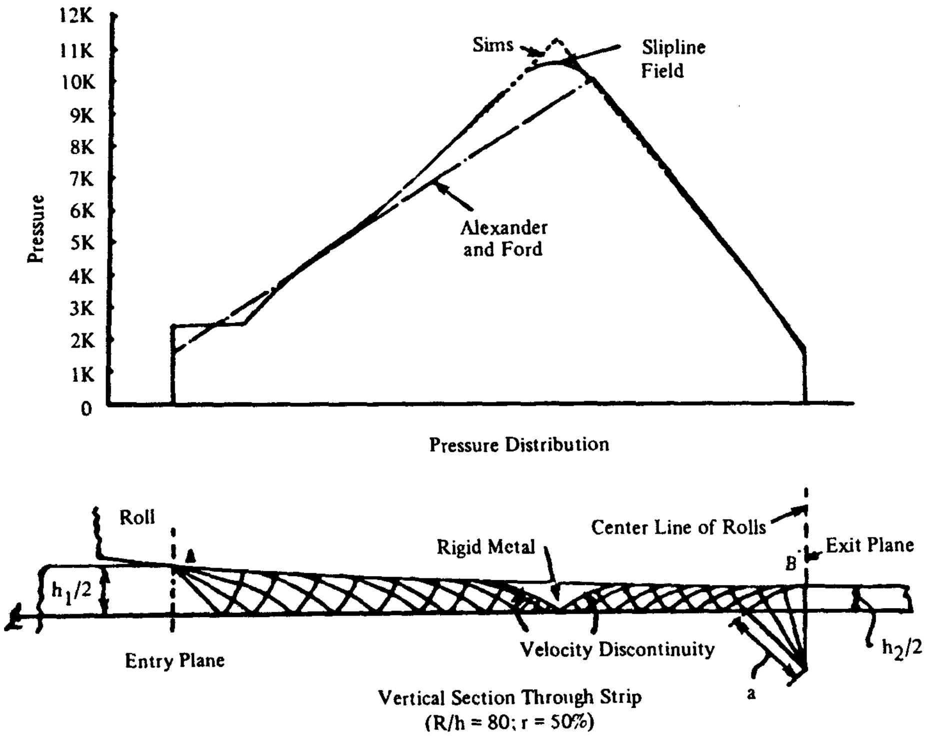

Fig. 3. Slip line fields for hot strip rolling, reprinted from Ref. [4], p. 690, by courtesy of Marcel Dekker, Inc., New York.

lacks uniqueness for constructing slip line field and it does not permit flexibility for treatment of boundary conditions. For example, in the case of strip rolling, it is difficult to consider slipping friction. Furthermore, the solutions are limited to rigid-perfectly plastic material under plane strain conditions.

Upper Bound Methods

The "kinematically admissible velocity field" is a field of generalized velocities (which may be strains, linear displacements, or angular rotations) which is kinematically compatible within itself and with the externally imposed displacements at the boundaries.

The upper bound theorem for strip rolling states that among all kinematically admissible velocity fields the actual one minimizes the expression [27],

$$J = \frac{2}{\sqrt{3}}\sigma_o \int_V \sqrt{(1/2)\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}\, dV + \int_\Gamma \tau \Delta v \, ds - \int_{st} T_i v_i \, ds \quad (8)$$

where J denotes the actual externally supplied power. The first term in the above equation expresses power for internal deformation over the volume of the deforming body. The second term includes shear power losses over the surfaces of velocity discontinuity including that at the boundary between the tool and the workpiece. The last term covers power supplied by predetermined surface tractions as, for example, front and back tensions in rolling.

In strip rolling, the neutral point is not a known prior; therefore, iterative methods can be used to minimize Eq. (8) and find the velocity field. Once the velocity field is known, the roll force and torque can be obtained.

These methods give only estimated upper limits of the required deformation force, according to kinematically admissible velocity fields. In addition, these methods lack the ability to deal with work hardening materials and to reveal detailed deformation information such as stress and strain distributions.

Slab Methods

This method considers the stresses on a plane perpendicular to the metal flow direction. A slab of infinitesimal thickness is selected in this plane at any arbitrary position in the deformed metal. The forces on the slab are balanced, which results in a differential equation of static equilibrium. By analytical or numerical integration of the differential equation and with the introduction of the boundary conditions, it is possible to determine forming forces, torques, neutral point, and stress, strain, strain rate, and temperature distributions.

