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Gravitational Interactions of the Proton and the Electron: The Possible Existence of a Massless Scalar Particle.

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Summary. — Assuming general covariance, we discuss the structure of gravitational interactions between the proton and the electron. There are 15 (3) different couplings between a massive spin-$\frac{1}{2}$ particle (a two-component neutrino) and the gravitational field, provided that interactions involve no higher derivatives than those appearing in the free Dirac and gravitational Lagrangians. In the weak-field limit the interactions may be divided into 4 classes: $(C, P, T) = (+, +, +)$, $(+, -, -)$, $(-, +, -)$ and $(-, -, +)$. Any deviation from Einstein's theory entails an asymmetric energy-momentum tensor, hence involving hypothetical *massless, spinless particles*, represented by a skewsymmetric tensor field whose source is the spin of the proton and the electron. We also discuss possible effects of this particle on the hyperfine structure of the hydrogen atom.

1. – Introduction and summary.

Gravitation is, together with electromagnetism, one of the fundamental interactions of matter and has the classical limit. In parallel with Maxwell's electromagnetic theory, Einstein's classical theory of gravitation has also attained remarkable success [1]. However, in contrast to the enormous de-

(1) I. I. SHAPIRO: Rapporteur's talk at the *VI International Conference on Gravitation and General Relativity at Copenhagen* (July 1971); I. I. SHAPIRO, M. E. ASH, R. P. INGALLS, W. B. SMITH, D. B. CAMPBELL, R. B. DYCE, R. F. JURGENS and G. H. PETTENGILL: *Phys. Rev. Lett.*, **26**, 1132 (1971); K. S. THORN and C. M. WILL: *Astrophys. Journ.*, **163**, 595 (1971); C. M. WILL: *Physics Today*, October Issue, p. 23 (1972) and references cited therein.

velopment of quantum electrodynamics, it is not yet clear how elementary particles interact with the gravitational field. For example, we do not know if gravitation conserves parity (P), charge conjugation (C), and time reversal (T) in elementary processes of particle physics [2]. The purpose of the present paper is to explore the structure of gravitational interactions between the proton and the electron.

The reason why we do not simply extrapolate Einstein's theory to the realm of particle physics is that this naive extension is without much theoretical and experimental justification. First of all, let us postulate that any microscopic gravitational theories must revert to Einstein's by means of the classical macroscopic limit. This « boundary condition » is obviously not enough to select one such theory; there are yet many possibilities. Take, for example, the matrix element of the symmetric conserved energy-momentum tensor, taken between spin-$\frac{1}{2}$ single-particle states (*e.g.* the proton or the electron) [3,4]

$$(1.1)\qquad \begin{aligned}(p_2|T_{kj}(0)|p_1) = \bar{u}(p_2)\{(i/2)\gamma_{(k}p_{j)}G_1(q^2) + \\ + iq_n\sigma_{n(k}p_{j)}G_2(q^2) + (q^2\delta_{kj} - q_kq_j)G_3(q^2) + \\ + [iq^2\gamma_{(k}p_{j)} + 2m\,q_{(k}p_{j)}]\gamma_5 G_4(q^2) + \\ + q_n\sigma_{n(k}p_{j)}\gamma_5 G_5(q^2) + i(q^2\delta_{kj} - q_kq_j)\gamma_5 G_6(q^2)\}u(p_1)\,,\end{aligned}$$

where $p = p_1 + p_2$, $q = p_2 - p_1$, and parentheses enclosing indices denote symmetrization. Evidently, the first term, multiplied by $G_1(q^2)$, may be derived from the perturbation expansion (a quantal version) of Einstein's theory; $G_1(q^2)$ is hence reasonably termed *Einstein form factor* by analogy with the Dirac form factor in quantum electrodynamics. However, the converse is not true; the possible existence of other Pauli-type form factors would not conflict at all with but would rather conform to the success of Einstein's theory in classical gravitational physics, because, first, they appear together with the *spin* of a particle involved, and second, they are accompanied with the derivatives of the gravitational field, which vanish in the long-range limit ($q \to 0$). Gravitational form factors, $G_A(q^2)$ ($A = 1, 2, \ldots, 6$), have not yet been measured due to the extreme weakness of gravitation. It now follows from these considerations that the beautiful achievements of Einstein's theory in classical physics cannot be taken as clear evidence of P, C and T conservation in quantal gravitational processes. *What theoretical grounds may we have in predicting the presence or absence of the Pauli-type form factors?*

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(2) J. LEITNER and S. OKUBO: *Phys. Rev.*, **136**, B 1542 (1964).
(3) K. HIIDA and Y. YAMAGUCHI: *Progr. Theor. Phys. Suppl.*, p. 261 (1965), Commemoration Issue for the XXX Anniversary of the Meson Theory by Dr. YUKAWA.
(4) H. PAGELS: *Phys. Rev.*, **144**, 1250 (1966); B. RENNER: *Phys. Lett.*, **33** B, 599 (1970).

There seem to be two available methods for answering the question we have posed:

A) the simple-minded method is to envisage, within the framework of special relativity, gravitational interactions which might cause violation of P, C and T;

B) the other one is to consider the structure of gravitational interactions invariant under a general co-ordinate transformation; some of the interactions would violate some of these discrete symmetries in the perturbation expansion where the P, C and T operations are well defined.

Here we make an important remark as to choice of the gravitational variable. In the first line of thought the latter may be taken as a rank-2 symmetric tensor field, subject to appropriate subsidiary conditions to choose a massless spin-2 particle, called graviton (3,5). In the method *B*), however, it should be noticed that fermions differ radically in their realization from bosons by the influence of gravitation. As is well known, the tensor representation of the proper Lorentz group $L_+^\uparrow$ may be extended to a representation of the general co-ordinate transformation group G, but the half-integer spin representation of $L_+^\uparrow$ cannot. Indeed, the latter is not contained in any finite-dimensional, linear representation of G. In sharp contrast with the well-defined Lorentz spinor being holomorphic to $L_+^\uparrow$, a « Riemann spinor » holomorphic to G does not exist. Since there is an abundant literature on mathematics of spinors in a Riemann space (6), we here refer to previous papers (7) which fit the present purpose; the gravitational field was therein introduced into the Dirac Lagrangian as the Yang-Mills field with 16 components $b_k{}^\mu(x)$ called *translation gauge field* associated with G. In particle physics the gauge field may be taken as more fundamental than the conventional metric tensor which plays the central role in Einstein's classical theory of gravitation, in the sense that the former can describe gravitational interactions of both integral and half-integral spin particle fields, while the latter can do those of only integral spin particle fields.

We shall limit our theoretical framework by postulating the following. First, we assume that the equations of motion for the proton and the electron may be derived from a single Lagrangian by means of Hamilton's principle. Obviously the proton is not such a simple entity as may be described by the equa-

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(5) S. Weinberg: *Phys. Rev.*, **138**, B 988 (1965).

(6) See, for instance, reviews on mathematics of spinors by W. L. Bade and H. Jehle: *Rev. Mod. Phys.*, **25**, 714 (1953); R. Penrose: *Ann. of Phys.*, **10**, 171 (1960); F. Cap, W. Majerotto, W. Raab and P. Unteregger: *Fort. d. Phys.*, **14**, 205 (1966), and references quoted therein. For nonlinear realization of spinors, see V. I. Ogievetskii and I. V. Polubarinov: *Sov. Phys. JETP*, **21**, 1093 (1965).

(7) K. Hayashi and T. Nakano: *Progr. Theor. Phys.*, **38**, 491 (1967); K. Hayashi: *Lett. Nuovo Cimento*, **5**, 529 (1972).

tion of motion. Nevertheless the latter would be a good approximation at least for a low energy proton, on the one hand, and higher-order radiative corrections would later be incorporated into the form factors, on the other hand. Second, we demand that:

a) the action integral be invariant under a general co-ordinate transformation;

b) the action integral be invariant under a *global* Lorentz transformation (see (2.38)-(2.41) for definition);

c) interaction Lagrangians depend linearly on at most the first derivative of a spinor field and the gravitational field (in other words no higher derivatives are included in interactions than those involved in the corresponding free Lagrangians);

d) a free gravitational Lagrangian be of bilinear form in the the first derivative of the gravitational field.

Our gravitational field exceeds the metric tensor in number. In order to avoid unnecessary confusion we explicitly state that our theoretical framework differs *in general* from the so-called *tetrad version* of Einstein's theory which assumes, instead of *b*), that [8]

b') the action integral be invariant under a *local* Lorentz transformation (see (2.38)-(2.41) with $\omega_{ij}(x) = -\omega_{ji}(x)$).

The tetrad version thus removes (as nonphysical) six additional degrees of freedom of the tetrads by the assumed local Lorentz invariance. The tetrads then have exactly the same number of components as the metric tensor. In fact the tetrad version is equivalent to Einstein's theory, being in perfect harmony with Einstein's idea that 10 components of the metric tensor, or equivalently of the tetrads, are in principle determined by the stress tensor (with 10 components) of macroscopic bodies through Einstein's equation of gravitation. The assumption *b*), on the contrary, does not eliminate from the outset the six additional degrees of freedom of the gauge field. *Whether or not these are physical then depends on how the proton and the electron interact with the gravitational field.* In this respect it is preferable to distinguish the gauge field from the tetrads. For a particular interaction our theory may acquire the local Lorentz invariance.

The reasons why we take *b*) rather than *b'*) are that i) there is no experimental evidence against the assumption *b*), nor against *b'*); ii) a more general theoretical framework than the tetrad version will be set up, where

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(8) K. Hayashi and A. Bregman: *Ann. of Phys.*, **75**, 562 (1973); K. Hayashi: *General Relativity and Gravitation* (*Berne*), **4**, 1 (1973).

the Pauli-type form factors may be derived; and finally iii) the dynamical assumption inherent in Einstein's theory and the physical significance of the local Lorentz invariance will be made much clearer. We emphasize that as postulated earlier our theory reverts to Einstein's by the macroscopic limit, and that *in parallel with this limit the global Lorentz invariance, assumed at quantal level, will be taken over by the local Lorentz invariance prevailing in Einstein's theory.* Since the six additional degrees of freedom of the gauge field may be excited by the spin of the proton and electron, it is clear that classical experiments can neither prove nor disprove the assumption *b*). The present experimental status will be described later on.

This paper begins in Sect. **2** with the enumeration of all the possible gravitational interactions, which amount to fifteen types. The analysis of these interactions leads to the *first conclusion* that for only one particular combination of the interactions, called Einstein's interaction, the energy-momentum tensor defined as a source of gravitation becomes symmetric, and for any other combinations of the interactions the energy-momentum tensor becomes asymmetric. Our theory acquires the local Lorentz invariance in the former case and has only global Lorentz invariance in the latter case. The one-to-one correspondence is thus established among them and schematically drawn as follows:

Einstein's interaction $\leftrightarrow$ symmetric energy-momentum tensor
$\leftrightarrow$ local Lorentz invariance,

deviation from Einstein's interaction $\leftrightarrow$ asymmetric energy-momentum tensor
$\leftrightarrow$ global Lorentz invariance.

In view of the first conclusion a free gravitational Lagrangian is then given in Sect. **3** in its most general form with four arbitrary parameters. The field equation of gravitation is derived from it by Hamilton's principle and a retarded solution to the linearized field equation is obtained. The analysis of the solution restricts the free parameters to only one, denoted by λ. The ensuing field equation splits in the weak-field limit into the symmetric and antisymmetric parts:

$$(1.2) \qquad \begin{cases} \Box S_{kj} = -\varkappa T_{(kj)}\,, \\ \Box A_{kj} = -\lambda T_{[kj]}\,, \end{cases}$$

where $S_{kj}=S_{jk}$ and $A_{kj}=-A_{jk}$ are linearized parts of the translation gauge field, subject to the divergence-free conditions, and $\varkappa^2=8\pi G/c^4$ (Einstein's gravitational constant). The *second conclusion* is that if there is any departure from Einstein's interaction, then there must be a skewsymmetric tensor field $A_{kj}(x)$, or, when quantized, a massless scalar particle, termed for short « deviaton ».