The starting point of the slab method is a major simplification of the material flow. It is assumed that the velocity along the slab cutting surfaces is constant, which means that the slab cutting surfaces will be planes during forming.

Figure 4 illustrates the pattern of deformation and the stresses acting on a transverse "slab" element at angle ϕ to the line of centers. The horizontal stress p is assumed to be uniformly distributed across the element, and a principle stress. The vertical stress q is also assumed to be a principal stress. The stress s is the pressure normal to the roll surface and the shear stress transmitted between roll and slab is τ.

The starting point of all methods under this heading is to develop the equation representing the horizontal equilibrium of forces in the roll gap. The forces on an elemental slab of material in the arc of contact are shown enlarged in the inset diagram of Figure 4. The horizontal stress, p, is assumed to be distributed uniformly over the vertical section, so that the horizontal force per unit width, f, is equal to $p \cdot h$. Consideration of the equilibrium of the elemental slab or slice of material leads to the basic first order differential equation first put forward by von Karman in 1925. With minor changes in the notation used by von Karman, his equation may be conveniently written for the current purpose as follows:

$$df/d\phi = 2R(s \sin\phi \pm \tau \cos\phi) \tag{9}$$

where the negative sign refers to conditions on the entry side of the neutral point, and the positive sign refers to conditions on the exit side. The neutral point occurs anywhere in the arc of contact, depending on the boundary conditions of the problem.

Many methods to solve Eq. (9) for both hot and cold rolling have been put forward in the last 65 years, beginning with the pioneering works of Siebel and von Karman in 1924 and 1925, as discussed in the review by Ford (1957) [7]. The most comprehensive of these was developed by Orowan (1943) [8], in which he accounted for all the various factors involved in arriving at an accurate solution. The complexity of his method undoubtedly caused later research workers, notably Bland and Ford (1948, 1952) [9] and Sims (1954) [10], to develop solutions based on simplifying assumptions. The assumptions allowed analytical expressions to be developed, thus avoiding most of the numerical integration involved in Orowan's method. Unfortunately, this inevitably led to a sacrifice in accuracy.

In order to produce greater precision in the analysis, Hitchcock (1935) [13] developed the equations to take into account the roll flattening due to local elastic deformation. In another attempt to produce greater precision in the analysis, Ford and his coworkers (1951, 1956) [11] and Bland and Sims (1953) [12] made modifications in the Bland and Ford's method for cold rolling by taking into account the entry and exit elastic arc of contact.

With the advent of the digital computer, the complexity of the basic differential equation describing strip rolling is no longer a barrier to its solution. Alexander (1972) [14] solved the equation using Orowan's approach. Lahoti, et al. (1978) [15] used a similar approach that took into account the inhomogeneity of deformation (suggested by Orowan) and allowed the variation of flow stresses as a function of strain, strain rate, and temperature. It coupled a heat transfer analysis module with the mechanics module. Venter, et al. (1980) [16] considered inhomogeneity of deformation and used a function developed by Orowan [8]. Wanheim, et al. (1986)

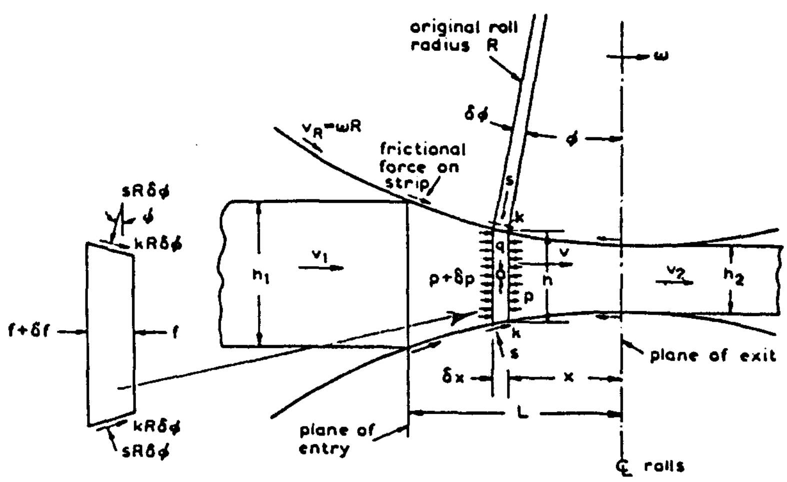

Fig. 4. Stress acting in the arc of contact during rolling, reprinted with permission from *Manufacturing Technology*, Vol. 2, by J. M. Alexander, R. C. Brewer, and G. W. Rowe, published in 1987 by Ellis Horwood Ltd., Chichester [2].