The linearized gravitational interactions are divided into 4 classes in Sect. **4** according to the degree of the reflexion symmetries; $(C, P, T) = (+, +, +)$, $(+, -, -)$, $(-, +, -)$ and $(-, -, +)$. This result leads to the *third conclusion* that for Einstein's interaction the Einstein form factor $G_1(q^2)$ may be given a theoretic basis, while for any other interactions the symmetric property of the energy-momentum tensor is incompatible with the nonvanishing Pauli form factors $G_2(0)$ and $G_5(0)$. In other words the possible existence of the latter two must involve *at least* one additional form factor $D_1(q^2)$ (all others, accompanied with momentum transfer, wil systematically be given in a separate paper), which should now be included in the matrix element of an asymmetric, conserved energy-momentum tensor in the form of

$$\bar{u}(p_2)\{(i/2)\gamma_{[k}p_{j]}D_1(q^2)\}u(p_1)\,, \tag{1.3}$$

to which a deviaton may couple with the strength λ. Needless to say, a graviton couples with the strength $\varkappa$ to the symmetric part (1.1); in particular the first term $G_1(q^2)$ gives rise to Newton's law in the long-range limit $(q \to 0)$ with $G_1(0) = 1$.

The second quantization of the skewsymmetric tensor field A_{kj} is carried out in Sect. **5**. The potential acting between a proton and an electron, generated by exchange of a deviaton according to (1.3) with $D_1(0) = 1$, is given by

$$V(\boldsymbol{r}) = (\hbar^2 c^2/8)(\lambda^2/4\pi)\cdot \cdot\{(8\pi/3)(\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^e)\delta(\boldsymbol{r}) - (1/r^3)[(\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^e) - (3/r^2)(\boldsymbol{\sigma}^p\cdot\boldsymbol{r})(\boldsymbol{\sigma}^e\cdot\boldsymbol{r})]\}\,. \tag{1.4}$$

As mentioned earlier, *whether or not the postulate b) holds at the quantal level depends on whether or not such a deviaton exists.* The best upper bound on the coupling constant of a deviaton to the proton or electron has been given from the precision measurements in quantum electrodynamics [9]:

$$\lambda^2/4\pi \leqslant 10^{-5}\hbar c/(\mathrm{GeV})^2\,. \tag{1.5}$$

The final Section is devoted to a brief discussion of neutrinos acting as a source of gravitation.

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(9) S. MIYAMOTO and T. NAKANO: *Progr. Theor. Phys.*, **45**, 295 (1971).

2. – Structure of gravitational interactions.

Our aim in this Section is to find all the possible gravitational interactions of a spin-$\frac{1}{2}$ field under the assumptions stated in the Introduction. We postulate a total Lagrangian composed of two parts (tensor densities will be written in boldface throughout this paper):

$$\boldsymbol{L} = \boldsymbol{L}' + \boldsymbol{L}_G , \tag{2.1}$$

and we will initially consider only the material Lagrangian $\boldsymbol{L}'$ involving the gravitational interaction, and do not specify, for the moment, the form of a free gravitational Lagrangian $\boldsymbol{L}_G$. In doing so, we shall obtain a condition that $\boldsymbol{L}_G$ must satisfy.

We start with the well-known Dirac Lagrangian

$$L_D = \tfrac{1}{2}(\bar{\psi}\gamma_k\psi_{,k} - \bar{\psi}_{,k}\gamma_k\psi) + m\bar{\psi}\psi , \tag{2.2}$$

where partial derivatives taken with respect to x_k are denoted by $,k$ and the gamma-matrices satisfy the anticommutation relations

$$\{\gamma_k, \gamma_l\} = 2\delta_{kl} ,$$

with Latin indices running from 1 to 4; the fourth component of x_k is pure imaginary, *i.e.* $x_4 = ict$, and we shall use units $c = \hbar = 1$. Now we demand that its action integral be invariant under a general co-ordinate transformation. The simplest solution to this problem was already given in detail in previous papers ([7]). Hence we here present a brief summary of the leading results obtained therein. Two steps are required for meeting our demand. First, the ordinary derivative appearing in L_D should be replaced by the *invariant* derivative as follows:

$$\psi_{,k} \to D_k\psi = b_k{}^\mu(x)\,\psi_{,\mu} , \tag{2.3a}$$

where a partial derivative taken with respect to the Riemann-space co-ordinate x^μ ($\mu = 0, 1, 2, 3$) is denoted by $,\mu$. The transformation properties of our gauge field $b_k{}^\mu(x)$ are similar to the conventional tetrads. It transforms, on the one hand, like a contravariant vector with respect to a general co-ordinate transformation as the position of the Greek index implies:

$$\bar{b}_k{}^\mu(\bar{x}) = (\partial\bar{x}^\mu/\partial x^\nu)\,b_k{}^\nu(x) , \tag{2.4}$$

and, on the other hand, like a Lorentz vector with respect to a Lorentz transformation acting on the Latin index,

$$\bar{b}_k{}^\mu(x) = \Lambda_{kl} b_l{}^\mu(x) , \tag{2.5a}$$

where the *constant* matrix Λ obeys the Lorentz-invariance condition

$$\Lambda^T \Lambda = I . \tag{2.5b}$$

The reason for this dual property characterized by Greek and Latin indices is easily understood by noting that $\psi_{,\mu}$ behaves like a covariant vector under a general co-ordinate transformation since a spinor field is a scalar in a Riemann space, and that the bilinear form $\bar{\psi}\gamma_k \psi_{,\mu}$ has therefore the mixed transformation properties, *i.e.* it behaves in the same way as $\psi_{,\mu}$ under a general co-ordinate transformation and like a Lorentz vector under a Lorentz transformation which defines the spin of a spinor field and has nothing to do with a general co-ordinate transformation (see (2.38)-(2.41) for the explicit specification). For further details we refer the readers to our previous papers. Secondly, the resultant Lagrangian should be multiplied by the determinant b, where

$$b = |\det\left(b_{k\mu}(x)\right)| ,$$

and the inverse gauge field $b_{k\mu}(x)$ is defined by

$$\begin{cases} b_k{}^\mu(x)\, b_{k\nu}(x) = \delta^\mu{}_\nu , \\ b_k{}^\mu(x)\, b_{l\mu}(x) = \delta_{kl} . \end{cases} \tag{2.6}$$

Thus, we obtain the modified Dirac Lagrangian

$$\boldsymbol{L}_M = b\{\tfrac{1}{2}(\bar{\psi}\gamma_k D_k \psi - D_k \bar{\psi}\cdot\gamma_k \psi) + m\bar{\psi}\psi\} . \tag{2.3b}$$

This is the simplest form for $\boldsymbol{L}'$, and we are now going to find the more general form with the help of (2.3*b*). This we do by following the analogy with the case of the electromagnetic interactions. The equations of motion derived from $\boldsymbol{L}_M$ are

$$\gamma_k D_k \psi + \tfrac{1}{2} V_k \gamma_k \psi + m\psi = 0 , \tag{2.7}$$

where

$$V_k = (b b_k{}^\mu)_{,\mu} . \tag{2.8}$$

The quadratic form of these equations is easily obtained by multiplying the dual operator

$$\gamma_m D_m + \tfrac{1}{2} V_m \gamma_m - m$$

from the left:

$$\{D_k D_k + i\sigma_{kl} D_k D_l - m^2 + V_k D_k + \tfrac{1}{2}(D_k V_k) + \\ + \tfrac{1}{4} V_k V_k + (i/2)\sigma_{kl}(D_k V_l)\}\,\psi = 0\,, \tag{2.9}$$

where $(D_k V_k)$ means the derivative acts on V_k only, not on a spinor field, and

$$\sigma_{kl} = (\gamma_k \gamma_l - \gamma_l \gamma_k)/2i\,. \tag{2.10}$$

In the presence of the minimal electromagnetic interaction, the Dirac equation is well known to be expressed as

$$\{\gamma_k(\partial_k - ieA_k) + m\}\,\psi = 0\,, \tag{2.11}$$

and its quadratic form becomes

$$\{\Box' + (e/2)\sigma_{kl}F_{kl} - m^2\}\,\psi = 0\,, \tag{2.12}$$

where $F_{kl} = A_{l,k} - A_{k,l}$ is the electromagnetic field strength, and $\Box'$ denotes $(\partial_k - ieA_k)^2$. Very interesting is to observe the parallelism between (2.9) and (2.12); $D_k D_k$ corresponds to the d'Alembertian operator in special relativity, and more significantly,

$$\sigma_{kl}(\partial_k - ieA_k)(\partial_l - ieA_l)\,\psi = (e/2)\,\sigma_{kl}F_{kl}\,,$$

$$\sigma_{kl}\,D_k D_l\,\psi = \sigma_{kl}\,C_{mkl}D_m\psi\,,$$

where

$$C_{mkl}(x) = -C_{mlk}(x) = b_k^{\ \mu}\, b_l^{\ \nu}(b_{m\mu,\nu} - b_{m\nu,\mu})\,. \tag{2.13}$$

We shall call this quantity the *gravitational field strength*. Our terminology may be justified by the above correspondence, and may be given a further basis by noting the fact that the quantity (2.13) behaves like a scalar under a general co-ordinate transformation. Our gravitational-field strength is, however, reducible under a Lorentz transformation, contrary to the electromagnetic field strength which belongs to a direct sum of the Lorentz-group representations $(1, 0) + (0, 1)$ and hence irreducible under a space reflexion.

To facilitate the argument, we decompose our field strength into the irreducible components with respect to a Lorentz transformation plus a space reflexion. This can be done with the help of the standard technique of Young tableau; we obtain the following three irreducible tensors.

1) The tensor belonging to the $(\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$ representations

$$T_{klm}(x) = \tfrac{1}{2}(C_{klm} + C_{lkm}) + \tfrac{1}{6}(\delta_{mk} V_l + \delta_{ml} V_k) - \tfrac{1}{3}\delta_{kl} V_m\,, \tag{2.14}$$

with the following symmetry properties:

$$\begin{cases} T_{klm}(x) = T_{lkm}(x)\,, \\ T_{klm}(x) + T_{lmk}(x) + T_{mkl}(x) = 0\,. \end{cases} \tag{2.15a}$$

Further, its trace vanishes, and hence it has only 16 components:

$$T_{kmm}(x) = 0 = T_{mmk}(x)\,. \tag{2.15b}$$

2) The vector of the $(\frac{1}{2}, \frac{1}{2})$ representation

$$V_k(x) = C_{mmk}(x) = (b b_k{}^{\mu})_{,\mu}\,. \tag{2.16}$$

3) The axial vector of the $(\frac{1}{2}, \frac{1}{2})$ representation

$$A_k(x) = (i/6)\,\varepsilon_{klmn} C_{lmn}(x)\,, \tag{2.17}$$

where ε_{klm} is the 4-dimensional Levi-Civita tensor; $\varepsilon_{1234} = 1$.

Thus, the field strength is now expressed in terms of these irreducible components

$$C_{klm}(x) = \tfrac{4}{3} T_{k[lm]} + \tfrac{2}{3}\delta_{k[l} V_{m]} + i\varepsilon_{klmn} A_n\,, \tag{2.18}$$

where square brackets enclosing indices indicate antisymmetrization.

With the field strength which is Lorentz tensor and invariant under a general co-ordinate transformation, we are ready to construct any kinds of the gravitational interactions invariant under arbitrary co-ordinate transformations. We enumerate a few examples. Any gravitational interaction of a spin-$\frac{1}{2}$ field takes the form of a product of the material part and the gravitational part. As for the latter, derivatives of the field strength could be derived by applying the invariant derivative (2.3a) successively to the field strength, *e.g.*

$$\begin{cases} D_i C_{klm} = b_i{}^{\mu} \partial_{\mu} C_{klm}\,, \\ D_i D_j C_{klm}\,, \end{cases} \tag{2.19}$$

and so forth, where the former contains the second derivatives of the gravitational field, while the latter the third derivatives. One may envisage a more complex form, *e.g.*

$$C_{klm} D_i C_{pqr} \tag{2.20}$$

which involves the first derivatives of the gravitational field quadratically. For the material part, we can also construct arbitrary kinds of the bilinear form in the spinor field, which are invariant under general co-ordinate transformations, by making use of the invariant derivatives, *e.g.*

$$\bar{\psi}\Gamma D_i\psi - D_i\bar{\psi}\cdot\Gamma\psi\,, \tag{2.21a}$$

$$\bar{\psi}\Gamma D_i D_j\psi - D_i D_j\bar{\psi}\cdot\Gamma\psi\,, \tag{2.21b}$$

and so on, where Γ denotes any set of the Dirac matrices 1, $i\gamma_5$, $i\gamma_k$, $i\gamma_k\gamma_5$, $i\sigma_{kl}$, $\sigma_{kl}\gamma_5$ with $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$.

Since there are too many possibilites in forming the invariant gravitational interactions, as seen above, it is useful to classify them into classes according to rank n of the derivatives applied to the gravitational field. Each such class may further be classified according to rank m of the derivatives acting on the spinor field. Now the third assumption specified in the Introduction enormously restricts the possible forms of the gravitational interaction. It requires that the rank of the derivatives appearing in the interaction Lagrangians be not higher than that appearing in the free material and gravitational Lagrangians, and that the derivatives of the spinor and gravitational fields should appear linearly in the interaction Lagrangian, respectively, if they are present there. For the material part, it then follows from the Dirac Lagrangian that m can only take the values 0 and 1. For the free gravitational part, n can also take the same values as m, because we can construct, as we shall do in the following Section, the most general form of the free gravitational Lagrangian in terms of our gravitational-field strengths. Here we note that this general form involves the Einstein's free gravitational Lagrangian as a special case, and that the apparent contradiction that the latter Lagrangian involves the second derivatives can be resolved by observing that these second derivatives appear only in the form of a four-divergence when expressed in terms of the gauge field $b_k{}^\mu(x)$. Thus we are led to the following classes of the gravitational interactions:

A) minimal interaction ($n = 0$):
only the case with $m = 1$ is allowed;

B) nonminimal interactions ($n = 1$):
*B*1) $m = 0$,
*B*2) $m = 1$.

The class *A*) consists of only the minimal-coupling Lagrangian $\boldsymbol{L}_M$ given by eq. (2.3*b*). No other couplings are allowed in *A*); one could yet presume $\xi\boldsymbol{L}_M + \zeta\boldsymbol{L}'_M$, where

$$\boldsymbol{L}'_M = b\tfrac{1}{2}(\bar{\psi}\gamma_5\gamma_k D_k\psi - D_k\bar{\psi}\cdot\gamma_5\gamma_k\psi) + m\bar{\psi}\psi\,.$$

However, for the choice $\xi^2-\zeta^2=1$, the resultant equations of motion reduce, in the special-relativistic limit, to the usual Dirac equation with a different representation of the gamma-matrices; otherwise these equations do not. Anyway, such a Lagrangian can be ruled out. Before writing explicitly the gravitational interactions of the class B), it is convenient to prepare some notation for bilinear forms of the spinor field. Now we use the following notations for the Hermitian bilinear forms coupled to the field strength, or more precisely its irreducible components which are of odd-rank Lorentz tensors. All the gamma-matrices are taken to be Hermitian throughout this paper. For nonderivative type

$$J_k(x)=i\bar{\psi}\gamma_k\psi\,, \tag{2.22a}$$

$$J_k^5(x)=i\bar{\psi}\gamma_5\gamma_k\psi\,, \tag{2.22b}$$

and four derivative type

$$J_{klm}(x)=(i/2)(\bar{\psi}\sigma_{kl}D_m\psi-D_m\bar{\psi}\cdot\sigma_{kl}\psi)\,, \tag{2.23a}$$

$$J_{klm}^5(x)=\tfrac{1}{2}(\bar{\psi}\sigma_{kl}\gamma_5 D_m\psi-D_m\bar{\psi}\cdot\sigma_{kl}\gamma_5\psi)\,, \tag{2.23b}$$

$$I_k(x)=(i/2)(\bar{\psi}D_k\psi-D_k\bar{\psi}\cdot\psi)\,,\qquad I_k^5(x)=\tfrac{1}{2}(\bar{\psi}\gamma_5D_k\psi-D_k\bar{\psi}\cdot\gamma_5\psi)\,. \tag{2.23c}$$

We note again that all these bilinear forms are Lorentz tensors invariant under a general co-ordinate transformation.

Now, for the class $B1$) we have the following four types:

$$\begin{cases} L_1=g_1bV_kJ_k\,,\\ L_2=g_2bV_kJ_k^5\,,\\ L_3=g_3bA_kJ_k\,,\\ L_4=g_4bA_kJ_k^5\,. \end{cases} \tag{2.24}$$

For the class $B2$) we have two interactions for the tensor part

$$\begin{cases} L_5=g_5bT_{mkl}J_{klm}\,,\\ L_6=g_6bT_{mkl}J_{klm}^5 \end{cases} \tag{2.25}$$

and eight interactions for the vector and axial vector field strengths

$$\begin{cases} L_7=g_7\,bV_kJ_{kmm}\,, & L_9=g_9\,bA_kJ_{kmm}\,,\\ L_8=g_8\,bV_kJ_{kmm}^5\,, & L_{10}=g_{10}bA_kJ_{kmm}^5\,.\\ L_{11}=g_{11}bV_kI_k\,, & L_{13}=g_{12}bA_kI_k\,,\\ L_{12}=g_{12}bV_kI_k^5\,, & L_{14}=g_{14}bA_kI_k^5\,, \end{cases} \tag{2.26}$$

All the coupling constants are real numbers. It should be noticed here that any permutations of Latin indices in (2.25) do not give rise to any new interactions owing to the symmetries exhibited in (2.15*a*) and in the definitions of the bilinear forms (2.23*a*) and (2.23*b*). This is also the case for the interactions (2.26). As for a two-component massless neutrino, there are only 3 Lagrangians, $\boldsymbol{L}_M$, $\boldsymbol{L}_1 \sim \boldsymbol{L}_2$, and $\boldsymbol{L}_3 \sim \boldsymbol{L}_4$, invariant under the « chiral projection » $\psi \to \{(1 \pm \gamma_5)/2\}\psi$ ($\sim$ means equivalence). We can now write the most general form of the gravitational-interaction Lagrangian in the form $\boldsymbol{L}' = \boldsymbol{L}_M + \boldsymbol{L}_{NM}$, where the index M means not only the basic material part but also the minimal interaction part, and the indices NM refer to the nonminimal interactions defined above.

We have so far not touched upon the connection between the gravitational interaction implied by Einstein's theory and ours listed above. Although his theory cannot directly be applied to a system involving a spin-$\frac{1}{2}$ particle, we shall transcribe it into our language. As is well known, Einstein's prescription to introduce the gravitational interaction amounts to replacing the ordinary derivative appearing in a material Lagrangian for an integer-spin field by the covariant derivative with respect to the Christoffel symbol $\Gamma^{\mu}{}_{\lambda\nu}$, which is denoted by a semicolon and usually referred to as the *minimal* gravitational interaction in the literature. However, it is not minimal but *nonminimal* in our framework, as we shall see in the following. First, let $\phi^{\mu}(x)$ be an arbitrary contravariant vector field in a Riemann space, to which we apply this covariant derivative:

$$\phi^{\mu}{}_{;\lambda} = \phi^{\mu}{}_{,\lambda} + \Gamma^{\mu}{}_{\nu\lambda}\phi^{\nu} \,. \tag{2.27a}$$

Secondly, as often adopted in the literature, we use the familiar relation between the metric tensor $g_{\mu\nu}$ and our gauge field

$$\begin{cases} g_{\mu\nu}(x) = b_{k\mu}(x)\, b_{k\nu}(x) \,, \\ g^{\mu\nu}(x) = b_k{}^{\mu}(x)\, b_k{}^{\nu}(x) \,. \end{cases} \tag{2.28}$$

If we rewrite the Christoffel symbol by means of this relation, and multiply both sides of (2.27*a*) by $b_k{}^{\lambda} b_{m\mu}$, we find

$$b_k{}^{\lambda} b_{m\mu} \phi^{\mu}{}_{;\lambda} = D_k \phi_m - D_{mlk}\phi_l \,, \tag{2.27b}$$

where D_k already appeared in (2.3*a*) and (2.19), and

$$\begin{cases} \phi_m \;\; = b_{m\mu}\phi^{\mu} \,, \\ D_{mlk} = \frac{1}{2}(C_{mlk} - C_{lmk} - C_{kml}) \,. \end{cases} \tag{2.29}$$

This formula can further be rewritten, in terms of the spin matrix S^1_{kl} for a vector field ϕ_m; its matrix elements are

$$(S^1_{kl})_{mn} = -i(\delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}) . \tag{2.30a}$$

It is easy to show that this matrix is a representation of the internal Lorentz group generators and hence satisfies the Lorentz-group commutation relations

$$[S_{kl}, S_{mn}] = i(\delta_{km}S_{ln} + \delta_{ln}S_{km} - \delta_{kn}S_{lm} - \delta_{lm}S_{kn}) . \tag{2.30b}$$

Thus, denoting the left-hand side of (2.27*b*) by $\mathscr{D}_k\phi_m$, we obtain from (2.27*b*)

$$\mathscr{D}_k\phi_m = D_k\phi_m - (i/2)D_{ijk}(S^1_{ij})_{mn}\phi_n , \tag{2.27c}$$

or equivalently, but more concisely,

$$\mathscr{D}_k\phi = D_k\phi - (i/2)D_{ijk}S^1_{ij}\phi , \tag{2.27d}$$

where ϕ_m is written as a column vector ϕ, on which the spin matrix acts. In passing we note that the second term on the right-hand side of (2.27*d*) vanishes for a particular affine connexion, $\Gamma^{\mu}{}_{\nu\lambda} = b_k{}^{\mu} b_{k\nu,\lambda}$, used in (2.27*a*). Now we are ready to apply the Einstein's gravitational coupling to a spinor field by using the form (2.27*d*); the latter form can be *algebraically* continued to the case of a spin-$\frac{1}{2}$ field by modifying the spin matrix only. Thus we have

$$\begin{cases} \mathscr{D}_k\psi = D_k\psi - (i/2)D_{ijk}S_{ij}\psi , \\ \mathscr{D}_k\bar{\psi} = D_k\bar{\psi} + (i/2)D_{ijk}\bar{\psi}S_{ij} , \end{cases} \tag{2.31}$$

where the spin matrix for a spinor field is well known:

$$S_{ij} = \tfrac{1}{2}\sigma_{ij} . \tag{2.32}$$

If we insert *Einstein's derivative* (2.31) into the free Dirac Lagrangian (2.2), we obtain

$$L_M - \tfrac{3}{4}bA_kJ^5_k = L_M + L_4 , \qquad \text{with } g_4 = -\tfrac{3}{4}. \tag{2.33}$$