[17] used an approach similar to Alexander's with their own general friction model. Alexander, et al. (1987) [18] again used a similar approach with the friction model of Wanheim and Bay, and provided a suitable treatment for the neutral zone as suggested by Tselikove (as mentioned in [18]).

The main considerations of the methods proposed by Orowan, Bland and Ford, Sims, and Alexander are discussed below.

Orowan's method. Orowan's method could be used for both hot and cold rolling. He made the following assumptions.

1. The frictional stress at the interface between roll and the material being deformed should be, $\tau = \mu s$ or $\tau = k$, whichever is minimum.
2. He recognized the fact that the yield stress of the material would vary during its passage through the arc of contact due to work hardening, temperature, and strain rate variation, and considered how these variations might be included in a comprehensive method.
3. With minimum mathematical assumptions, he developed his classical homogeneous graphical method for the numerical computation of the solution to Eq. (9).
4. He discussed the inhomogeneity of the deformation which would lead to a departure from the simple situation often assumed (plane sections remaining plane) and introduced a complicated adjusting factor by which the inhomogeneity could be taken into account. In fact, as discussed in [3], his method for accounting the inhomogeneity of the deformation is based on an analysis which is not strictly applicable to the complicated situations that exist in the roll gap.

Bland and Ford's method. This method is specifically designed for cold rolling. To get the analytical solution of Eq. (9), the following major assumptions are made.

1. The frictional stress at the interface between the roll and the material being deformed is always $\tau = \mu s$.
2. The yield strength of the strip in the roll gap is considered to be a constant, which is equal to the mean or average yield strength of the strip before and after rolling.
3. The radial pressure s at any point along the arc of contact is equal to its vertical component q at that point.
4. Mathematical approximations $\cos\phi = 1 - \phi^2/2$ and $\sin\phi = \phi$ are used in the calculations.

Sims' method. This method is specifically designed for hot rolling. To get the analytical solution of Eq. (9), the following major assumptions are made.

1. The frictional stress at the interface between the roll and the material being deformed is always equal to the shear flow stress.
2. The yield strength of strip in the roll gap is considered to be a constant, which is equal to the mean or average yield strength of the strip before and after rolling.
3. In solving the differential equation, a previously developed equation is used, which is based on the assumption that the rolling process can be compared with the deformation between rough inclined plates.
4. Mathematical approximations $1 - \cos\phi = \phi^2/2$, $\cos\phi = 1$, and $\tan\phi = \sin\phi = \phi$ are used in the calculations.
5. In calculating the roll force, the normal roll pressure s is equal to the vertical component q.

Alexander's method. To get solutions for both hot and cold rolling problems, Alexander developed the numerical solutions to Eq. (9) following the first three assumptions of Orowan's method and using the approach suggested by Orowan. He used the fourth order Runga-Kutta method for the integration of the differential equations. He developed equations for the roll force and torque calculations and numerically solved the integrals using trapezoidal rule and Simpson's rule.

Finite Elemental Methods

In the last several years, with the development of computers, the well-known finite element methods [6] have been developed. The finite element methods are a series of numerical techniques that solve boundary value problems, initial value problems, and eigenvalue problems. Currently, these methods offer one of the most flexible and comprehensive theoretical tools for the analyses of metal forming processes.