We shall call this the *Einstein gravitational interaction Lagrangian*. Here we note that the extension from (2.27*d*) to (2.31) is of course not unique. Nevertheless, we say that the latter corresponds to the Einstein gravitational coupling in view of the following facts. The energy-momentum tensor defined as the source of gravitation is defined by

$$T_{kl} = -(\delta L'/\delta b_{k\mu})\, b_{l\mu} , \tag{2.34}$$

where $(\delta/\delta b_{k\mu})$ stands for the variational derivative with respect to $b_{k\mu}$. For $\boldsymbol{L}'=\boldsymbol{L}_M$, it follows immediately that the energy-momentum tensor is not symmetric in this case:

$$\boldsymbol{T}^M_{kl}=(b/2)(\bar{\psi}\gamma_l D_k\psi - D_k\bar{\psi}\cdot\gamma_l\psi)-\delta_{kl}\boldsymbol{L}_M\,. \tag{2.35}$$

In the special relativistic limit specified by $b_k{}^\mu\to\delta_k{}^\mu$, this energy-momentum tensor reduces to the well-known canonical energy-momentum tensor which is obviously not symmetric:

$$\boldsymbol{T}^M_{kl}\to T^C_{kl}=\tfrac{1}{2}(\bar{\psi}\gamma_l\psi_{,k}-\bar{\psi}_{,k}\gamma_l\psi)-\delta_{kl}L_D\,, \tag{2.36a}$$

$$=(\delta L_D/\delta\psi)\psi_{,k}+\bar{\psi}_{,l}(\delta L_D/\delta\bar{\psi})-\delta_{kl}L_D\,. \tag{2.36b}$$

On the other hand, when we choose the Lagrangian (2.33) as $\boldsymbol{L}'$, we can obtain the symmetric energy-momentum tensor in this case and observe the following transition at the special relativistic limit:

$$\boldsymbol{T}_{kl}=-\left\{\frac{\delta}{\delta b_{k\mu}}(\boldsymbol{L}_M-(3b/4)A_kJ^5_k)\right\}b_{l\mu}\to T^B_{kl}=\tfrac{1}{2}(T^C_{kl}+T^C_{lk})\,, \tag{2.37}$$

where T^B_{kl} coincides with Belinfante's symmetric energy-momentum tensor for a spinor field. This calculation is rather lengthy, but was already given in detail in our previous papers (7), hence we here quoted only the result. We may now better understand the role played by the axial vector coupling $A_kJ^5_k$ in Einstein's gravitational interaction; it is, among others, just to counteract the antisymmetric part of $\boldsymbol{T}^M_{kl}$, thus rendering the expression (2.37) symmetric. The proof of such a cancellation by calculating explicitly the energy-momentum tensor is, however, complicated, because the equations of motion for a spinor field must be used repeatedly to reduce the complicated expression for the energy-momentum tensor.

We shall now supply a more powerful and simpler method to prove the symmetry property of the energy-momentum tensor in the following. With this method we shall obtain one of the most significant results derived in this paper. First, our Lagrangian $\boldsymbol{L}'$ is, by construction, invariant under an *internal* Lorentz transformation, or equivalently its infinitesimal form specified by

$$x^\mu\to\bar{x}^\mu=x^\mu\,, \tag{2.38}$$

$$b_k{}^\mu(x)\to\bar{b}_k{}^\mu(x)=b_k{}^\mu(x)+(i/2)\omega_{ij}(S^1_{ij})_{kl}\,b_l{}^\mu(x)\,, \tag{2.39}$$

$$\psi(x)\to\psi'(x)=\psi(x)+(i/2)\omega_{ij}S_{ij}\psi(x)\,, \tag{2.40}$$

$$\omega_{ij}=-\omega_{ji}\quad\text{(infinitesimal constant parameters)}, \tag{2.41}$$

with the spin matrices S^1_{ij} and S_{ij} as given by (2.30a) and (2.32), respectively; the transformation (2.39) was already mentioned in (2.5a). If we insert these transformations into the invariance conditions of $\boldsymbol{L}'$ (see (2.22) of ref. [7]), we can derive the following relation, using the equations of motion for a spinor field:

(2.42) $$\boldsymbol{T}_{kl} - \boldsymbol{T}_{lk} = (\boldsymbol{S}^{\mu}{}_{kl} + {}^{NM}\boldsymbol{S}^{\mu}{}_{kl})_{,\mu}\,,$$

where the energy-momentum tensor $\boldsymbol{T}_{kl}$ is of course given by (2.34), and the spin angular-momentum tensor consists of two parts:

(2.43a) $$\boldsymbol{S}^{\mu}{}_{kl} = (\partial \boldsymbol{L}'/\partial\psi_{,\mu})\, iS_{kl}\psi\,,$$

(2.44a) $${}^{NM}\boldsymbol{S}^{\mu}{}_{kl} = (\partial \boldsymbol{L}_{NM}/\partial b_{m\nu,\mu})(iS^1_{kl})_{mn}\, b_{n\nu}\,.$$

A simple calculation yields for $\boldsymbol{L}' = \boldsymbol{L}_M$,

(2.43b) $$\boldsymbol{S}_{mkl} = b_{m\mu}\boldsymbol{S}^{\mu}{}_{kl} = (b/2)\,\varepsilon_{mkln}\,\bar{\psi}\gamma_5\gamma_n\psi\,,$$

where we have used the anticommutation relation

$$\{\gamma_k, \sigma_{lm}\} = (2/i)\,\varepsilon_{klmn}\gamma_5\gamma_n\,.$$

Clearly, this relation (2.42) may be regarded as a simple generalization of the Tetrode formula in special relativity [10]

(2.45) $$T^c_{kl} - T^c_{lk} = S_{mkl,m}\,.$$

Here T^c_{kl} is the canonical energy-momentum tensor (2.36) and S_{mkl} the canonical spin angular-momentumtensor

(2.46) $$S_{mkl} = (\partial L_{\mathrm{D}}/\partial\psi_{,m})\, iS_{kl}\psi\,.$$

Now it is quite easy to show the expression inside parentheses on the right-hand side of (2.42) vanishes for the Einstein's gravitational interaction Lagrangian. The fact that the equation of motion for a spinor field is already taken into account in the relation (2.42) enormously simplifies the argument by which it is decided whether the energy-momentum tensor defined by (2.34) is symmetric or not. The expression (2.44a) can further be simplified by noting that $\boldsymbol{L}_{NM}$ is a functional of the gravitational field strength C_{klm}:

(2.44b) $${}^{NM}\boldsymbol{S}_{mkl} = b_{m\mu}{}^{NM}\boldsymbol{S}^{\mu}{}_{kl} = 2\left(\frac{\partial \boldsymbol{L}_{NM}}{\partial C_{klm}} - \frac{\partial \boldsymbol{L}_{NM}}{\partial C_{lkm}}\right).$$

[10] H. TETRODE: *Zeits. Phys.*, **48**, 52 (1928); **49**, 858 (1928).

It follows immediately from this formula that (2.44b) does not vanish for $\boldsymbol{L}_{NM} = \boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{L}_5, \boldsymbol{L}_6, \boldsymbol{L}_7$ and $\boldsymbol{L}_8$, and that *the totally antisymmetric contributions arise from* $\boldsymbol{L}_3, \boldsymbol{L}_4, \boldsymbol{L}_9$, *and* $\boldsymbol{L}_{10}$, *which contain only the axial vector part of the gravitational field strength*; we here write them down, by suppressing indices NM for the convenience of typography:

$$(2.47)\quad \begin{cases} \boldsymbol{S}_{mkl} = (2g_3/3)\, b\varepsilon_{klmn}\bar{\psi}\gamma_n\psi & \text{for } \boldsymbol{L}_3, \\ \boldsymbol{S}_{mkl} = (2g_4/3)\, b\varepsilon_{klmn}\bar{\psi}\gamma_5\gamma_n\psi & \text{for } \boldsymbol{L}_4, \\ \boldsymbol{S}_{mkl} = -(2ig_9/3)\, b\varepsilon_{klmn}J_{njj} & \text{for } \boldsymbol{L}_9, \\ \boldsymbol{S}_{mkl} = -(2ig_{10}/3)\, b\varepsilon_{klmn}J^5_{njj} & \text{for } \boldsymbol{L}_{10}. \end{cases}$$

After having investigated all the gravitational interactions, we are now led to the following conclusion: *All the gravitational interactions enumerated in this Section, except for Einstein's gravitational interaction* (2.33), *give rise to an asymmetric energy-momentum tensor as the source of gravitation.*

The above conclusion imposes a serious condition on a free gravitational Lagrangian $\boldsymbol{L}_G$, namely $\boldsymbol{L}_G$ should be so chosen that its variational derivative with respect to $b_{k\mu}(x)$, multiplied by $b_{l\mu}(x)$, is not symmetric when some gravitational interactions other than the Einstein gravitational interaction are present. Hence in this case $\boldsymbol{L}_G$ cannot be the Riemann scalar (expressed in terms of $b_{k\mu}(x)$ by (2.28)). In other words, we would have to modify drastically Einstein's theory of gravitation at the very root.

We close this Section by remarking that any deviations from the Einstein gravitational interaction must inevitably lead us to a more general form for the free gravitational Lagrangian than the conventional Einstein one, *i.e.* the Riemann scalar.

3. – Structure of the free gravitational Lagrangian.

As was shown in the preceding Section, we have to construct a more general free gravitational Lagrangian than Einstein's when we consider those gravitational interactions other than Einstein's as given by (2.33). In this Section we shall construct the most general form for $\boldsymbol{L}_G$ under the four assumptions specified in the Introduction. Particularly, in view of the third assumption demanding that our $\boldsymbol{L}_G$ should consist of at most first derivatives of the gauge field $b_k{}^\mu(x)$, we shall further postulate that $\boldsymbol{L}_G$ should depend upon derivatives of the gravitational field quadratically, if these are present. With these assumptions in mind, we are now led to the most general form for a free gravitational Lagrangian with six arbitrary coefficients

$$(3.1)\qquad \boldsymbol{L} = b(\alpha T^2_{klm} + \beta V^2_k + \gamma A^2_k + \delta + \eta V_k A_k + i\zeta\varepsilon_{klmn}T_{jkl}T_{jmn}),$$

where the field strength and its irreducible components were all given in Sect. **2.** We notice here that this Lagrangian is obviously invariant under the internal Lorentz transformation defined by (2.38)-(2.41). The last member in (3.1) can be absorbed into the fifth term by virtue of identity

$$(\tfrac{3}{2})(bb_m{}^\mu A_m)_{,\mu} = b\big(V_k A_k - (2i/9)\,\varepsilon_{klmn} T_{jkl} T_{jmn}\big)\,,$$

hence we shall drop it hereafter. To permit ready comparison with the celebrated Einstein's theory of gravitation, we put our Lagrangian into the more convenient form (see Appendix A for derivation)

$$(3.2)\quad \boldsymbol{L}_G = \boldsymbol{L}_{\mathrm{E}}/2\varkappa^2 + b\{(\alpha+\beta)V_k^2 - (9/4\lambda^2)A_k^2 + \eta V_k A_k\} - (bb_m{}^\mu V_m)_{,\mu}/\varkappa^2\,.$$

Here we denote the Einstein's gravitational Lagrangian by $\boldsymbol{L}_{\mathrm{E}}$, *i.e.* $\boldsymbol{L}_{\mathrm{E}} = \sqrt{-g}\cdot\cdot(R+2\bar{\lambda})$, where R is of course the Riemann scalar now expressed in terms of our gravitational variable $b_k{}^\mu(x)$ and $\bar{\lambda}$ the cosmological constant adjusted as $\bar{\lambda} = \delta/3\alpha$. (See Appendix A for the explicit form of R.) $\varkappa$ is Einstein's gravitational constant, related to one of our parameters as

$$(3.3)\quad \begin{cases} 1/\varkappa^2 = 3\alpha\,, \\ \varkappa^2/4\pi = 2G/c^4 = 1.34\cdot 10^{-38}\hbar c/(\mathrm{GeV})^2\,, \end{cases}$$

where G denotes Newton's gravitational constant, and

$$(3.4)\quad 1/\lambda^2 = \alpha - 4\gamma/9\,.$$

Variation of a total Lagrangian (2.1) with respect to $b_{k\mu}(x)$ gives rise to the field equations, upon multiplying these by $b_{l\mu}(x)$:

$$(3.5)\quad \boldsymbol{G}_{kl} - \varkappa^2 \boldsymbol{B}_{kl} = -\varkappa^2 \boldsymbol{T}_{kl}\,,$$

where the first member on the left is the familiar Einstein's tensor now written in terms of our gravitational field $b_k{}^\mu(x)$

$$(3.6)\quad \boldsymbol{G}_{kl} = -\tfrac{1}{2}(\delta \boldsymbol{L}_{\mathrm{E}}/\delta b_{k\mu})\, b_{l\mu} = b(R_{kl} - \tfrac{1}{2}\delta_{kl}R - \delta_{kl}\bar{\lambda}) = \boldsymbol{G}_{lk}\,.$$

The explicit form for the contracted curvature tensor R_{kl} is given in Appendix A. The source of gravitation $\boldsymbol{T}_{kl}$ was already defined by (2.34), and

$$(3.7)\quad \boldsymbol{B}_{kl} = -(b_m{}^\mu \boldsymbol{F}_{klm})_{,\mu} + \{\tfrac{1}{2}C_{lmn}\boldsymbol{F}_{kmn} - C_{mnk}\boldsymbol{F}_{mnl}\} + \delta_{kl}\boldsymbol{L}'_G\,,$$

where $\boldsymbol{L}'_G$ stands for the second term of (3.2), enclosed by curly brackets, and

$$(3.8)\quad \boldsymbol{F}_{klm} = 4b\{(\alpha+\beta)\,\delta_{k[l}V_{m]} + (9/4\lambda^2)\,i\varepsilon_{klmn}A_n/6 + (\eta/2)(\delta_{k[l}A_{m]} - (i/6)\,\varepsilon_{klmn}V_n)\}.$$

As is clear from these field equations, the energy-momentum tensor can become symmetric if and only if the free parameters are subject to the following conditions:

$$\alpha + \beta = 0\,, \tag{3.9}$$

$$\alpha - 4\gamma/9 = 0\,, \tag{3.10}$$

$$\eta = 0\,. \tag{3.11}$$

In other words, when $\boldsymbol{T}_{kl}$ has a skewsymmetric part, the free parameters cannot take these special values given by (3.9)-(3.11). As will be shown in the following Section, P, C and/or T violating gravitational interactions may be obtained, in the weak field approximation, from the nonminimal gravitational interactions of the class B) defined in Sect. **2**. Hence, one should add the second member in (3.2) to the Einstein's free gravitational Lagrangian if one wishes to discuss the violation of any of P, C and T.

Now we are going to investigate to what extent the arbitrary parameters can be chosen beyond the special values (3.9)-(3.11), by considering some physical «boundary» conditions. This we do first by using the linear approximation

$$b_k{}^\mu(x) \to \delta_k{}^m + a_k{}^m(x)\,, \qquad b_{k\mu}(x) \to \delta_{km} + a_{km}(x)\,, \tag{3.12a}$$

where the linearized fields $a_k{}^m(x)$ and $a_{km}(x)$ are related to each other by (2.6), *i.e.*

$$a_k{}^m(x) = -a_{mk}(x)\,. \tag{3.12b}$$

We note that distinction between Greek and Latin indices is no longer necessary in the linear approximation. We also note in passing that the determinant b becomes $1 + a_{mm}(x)$ in the weak-field approximation. We then split the field equations into the symmetric and antisymmetric parts

$$\boldsymbol{G}_{kl} - \varkappa^2 \boldsymbol{B}_{(kl)} = -\varkappa^2 \boldsymbol{T}_{(kl)}\,, \tag{3.13}$$

$$\boldsymbol{B}_{[kl]} = \boldsymbol{T}_{[kl]}\,, \tag{3.14a}$$

where

$$\boldsymbol{B}_{[kl]} = -(b_m{}^\mu \boldsymbol{F}_{[kl]m})_{,\mu}\,, \tag{3.14b}$$

because the second term of (3.7), enclosed by curly brackets, is a symmetric tensor. Applying the linear approximation to (3.13), we obtain

$$\Box S_{kl} + \varkappa^2(\alpha + \beta)(\partial_k \partial_l - \delta_{kl}\Box) S_{mm} + (i/3)\,\eta\varkappa\lambda\varepsilon_{mnj(k}\partial_{l)}\partial_j A_{mn} = -\varkappa \boldsymbol{T}_{(kl)}\,, \tag{3.15}$$

where we suppressed the cosmological term multiplied by $\bar{\lambda}$, and we used the following notations and the divergence-free conditions: $\Box$ is the d'Alembertian operator and

$$\varkappa S_{kl} = a_{(kl)} - \tfrac{1}{2}\delta_{kl} a_{mm} , \tag{3.16}$$

$$\partial_l S_{kl} = 0 . \tag{3.17}$$

Likewise, we obtain from (3.14)

$$\Box A_{kl} - (i/3)\eta\lambda^2 \varepsilon_{mnj[k}\partial_{l]}\partial_j A_{mn} = -\lambda T_{[kl]} , \tag{3.18}$$

where we adopted the notation

$$\lambda A_{kl} = a_{[kl]} , \tag{3.19}$$

and also the divergence-free condition

$$\partial_l A_{kl} = 0 . \tag{3.20}$$

At this point we emphasize that there is no *a priori* reason to identify λ with Einstein's gravitational constant, hence we shall leave it as a free parameter, assuming it so small that the linear approximation may hold for a skew-symmetric field (this assumption will be verified in Sect. **5**). The parity-violating term appears in both sets of field equations, (3.15) and (3.18), thus rendering them coupled equations, while the vector part V_k^2 and the axial vector part A_k^2 contributes to the symmetric and antisymmetric parts of the field equations, respectively. If we take the trace of (3.15), we get

$$\Box S_{mm} = \varkappa(\alpha/\beta) T_{mm} , \tag{3.21}$$

which further simplifies (3.15) as

$$\Box S_{kl} + \varkappa^2(\alpha+\beta)\partial_k\partial_l S_{mm} + i\eta\varkappa\lambda\varepsilon_{mnj(k}\partial_{l)}\partial_j A_{mn}/3 = -\varkappa\left(T_{(kl)} - \delta_{kl}\frac{\alpha+\beta}{3\beta}T_{mm}\right) . \tag{3.22}$$

It is worth noting that, for the static gravitational source to which only the $k = l = 4$ components can contribute, (3.22) becomes identical with the Poisson equations except for the additional factor dependent on the arbitrary parameters

$$\Delta S_{00}(x) = -\varkappa\frac{2\beta-\alpha}{3\beta}T_{00}(x) , \tag{3.23}$$

where $S_{00} = -S_{44}$ and $T_{00} = -T_{44}$.

Apart from the linear approximation, one can determine the arbitrary parameters involved in $\boldsymbol{L}_G$ by comparing a solution of our field equations with the exact solution to Einstein's equation of gravitation. It was shown in our previous paper (7) that one has to set the particular relation (3.9) for the parameters so as to reach the well-known Schwarzschild's solution of the static, spherically symmetric gravitational field. It was also pointed out there that it is not necessary to take the special values (3.10) and (3.11) for that purpose, because all the components of the axial vector field $A_k(x)$ vanish in the case of the static and spherically symmetric gravitational field.

However, we can discuss the physical meaning of η from a different theoretical side. Recently the retarded solutions to (3.18) and (3.22) have been given in their most general form by SETO (11). First, for $\eta = 0$ the retarded solution to (3.18) is well known; it is given by the formula (B.3) in Appendix B, where S_{kl}, $T_{(kl)}$ and $\varkappa$ are now replaced by A_{kl}, $T_{[kl]}$ and λ, respectively. In this case the retarded solution to (3.22) was already given in a previous letter (12) and hence we quote the result obtained therein: For $\alpha + \beta \neq 0$ the S_{kj}-field propagates not only on the light-cone but also *inside* the light-cone, thus in conflict with Huygens' principle. Obviously, for $\alpha + \beta = 0$ the solution is the standard one given by (B.3). Second, for $\eta \neq 0$, the retarded solution to (3.18) takes the form of

$$(3.18a) \qquad A_{kj}(x) = \frac{\lambda}{2\pi} \frac{1}{[1 + (\eta\lambda^2/3)]^2} \int \theta(y^0)\,\delta(y^2)\,\mathrm{d}^4y \cdot \cdot \{T_{[kj]}(x-y) + (\eta\lambda^2/3)\tilde{T}_{[kj]}(x-y)\}\,,$$

where $\tilde{T}_{[kl]} = \varepsilon_{klmn}T_{[mn]}/2i$. The divergence-free condition (3.20) imposes the condition on the energy-momentum tensor

$$(3.18b) \qquad \partial_j\{T_{[kj]} + (\eta\lambda^2/3)\tilde{T}_{[kj]}\} = 0\,.$$

The solution (3.18a), together with the condition (3.18b), is inserted into (3.22) only to give the complete retarded solution ($\partial_j = \partial/\partial x_j$)

$$(3.22a) \qquad S_{kj}(x) = \frac{\varkappa}{2\pi}\int\theta(y^0)\,\delta(y^2)\,T_{(kj)}(x-y)\,\mathrm{d}^4y + \\ + \frac{\varkappa}{4\pi^2}\int\theta(y^0)\,\delta(y^2)\,\theta(z^0)\,\delta(z^2)\,\mathrm{d}^4y\,\mathrm{d}^4z\{\partial_j\,\partial_m T_{[mk]}(x-y-z) + \partial_k\,\partial_m T_{[mj]}\} + \\ + \frac{\varkappa}{2\pi}\frac{\varkappa^2(\alpha+\beta)}{1-3\varkappa^2(\alpha+\beta)}\left\{\delta_{kj}\int\theta(y^0)\,\delta(y^2)\,T_{mm}(x-y)\,\mathrm{d}^4y + \right. \\ \left. + \frac{1}{2\pi}\int\theta(y^0)\,\delta(y^2)\,\theta(z^0)\,\delta(z^2)\,\partial_k\partial_j T_{mm}(x-y-z)\,\mathrm{d}^4y\,\mathrm{d}^4z\right\}.$$

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(11) The author is indebted to Dr. N. SETO for illuminating discussions of the solution in private communications when this work appeared in the form of preprint.

(12) K. HAYASHI: *Lett. Nuovo Cimento*, **5**, 739 (1972).

If the energy-momentum tensor is conserved, $\partial_k T_{kj} = 0$, then the divergence-free condition (3.17) is satisfied. In this case the S_{kj}-field propagates inside the light-cone even for $\alpha + \beta = 0$, in conflict with Huygens' principle. The trouble lies in the second term of (3.22*a*), where a skewsymmetric part of the energy-momentum tensor acts as a source of the S_{kj}-field. Evidently, this infamous term vanishes for $\eta = 0$, although η does not appear therein explicitly.