The finite element methods for metal forming processes can be divided into two groups which are rigid-plastic [5, 19, 20, 22, 24] and elastic-plastic [5, 23, 25, 26]. In both of the methods, solutions can be obtained for both isothermal and nonisothermal deformations. The starting point is to develop a functional, considering the calculus of variation, the principle of virtual work, and the convexity of yield surfaces. Apart from the basic assumptions made in the yield criteria, no other assumptions are made. The only approximations are in the numerical methods, which cannot be avoided. Derivations and descriptions óf these functionals can be found in the references mentioned above. A review of finite element methods for metal forming is given in [28]. After developing the functional, the method can be subdivided into six basic steps as follows:

1. Discretize and select element type. This involves dividing the body into an equivalent system of finite elements with associated nodes and choosing the most appropriate element type.
2. Specify the approximation equation. This involves choosing a velocity (temperature) function within each element.
3. Develop the system of equations. The variational functional derived from the virtual work principle for the system is written in terms of nodal velocities (temperatures) and is minimized. This generates one equation for each unknown nodal value (velocity, temperature).
4. Assemble the element equation using the method of superposition. The individual element equations generated in step 3 are added together using a method of superposition, whose base is nodal force or energy equilibrium. The result is global equations for the whole body or structure. Implicit in this method is the concept of continuity or compatibility, which requires that the body remains together and that no tearing occurs anywhere in the body.
5. Solve for generalized velocities (temperatures). The global stiffness equations, modified to account for the boundary conditions, are a set of simultaneous algebraic equations. There are different methods available to solve the equations such as the elimination method (Gauss method) or an iterative method (such as Gauss-Siedel method).
6. Calculate quantities of interest. These quantities are usually related to the derivative of the parameter. In the current case, these are stresses, strains, strain rates, temperatures, force, and torque.

The finite element analysis for the plastic deformation of metals is much more complicated than that for elastic deformation. The functionals, to solve any metal forming problem using finite element methods, considering rigid-plastic or elastic-plastic material behavior, generate a highly nonlinear set of equations. Therefore, iterative methods are needed to solve the equations. Apart from this, more complications come in the formulation to deal with the changing boundary conditions and the yield surface changes at every point as a function of work hardening, strain rate, and temperature. To take these variables into account, it is necessary to deform the object (strip in the case of rolling) step by step and the finite element mesh gets distorted. After a certain amount of distortion, it becomes necessary to remesh the object, and interpolate the strains and temperature history to the new nodal points. Consideration of such factors makes the formulation complicated and necessitates a great deal of computer time.

The rigid-plastic formulations are simpler than elastic-plastic formulations because there is no need to treat the elements separately for elastic and plastic deformations. Consideration of this factor saves a great deal of computer time. The main advantage of elastic-plastic formulations over rigid-plastic formulations is the prediction of residual stresses in the object after deformation. With the prediction of residual stresses, it becomes possible to treat the object for spring-back action.

Concluding Comparison

The major disadvantage of the slip line methods and upper bound methods is the assumption of rigid-perfectly plastic material behavior. Other limitations are mentioned under the description of these methods. The assumptions in these two methods may oversimplify the actual solutions and limit their value in practical applications.

The major assumption of the slab method is that the slab cutting surfaces will be planes during rolling, that is, homogeneous deformation of the slabs. But, this assumption is not as drastic as the assumptions used in the slip line method and upper bound method. This method has been used most to solve strip rolling problems. Also in this method, heat transfer analysis can be coupled with the mechanics analysis to simulate the true behavior. It seems that the enhanced slab method (slab method with recent developments) is suitable for industrial use.

So far, finite element methods have not been used much in the area of rolling. But, the successful application of these methods in solving the other metal forming problems (forging and extrusion) shows bright prospects in this area. In finite element methods, no assumptions are made apart from the basic assumptions in defining the yield criterion and neglecting the elastic deformations while assuming the rigid-plastic material behavior. A comparison of elastic-plastic and rigid-plastic methods is given under the description of finite element methods. The advantages of finite element methods are as follows: (a) a minimum number of assumptions are made for material and deformation behaviors, and therefore it can be expected that these methods will give better results; and 2) they give detailed distributions of stress, strain, strain rate, and temperature; these variables affect the microstructure of strip. Finite element methods have disadvantages as they need a large computer memory, CPU time, and personnel time.

Based on the above facts, it is recommended that a particular method should be selected considering the following factors: (a) application of the results, (b) accuracy of the results, and (c) requirements of computer and personnel time. The following types of results can be obtained: (a) forces required for rollling, (b) torques required for rolling, (c) entry and exit speed of the strip, (d) stress, strain, strain rate, and temperature distribution within the roll gap, and (e) residual stresses in the strip after rolling. In general, an enhanced slab method for preliminary analysis and a finite element method for detailed analysis are recommended.

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