To sum up, we shall take $\eta = 0$ and $\alpha + \beta = 0$ in order to retain Huygens' principle on the one hand and Schwarzschild's solution on the other hand. We make a short remark here. If we take free Dirac particles (subject to the Dirac Lagrangian (2.2)) as a source of gravitation, then the energy-momentum tensor is given by the canonical one (2.36), whose skewsymmetric part is conserved, $\partial_j T_{[kj]} = 0$, but its dual is not, $\partial_j \tilde{T}_{[kj]} \neq 0$. Thus, (3.18*b*) requires $\eta = 0$.

Consequently, we are left with a single free parameter λ, which cannot be zero when the energy-momentum tensor of matter fields is not symmetric. Finally, we obtain the following two sets of the field equations in the weak-field limit:

$$\Box S_{kj} = -\varkappa T_{(kj)}, \tag{3.24}$$

$$\Box A_{kj} = -\lambda T_{(kj)}, \tag{3.25}$$

where the linearized field variables are both subject to the divergence-free condition, (3.17) and (3.20). Needless to say, the first set is the indispensable ingredient of any reasonable gravitational theory which must be able to account for Newton's law of gravitation.

The conclusion we draw is therefore the following. When the energy-momentum tensor defined as a source of gravitation is not symmetric, the field equations for gravitation must inevitably be supplemented, for consistency, by (3.14) with the conditions (3.9) and (3.11). In the weak field limit, the familiar field equations for the linearized gravitational field S_{kj}, (3.24), must inevitably, for consistency, be accompanied by a new set of the field equations (3.25) for a *skewsymmetric tensor field* A_{kj}, whose source is an antisymmetric part of the energy-momentum tensor, related to the spin of the source particles involved by the Tetrode formula, as seen by (2.45). Such a field, however, has not yet been observed.

4. – Classification of gravitational interactions by P, C and T.

In this Section we shall discuss the discrete symmetries such as parity (P), charge conjugation (C) and time reversal (T) in the weak-field limit where we can apply the usual quantum field theory.

First, with the help of the weak-field approximation, we obtain the linearized field strength

$$C_{klm}(x) = a_{kl,m} - a_{km,l}\,, \tag{4.1}$$

and thereby its irreducible parts with the correct choice of the linearized field variables, (3.16) and (3.19)

$$V_k(x) = -\varkappa\,\tfrac{1}{2} S_{mm,k}\,, \tag{4.2}$$

$$A_k(x) = \lambda(i/3)\,\varepsilon_{klmn} A_{lm,n}\,. \tag{4.3}$$

Inserting these expressions into $\boldsymbol{L}_1, \ldots, \boldsymbol{L}_4$ and $\boldsymbol{L}_7, \ldots, \boldsymbol{L}_{10}$, we get the explicit forms for these gravitational interactions in the linear approximation. For the irreducible tensor $T_{klm}(x)$, on the other hand, it has a rather complicated expression, but can be simplified when coupled to the bilinear forms J_{klm} or J^5_{klm} which are antisymmetric with respect to the first pair of indices:

$$T_{mkl} J_{klm} = \varkappa\tfrac{3}{2} S_{mk,l} J_{klm} - \lambda A_{kl,m} J_{k(lm)} - V_k J_{kmm}\,, \tag{4.4}$$

$$T_{mkl} J^5_{klm} = \varkappa\tfrac{3}{2} S_{mk,l} J^5_{klm} - \lambda A_{kl,m} J^5_{k(lm)} - V_k J^5_{kmm}\,. \tag{4.5}$$

Thus, we now find the linearized forms for all the gravitational interactions enumerated in Sect. **2**, which will not be written in boldface but in usual capital letters. In particular, the minimal gravitational interaction becomes

$$\boldsymbol{L}_M \to L_D - a_{kl} T^C_{kl}\,, \tag{4.6}$$

where T^C_{kl} is the *nonsymmetric* canonical energy-momentum tensor defined by (2.36), while the Einstein gravitational interaction is linearized as

$$\boldsymbol{L}_M - \tfrac{3}{4}\, b A_k J^5_k \to L_D - a_{(kl)} T^B_{kl}\,, \tag{4.7}$$

where T^B_{kl} is Belinfante's *symmetric* energy-momentum tensor given by the second line of (2.37) for a spinor field. As has often been remarked in previous Sections, a skewsymmetric field would couple to a spinor field if the gravitational interaction deviates from Einstein's (4.7).

Now we are going to decide whether or not these interactions conserve $\boldsymbol{P}$, C and/or T. For this purpose, we shall henceforth take the linearized fields $S_{kl}(x)$ and $\boldsymbol{A}_{kl}(x)$ as quantized free Hermitian fields representing neutral massless spin-2 and -0 particles, respectively. First, we may fix the charge-conjugation parity of these fields as $+$, in view of the fact that the gravitational coupling was introduced in the symmetric way for the electron and its antiparticle, as seen from the Dirac equation (2.7) and its adjoint:

$$(4.8)\qquad \begin{cases} \mathscr{C}S_{kl}(x)\mathscr{C}^{-1} = +S_{kl}(x)\,, \\ \mathscr{C}A_{kl}(x)\mathscr{C}^{-1} = +A_{kl}(x)\,, \end{cases}$$

where $\mathscr{C}$ is a unitary operator for charge conjugation. Then, the gravitational interactions given by (4.6) and (4.7) obviously conserve C, because the canonical energy-momentum tensor (2.36) has the charge-conjugation parity $+$. Next, we postulate the time-reversal property under Wigner's time inversion as

$$(4.9)\qquad \begin{cases} S_{kl}(\boldsymbol{x}, t) \to \varepsilon_k \varepsilon_l S_{kl}(\boldsymbol{x}, -t)\,, \\ A_{kl}(\boldsymbol{x}, t) \to \varepsilon_k \varepsilon_l A_{kl}(\boldsymbol{x}, -t)\,, \end{cases}$$

where $\varepsilon_k = (---+)$. Finally, for the parity operation, we have

$$(4.10)\qquad \begin{cases} S_{kl}(\boldsymbol{x}, t) \to \varepsilon_k \varepsilon_l S_{kl}(-\boldsymbol{x}, t)\,, \\ A_{kl}(\boldsymbol{x}, t) \to \varepsilon_k \varepsilon_l A_{kl}(-\boldsymbol{x}, t)\,, \end{cases}$$

which guarantee the invariance of the commutation rules and the gravitational Lagrangians (4.6) and (4.7) under P. With these transformation properties in mind, we convince ourselves that the linearized form of the minimal gravitational interaction and Einstein's gravitational interaction do conserve P, C and T, separately. In other words, this is trivial because we have required, in view of the field equations (3.24) and (3.25), that $S_{kl}(x)$ and $A_{kl}(x)$ transform like the canonical energy-momentum tensor under C, P and T.

As for a spinor field, the transformation properties of the bilinear forms, I_k, I^5_k, J_k, J^5_k, J_{klm} and J^5_{klm}, defined by (2.20)-(2.23) are easily derived by using the standard method appearing in the text-book; here we note that the invariant derivative D_k involved in the bilinear forms, J_{klm} and J^5_{klm}, should henceforth be replaced by the usual partial derivative in special relativity, because the field strengths coupled to these bilinear forms carry $\varkappa$ and λ. We summarize the results in Table I.

TABLE I. – *The transformation properties of the bilinear forms under C, P and T: $\varepsilon_k = (---+)$.*

	C	P	T
J_k, I_k	$-$	ε_k	ε_k
J^5_k	$+$	$-\varepsilon_k$	ε_k
T^C_{kl}	$+$	$\varepsilon_k \varepsilon_l$	$\varepsilon_k \varepsilon_l$
J_{klm}	$+$	$\varepsilon_k \varepsilon_l \varepsilon_m$	$-\varepsilon_k \varepsilon_l \varepsilon_m$
J^5_{klm}	$+$	$-\varepsilon_k \varepsilon_l \varepsilon_m$	$\varepsilon_k \varepsilon_l \varepsilon_m$
I^5_k	$-$	$-\varepsilon_k$	$-\varepsilon_k$

Now, it is easy to classify all the linearized gravitational interactions according to the degree of space-time reflection and of particle-antiparticle conjugation. The final results are shown in Table II.

TABLE II. – *The status of the reflexion symmetries C, P and T is tabulated with rank m of derivative acting on a spinor field involved.* The presence of the symmetric field S_{kl} and antisymmetric one A_{kl} is indicated. *Both* means the presence of both fields.

m	Interaction	C	P	T	S_{kl}	A_{kl}	Both
0	L_M, L_4	+	+	+		L_4	L_M
1	L_5, L_7, L_{10}	+	+	+	L_7	L_{10}	L_5
0	L_3	—	—	+		L_3	
1	L_{12}, L_{13}	—	—	+	L_{12}	L_{13}	
0	L_2	+	—	—	L_2		
1	L_6, L_8, L_9	+	—	—	L_8	L_9	L_6
0	L_1	—	+	—	L_1		
1	L_{11}, L_{14}	—	+	—	L_{11}	L_{14}	

5. – A massless scalar particle.

We have so far not discussed the physical meaning of the modified Dirac equation (2.7), but implicitly taken this as the equation for a spin-$\frac{1}{2}$ particle, *e.g.*, the electron, in the presence of gravitation. The reason for this is partly because we have already shown in our previous paper (7) that (2.7) indeed reduces, in the nonrelativistic limit, to the Schrödinger equation with the Newtonian potential. One could of course supply a more exact argument to this problem, by considering relativistic corrections and spin-dependent terms peculiar to the Dirac equation, for instance, by employing the well-known Pauli approximation which has successfully been applied to the Dirac equation (2.11) (to be more precise, its quadratic form (2.12)) for a single electron in a fixed Coulomb potential.

We here touch briefly this problem so as to convince the readers that (2.7) actually shows a parallelism with the corresponding electromagnetic Dirac equation (2.11). Fortunately, our quadratic equation (2.9) can be enormously simplified by the linear approximation (3.12). In fact, only the first four terms survive in this case:

$$(D_kD_k + i\sigma_{kl}D_kD_l - m^2 + V_kD_k)\psi = 0 \,. \tag{5.1}$$

In the case of a prescribed external static gravitational field, *i.e.* the Newtonian potential due to a macroscopic body of mass M

$$\phi(r) = -GM/r \,, \tag{5.2}$$

related to our field variable S_{44} as (see Appendix B for derivation)

$$S_{44}(r) = 2\phi(r)/c^2\,, \tag{5.3}$$

(5.1) becomes

$$\left\{E + \frac{\hbar^2}{2m}\Delta - m\phi + \left(1 - \frac{2\phi}{c^2}\right)\frac{E^2}{2mc^2} - \frac{2\phi}{c^2}E + \frac{2\phi}{c^2}\frac{\hbar^2}{2m}\Delta + \right.$$
$$\left. + \frac{i\hbar^2}{2mc^2}\left[\frac{W}{c\hbar}(\nabla\phi\cdot\boldsymbol{\alpha}) + \boldsymbol{\Sigma}\cdot(\nabla\phi\times\nabla)\right]\right\}\psi = 0\,, \tag{5.4}$$

where we explicitly write c, the velocity of light, and $\hbar$, Planck's constant divided by 2π; we denote the total energy by $W = mc^2 + E$ and use the conventional notations for the Dirac spin matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} \cdot & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \cdot \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \cdot \\ \cdot & \boldsymbol{\sigma} \end{pmatrix}, \tag{5.5}$$

($\boldsymbol{\sigma}$ are the Pauli spin matrices).

Likewise, in the case of a fixed Coulomb potential

$$\varphi(r) = e/r, \tag{5.6}$$

the quadratic form (2.12) becomes

$$\left\{E + \frac{\hbar}{2m}\Delta + e\varphi + \frac{(E + e\varphi)^2}{2mc^2} - \frac{ie\hbar}{2mc}(\nabla\varphi\cdot\boldsymbol{\alpha})\right\}\psi = 0\,, \tag{5.7}$$

where the vector potential is neglected, and $-\nabla\varphi$ means an electric field. These two equations are similar in form in the first three terms, which are nothing but the ordinary Schrödinger equation. The relativistic correction due to the velocity dependence of mass appears in the fourth and fifth terms in our eq. (5.4), while it appears only in the fourth term in (5.7). The last two members in (5.4) along with the last one in (5.7) are of course peculiar to the Dirac theory, because these terms involve the Dirac spin matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}$. Further discussion of (5.4) can be given straightforwardly; for instance, (5.4) will be represented in terms of only the large component spinor, and thereby given a physical meaning more clearly. When we take into account those nonminimal interactions which violate C, P and/or T and involve the symmetric field (see Table II), there appear additional contributions to (5.4), which will be discussed in detail elsewhere. There is a preliminary but interesting argument on this subject ([3]); we note the equation therein discussed is different from (5.4) which is derived from the generally covariant Dirac equation (2.7).

Now we shift our attention to a skewsymmetric tensor field $A_{kl}(x)$. This field cannot be generated by unpolarized macroscopic bodies, as is clear from the field equations (3.25) it satisfies and from the Tetrode formula (2.45) which relates its source, an antisymmetric part of the energy-momentum tensor, to the spin of the spinor field. Thus, we have to consider this field as a quantized field in order to see its behaviour. If we assume its coupling constant λ to be fairly small as compared to unity, we will be able to work in perturbation theory, where only a free quantized field $A_{kl}(x)$ appears. Before proceeding to quantize our skewsymmetric field, we here determine the number of its independent components. It suffices for this purpose to consider the field equations (3.25) without a source term and the supplementary condition (3.20) at the classical level:

$$\Box A_{kl}(x) = 0 \,, \tag{5.8a}$$

$$\partial_l A_{kl}(x) = 0 \,, \tag{5.8b}$$

$$A_{kl}(x) = -A_{lk}(x) \,. \tag{5.8c}$$

The divergence-free condition (5.8b) lowers the number of independent variables from 6 to 3. We can, however, envisage a gauge transformation specified by

$$A_{kl}(x) \to A_{kl}(x) + f_k(x)_{,l} - f_l(x)_{,k} \,, \tag{5.9}$$

under which the set of equations (5.8) is required to be invariant. This condition is met if a set of the gauge functions $f_k(x)$ satisfies the relations

$$\Box f_k(x) = 0 \,, \tag{5.10a}$$

$$\partial_k f_k(x) = 0 \,, \tag{5.10b}$$

which are also subject to a gauge transformation given by

$$f_k(x) \to f_k(x) + g(x)_{,k} \,, \tag{5.11}$$

with

$$\Box g(x) = 0 \,. \tag{5.12}$$

Thus, the gauge functions $f_k(x)$ have only 2 independent components. With a suitable choice of the latter ones, the number of independent components of $A_{kl}(x)$ reduces further from 3 to 1. Thus, we conclude that there is only *one physical particle* associated with A_{kl}, which is a *neutral massless scalar particle.* In fact, one can easily show that there survives only one variable, *e.g.* $A_{12}(x) = -A_{21}(x)$ for a plane wave propagating along the z-axis; $A'_{12}(x') = A_{12}(x)$ for a rotation about the z-axis. In passing we note that the gauge transformation (5.9) is nothing but an antisymmetric part of the linearized form of (2.4). For an infinitesimal general co-ordinate transformation

$$x^\mu \to x^\mu - \lambda f^\mu(x)\,, \tag{5.13}$$

(2.4) becomes

$$b_k{}^\mu(x) \to b_k{}^\mu(x) - \lambda f^\mu(x)_{,\nu}\, b_k{}^\nu(x)\,. \tag{5.14}$$

If we apply the linear approximation (3.12) to this, we obtain the desired result (5.9).

Now, we express our skewsymmetric field in terms of the creation and annihilation operators $a(\boldsymbol{p}; h)$ and $a^\dagger(\boldsymbol{p}; h)$ of a neutral massless scalar particle with helicity $h = 0$, and a fictitious neutral massless spin-1 particle with helicity $h = \pm 1$, including another set of particles which have the same nature as the former:

$$\begin{aligned} A_{kl}(x) = (2\pi)^{-\frac{3}{2}} \int \frac{\mathrm{d}^3 p}{(2p_0)^{\frac{1}{2}}} \cdot \\ \cdot \sum_h \{e_{kl}(\boldsymbol{p}; h)\, a(\boldsymbol{p}; h) \exp[ipx] + e^\dagger_{kl}(\boldsymbol{p}; h)\, a^\dagger(\boldsymbol{p}; h) \exp[-ipx]\}\,, \end{aligned} \tag{5.15}$$

where helicity sum over h takes ± 1 and 0 for one set of massless spin-1 and -0 particles, and also ± 1 and 0 for another set of massless spin-1 and -0 particles, and the symbol † means complex conjugation for c-number quantities and Hermitian adjoint for operators. The various conditions imposed on $A_{kl}(x)$ are now shifted to the « polarization tensors » $e_{kl}(\boldsymbol{p}; h)$:

$$e_{kl}(\boldsymbol{p}; h) = -e_{lk}(\boldsymbol{p}; h)\,, \tag{5.16a}$$

$$p_l\, e_{kl}(\boldsymbol{p}; h) = 0\,, \tag{5.16b}$$

and for the gauge transformation (5.9)

$$e_{kl}(\boldsymbol{p}; h) \to e_{kl}(\boldsymbol{p}; h) + ip_l\, e_k(\boldsymbol{p}; h) - ip_k\, e_l(\boldsymbol{p}; h)\,, \tag{5.16c}$$

where $e_k(\boldsymbol{p}; h)$ are the « polarization vectors » involved in a free quantized field $f_k(x)$ satisfying (5.10). We could of course take a different representation of a free-field operator, for instance, in terms of only a single physical scalar particle involved in $A_{kl}(x)$, as deduced from a previous argument on the number of independent field variables. Then we would have to worry about the Lorentz invariance of the S-matrix. We have, however, learned from the Gupta-Bleuler formalism of quantum electrodynamics that this problem can be circumvented by introducing some unphysical particles with indefinite metric (timelike *vs.* longitudinal photons) with the modified version of the Lorentz condition applied to Hilbert space so as to admit only the physical particles in the asymptotic states. With this in mind, we have expressed our field operator $A_{kl}(x)$ as in (5.15). We would have to construct the polarization tensor explicitly when the particles associated with $A_{kl}(x)$ appear as external particles in a Feynman diagram. On the contrary, it is here not necessary because we are only interested in the scattering problem between spin-$\frac{1}{2}$ particles, more specifically, between the proton and the electron, where A_{kl} is exchanged in the lowest-order perturbation. We need instead the propagator function for our skewsymmetric field; for this purpose, it is no longer necessary to specify the polarization tensor, as we will show below. What we need later in some calculations is the time-ordered product of A_{kl} with itself taken between vacuum states:

$$(5.17)\qquad \langle T\{A_{kl}(x)A_{mn}(y)\}\rangle_0 = \frac{-i}{(2\pi)^4}\int \frac{\mathrm{d}^4 p}{p^2 - i0} P_{klmn}(p)\exp[ip(x-y)]\,,$$

where

$$(5.18)\qquad P_{klmn}(p) = \sum_h e_{kl}(\boldsymbol{p};h)\,e^{\dagger}_{mm}(\boldsymbol{p};h) = P_{mnkl}(p)\,.$$

In a previous paper ([13]) we discussed the method of projection operators for high-spin propagators. Here we apply this method *mutatis mutandis*. First, the numerator of the propagator function in momentum space must be antisymmetric with respect to k and l as well as m and n in (5.18) because of (5.16*a*). Secondly, (5.16*b*) now requires that contraction of four-momentum with any of the indices in P_{klmn} vanish, for instance

$$(5.19)\qquad p_k P_{klmn}(p) = 0\,.$$

Finally, P_{klmn} ought to be gauge invariant in the sense of (5.16*c*). Nevertheless, we transcribe this condition into the gauge invariance of the scattering amplitudes under (5.16*c*), hence the numerator of the propagator is not necessarily gauge invariant by itself. In other words, gauge-variant terms therein are expected to be counteracted when coupled to the material part (bilinear forms of a spinor field, in the present case). It will be shown that this cancellation indeed occurs at least in the scattering amplitudes to be considered below. With these properties mentioned above, let us construct the numerator; using an unspecified symmetric tensor d_{kl}, we write

$$(5.20)\qquad P_{klmn} = d_{km}d_{ln} - d_{kn}d_{lm}\,,$$

([13]) K. HAYASHI: *Progr. Theor. Phys.*, **41**, 214 (1969).

which is so chosen that it is antisymmetric under interchange of k and l as well as of m and n. Next, let us determine the explicit form for d_{kl}. Owing to Lorentz invariance, this can be written as

$$d_{kl} = a\delta_{kl} + bp_k p_l/p^2 , \tag{5.21}$$

with two arbitrary coefficients, which must satisfy the following relation:

$$a + b = 1 \tag{5.22}$$

in view of the condition (5.19). As was shown in ref. ([13]), the numerator of the propagator is the projection operator in the case of a massive particle. We here postulate that this is also the case for our present problem. Regarding P_{kmln} as the (km)-(ln) element of matrix P, we find that the condition for the projection operator takes the form

$$P_{kilj}P_{imjn} = a^4 P_{kmln} = P_{kmln} . \tag{5.23}$$

Thus, a can take the values, ± 1 and $\pm i$. On the other hand, trace of the projection operator is its multiplicity, the number of all the degrees of freedom involved in A_{kl}, *i.e.* $6 =$ all the helicity states of two massless spin-1 particles and of two massless spin-0 particles, hence we have

$$P_{klkl} = 6a^2 = 6 . \tag{5.24}$$

Hence a is either $+1$ or -1, but the physical results derived by the use of the propagator with (5.20) do not depend on the sign of a. For simplicity, we shall take a as $+1$ hereafter. Thus, we conclude that (5.17) is the propagator for a massless scalar particle associated with the A_{kl}-field, Needless to say, the building block d_{kl} of our propagator is, with the above choice of the coefficients, the projection operator for a spin-1 particle in the sense that d_{kl} selects only the pure P-wave component, *i.e.*

$$\begin{cases} d_{km}d_{ml} = d_{kl} , \\ d_{mm} = 3 , \end{cases} \tag{5.25}$$

where

$$d_{kl} = \delta_{kl} - p_k p_l/p^2 . \tag{5.26}$$

Before closing the argument on the propagator, we note that its numerator (5.20) can further be simplified in the actual calculations of the scattering amplitudes in the following form, as can readily be verified in an explicit example:

$$P_{klmn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm} . \tag{5.27}$$

We conclude this Section with a brief discussion of the possibility that gravitational interactions violating some of P, C and T might cause the energy-level shift of the hyperfine structure of the hydrogen atom, one of the simplest quantum-mechanical systems. As for the usual linearized gravitational field denoted by $S_{kl}(x)$ in this paper, it follows immediately from the extreme smallness of Einstein's gravitational constant (3.3) that it is almost hopeless to discuss the physical effects due to the possible violation of these discrete symmetries, although the latter cannot be ruled out from the present accuracy of the available data ([2]). We then turn our attention to a skewsymmetric field denoted by $A_{kl}(x)$ in this paper, which would have to exist in the case of violation of C, P and/or T. It is worth noting that this new field represents a *deviation* from Einstein's theory of gravitation, irrespective of P, C and T conservation. Accordingly, its quantum, a massless scalar particle, will be called for short a «deviaton.» We have so far assumed that a departure, if any, from Einstein's theory might occur via the violation of the discrete symmetries. MIYAMOTO and NAKANO took the alternative way, *viz.* to depart from Einstein's theory by retaining the separate conservation of C, P and T ([9]). They studied the simplest quantal system of the hydrogen atom in order to see if a deviaton affects its hyperfine structure. The method of deriving a potential from a relevant scattering amplitude has been well known ([14,15]), and hence we quote only the result: The elastic scattering amplitude via one-deviaton exchange between a proton and an electron, where L_M contributes to both proton and electron vertices, is given at the nonrelativistic limit by

$$T(\boldsymbol{q};\, L_M, L_M) = (\lambda^2/64\pi^3)(\boldsymbol{q}\times\boldsymbol{\sigma}^p)(\boldsymbol{q}\times\boldsymbol{\sigma}^e)/\boldsymbol{q}^2\,, \tag{5.28}$$

from which the potential is derived:

$$V(\boldsymbol{r}) = (\lambda^2/32\pi)\{(8\pi/3)(\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^e)\,\delta(\boldsymbol{r}) - (1/r^3)[(\boldsymbol{\sigma}^p\cdot\boldsymbol{\sigma}^e) - (3/r^2)(\boldsymbol{\sigma}^p\cdot\boldsymbol{r})(\boldsymbol{\sigma}^e\cdot\boldsymbol{r})]\}\,. \tag{5.29}$$

The present accuracy of the precision measurements in quantum electrodynamics gives the upper bound on the unknown coupling ([9])

$$\lambda^2/4\pi \leqslant 10^{-5}\hbar c/(\mathrm{GeV})^2\,. \tag{5.30}$$

It is not yet clear at the moment if such a deviaton exists with the coupling strength obeying this condition. However, one thing is clear: Whether or not the gravitational interaction between the proton and the electron conserves P, C and T is also an open question at the moment. There are no theoretically reasonable grounds to put $G_2(0) = 0 = G_5(0)$, which appeared in Eq. (1.1).

([14]) S. S. SCHWEBER: *An Introduction to Relativistic Quantum Field Theory* (New York, 1960), p. 580.

([15]) N. HOSHIZAKI and S. MACHIDA: *Progr. Theor. Phys.*, **24**, 1325 (1960).

6. – Discussion.

Most particle physicists know that the weaker the interaction strength gets the greater the reflection symmetries such as parity, charge conjugation, and time reversal are violated. Motivated by this *empirical* rule, we attempted to answer the inquiry, whether or not gravitation, the weakest of all the forces we know, does also enjoy the violation of these discrete symmetries. There is no *a priori* basis whatsoever for applying it to gravitation, which indeed differs significantly from the former interactions in its great universality and its infinite range of force (the classical limit), subject to a macroscopic inverse-square law.

The extreme weakness of gravitation prevents us from having a clear answer to the question posed. From the experimental side almost nothing can be said about it for the moment. We had therefore to approach the question mainly from the theoretical side. With general covariance as the guiding principle we find an unexpected situation that there must be a *massless scalar particle*, termed for short a «deviaton» if Nature prefers a deviation from Einstein's theory of gravitation. Obviously, the gravitational interactions violating any of C, P and T do deviate from the theory. Unfortunately, we have to express our inability to present any means of detecting such a deviaton —neutral, massless and spinless. Contrary to the «common sense» that the masslessness of a particle exchanged between, say, the nucleons implies an infinite range of force, this new deviaton, if yet unobserved, does not give rise to a long-range force, nor any macroscopic force, instead exhibiting rather peculiar properties. First, its force, when interpreted in terms of potential, depends on the spin of the proton and electron which generates it. Second, the deviaton couples to matter with a singular coupling involving the derivative of its own field (a skewsymmetric tensor field) when the gravitational interaction violates any of C, P and T. This means that a resulting potential between spin-$\frac{1}{2}$ particles would become highly singular, for instance, involving the delta-function and/or its derivative, thereby affecting the material wave function only at the origin.

We have so far not touched upon the neutrino as a source of gravitation. As far as we know the neutrino (antineutrino) appears with left (right) helicity alone in the weak interactions. If this is also the case for the gravitational interaction, they obviously violate parity (P) and charge conjugation (C), nevertheless conserving CP. Furthermore, if these neutrinos are a *unique* agency through which the violation of P and C can occur, then we will be able to discuss, for instance, P-violation without any need of deviatons and hence we would have to withdraw most of our conclusions drawn in this paper. However, as is well known, strong P- and C-violation effects are observed in the nonleptonic weak decay processes such as $\Lambda \to p\pi^-$, hence the neutrinos are not the unique cause of the violation of P and C. Alternatively, we can even speculate that owing to its great universality the gravitational field would presumably couple to the neutrino with right helicity and the antineutrino with left helicity which do not participate in the weak interactions, thus conserving P and C.

We close this paper with our hope that a more definite answer will be given as to the presence or absence of a deviaton, thereby verifying or denying a deviation from Einstein's gravitational theory in the realm of particle physics.

* * *

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APPENDIX A

The Riemann tensor and its contracted tensors in terms of the tetrads.

We shall obtain the Riemann tensor in terms of the tetrads by applying the commutator of the covariant derivative with respect to the Christoffel symbol to an arbitrary contravariant vector field A^α:

$$(A^\alpha{}_{;\mu})_{;\nu} - (A^\alpha{}_{;\nu})_{;\mu} = R^\alpha{}_{\lambda\mu\nu} A^\lambda \,, \tag{A.1}$$

where the Riemann tensor is defined by

$$R^\alpha{}_{\lambda\mu\nu} = \Gamma^\alpha{}_{\lambda\mu,\nu} - \Gamma^\alpha{}_{\lambda\nu,\mu} + \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\lambda\mu} - \Gamma^\alpha{}_{\beta\mu}\Gamma^\beta{}_{\lambda\nu} \,. \tag{A.2}$$

Substituting $b_k{}^\mu(x)$ for A^μ and transforming Greek indices to Latin ones, we obtain the desired tensor as follows:

$$\begin{aligned} R_{klmn} &= b_{k\alpha} b_l{}^\lambda b_m{}^\mu b_n{}^\nu R^\alpha{}_{\lambda\mu\nu} = \\ &= -\left(D_n D_{klm} - D_m D_{kln} + D_{jkn} D_{jlm} - D_{jln} D_{jkm} + D_{klj}(D_{jmn} - D_{jnm})\right) \,, \end{aligned} \tag{A.3}$$

where

$$D_k = b_k{}^\mu(x)\,\partial_\mu \,, \tag{A.4}$$

$$D_{klm} = \tfrac{1}{2}\,(C_{klm} - C_{lkm} - C_{mkl}) \tag{A.5}$$

and C_{klm} is defined by (2.13). Now it is easy to derive the contracted Riemann tensor R_{kl} corresponding to $R^{\alpha}{}_{\lambda\alpha\nu} = R_{\lambda\nu}$:

$$(A.6)\qquad bR_{kl} = bR_{mkml} = b_k{}^{\mu} b_l{}^{\nu}(-g)^{\frac{1}{2}} R_{\mu\nu} = = (bb_m{}^{\mu} C_{(kl)m})_{,\mu} + bb_{(k}{}^{\mu} V_{l),\mu} + bC_{mnk} C_{(mn)l} - \tfrac{1}{4} C_{kmn} C_{lmn}$$

$$(V_l = C_{mml}) ,$$

where the determinant b is given above (2.6) and g is the determinant det $(g_{\mu\nu})$; small parentheses enclosing Latin indices indicate symmetrization. Finally, we obtain the Riemann scalar $R = R_{mm} = g^{\mu\nu} R_{\mu\nu}$:

$$(A.7)\qquad bR = b\{\tfrac{1}{4} C_{klm} C_{klm} - \tfrac{1}{2} C_{klm} C_{mkl} + V_m V_m\} + (2bb_m{}^{\mu} V_m)_{,\mu} .$$

Appendix B

Relation between the Newton potential and the linearized gravitational field variable.

In Einstein's theory of gravitation, the scalar potential of a gravitational field is defined by the metric tensor

$$(B.1)\qquad g_{00}(x) = -(1 + 2\phi/c^2) .$$

If we use the relation between the metric tensor and the tetrads (2.28) with the linear approximation (3.12), and also take the nonrelativistic limit where only $a_{44}(x) = -a_{00}(x)$ can contribute, we obtain

$$(B.2)\qquad g_{00}(x) = -1 + 2a_{00}(x) .$$

On the other hand, the *correct* gravitational field variable is not $a_{kl}(x)$ but $S_{kl}(x)$ in the sense that the latter satisfies the wave equation (3.24) subject to the divergence condition (3.17): A retarded solution to this set of the field equations is well known:

$$(B.3)\qquad S_{kl}(\boldsymbol{x}, t) = \frac{\varkappa}{4\pi} \int \mathrm{d}\boldsymbol{x}' \, \frac{T_{(kl)}(\boldsymbol{x}', t - r/c)}{|\boldsymbol{x} - \boldsymbol{x}'|} .$$

If we employ (3.16) in (B.1) and (B.2) in view of the Poisson equation

$$(B.4)\qquad \Delta\phi = 4\pi G M \delta^3(\boldsymbol{x}) ,$$

and our equation derived from (3.24) with (B.3)

$$(B.5)\qquad \Delta S_{00} = -\varkappa M c^2 \delta^3(\boldsymbol{x}) ,$$

we obtain the desired result

$$S_{00}(x) = 2a_{00}(x) = -2\phi/c^2 \,, \tag{B.6}$$

and reproduce the well-known relation

$$\varkappa^2 c^4 = 8\pi G \,. \tag{B.7}$$

● RIASSUNTO (*)

Nell'ipotesi della covarianza generale si discute la struttura delle interazioni gravitazionali fra protone ed elettrone. Vi sono 15 (3) diversi accoppiamenti fra una particella di spin $\frac{1}{2}$ con massa (un neutrino a due componenti) ed il campo gravitazionale, purché queste interazioni non coinvolgano derivate superiori a quelle che compaiono nei lagrangiani di Dirac e gravitazionale liberi. Nel limite del campo debole si possono suddividere le interazioni in 4 classi: $(C, P, T) = (+, +. +,)$, $(+, -, -)$, $(-, +, -)$ e $(-, -, +)$. Ogni deviazione della teoria di Einstein comporta un tensore energia-impulso asimmetrico, quindi coinvolgendo ipotetiche *particelle senza massa e senza spin*, rappresentate da un campo tensoriale a simmetria obliqua la cui sorgente è lo spin del protone e dell'elettrone. Si discutono anche i possibili effetti di questa particella sulla struttura iperfine dell'atomo d'idrogeno.

(*) *Traduzione a cura della Redazione.*

Гравитационные взаимодействия протона и электрона. Возможное существование скалярной частицы с нулевой массой.

Резюме (*). — Предполагая общую ковариантность, мы обсуждаем структуру гравитационных взаимодействий между протоном и электроном. Существуют 15 (3) различных связей между массивными частицами со спином $\frac{1}{2}$ (двух-компонентным нейтрино) и гравитационным полем, при условии, что эти взаимодействия не содержат более высоких производных, чем те, которые появляются в свободном дираковском и гравитационном лагранжианах. В пределе слабого поля рассматриваемые взаимодействия могут быть разделены на четыре класса: $(C, P, T) = (+, +, +)$, $(+, -, -)$, $(-, +, -)$ и $(-, -, +)$. Любое отклонение от теории Эйнштейна приводит к асимметричному тензору энергии-импульса, следовательно, включает гипотетические *безмассовые, бесспиновые* частицы, которые описываются кососимметрическим тензорным полем, источник которого представляет спин протона и электрона. Мы также обсуждаем возможное влияние такой частицы на сверхтонкую структуру атома водорода.

(*) *Переведено редакцией.*