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INTUITIONISTIC FUZZY IMPLICATION OPERATORS: EXPRESSIONS AND PROPERTIES

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ABSTRACT. The expressions of 32 fuzzy coimplication operators(FCO) and 32 intuitionistic fuzzy implication operators(IFIO) are given in this paper. The Co-D-P properties which the FCOs should satisfy are presented. The FCOs and IFIOs'situation of satisfying the properties which they should satisfy, respectively, are discussed in details.

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1. Introduction

Study in intuitionistic fuzzy sets and application of intuitionistic fuzzy control have been developed quickly since the definition of intuitionistic fuzzy sets was introduced by Atanassov in 1983. Technology of intuitionistic fuzzy control has been applied to many fields including medical field[9-12]. But the basic theory of intuitionistic fuzzy control is inferior to its application, especially the theory of intuitionistic fuzzy reasoning. Since Zadeh[2] introduced the compositional rule of inference (CRI), many researchers have take advantage of fuzzy implication operators to represent the relation between two variables linked together by means of an *if-then* rule. In intuitionistic fuzzy reasoning theory, intuitionistic fuzzy implication operators play the same important role. However how many intuitionistic implication operators can be used in reality? What properties do they satisfy? Which one is t-norm and t-conorm? etc.. In this paper, we will focus on the expressions of intuitionistic fuzzy implication operators and their properties.

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The paper is arranged as follows. In Section 2, the expressions of 32 fuzzy coimplication operators are given. It is verified whether they satisfy the 14 properties in common use. Section 3 introduces 32 intuitionistic fuzzy implication operators by making use of fuzzy implication, coimplication and aggregation operators. Their properties are discussed in detail.

2. Definition and notations

The notation of intuitinistic fuzzy set was introduced in [1] as a generalization of the notation of fuzzy set.

Let X be an ordinary finite non-empty set. An intuitionistic fuzzy set in X is an expression A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\},$$

where $\mu_A(x) : X \to [0,1]$, $\nu_A(x) : X \to [0,1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all x in X.

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the membership degree and the nonmembership degree of the element x in A.

For convenience of natation, we abbreviate "intuitionistic fuzzy set" to $\mathcal{IFS}$ and represent $\mathcal{IFS}(X)$ as the set of all the $\mathcal{IFS}$ in X.

$\mathcal{IFS}$ theory has the virtue of complementing fuzzy sets, that are able to model vagueness, with an ability to model uncertainty as well. In $\mathcal{IFS}$ theory the value $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, the intuitionistic fuzzy index, denotes a measure of non-determinacy (or undecidedness).

The operations equality, inclusion, complement, intersection, union,etc., for $\mathcal{IFS}$ of an ordinary set are defined in terms of their membership and nonmembership functions. That is, for every $A, B \in \mathcal{IFS}(X)$,

- $\diamond$ $A \leq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all x in X.
- $\diamond$ $A \preceq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for all x in X.
- $\diamond$ $A = B$ if and only if $A \leq B$ and $B \leq A$.
- $\diamond$ $A \cap B = \{< x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) > | x \in X\}$.
- $\diamond$ $A \cup B = \{< x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) > | x \in X\}$.
- $\diamond$ The complementary of an $\mathcal{IFS}$ is $A_c = \{(x, \nu_A(x), \mu_A(x)) | x \in X\}$.

For convenience, let us take the following set:

$$L^* = \Big\{\tilde{x} = (x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\Big\}.$$

For every $\tilde{x}, \tilde{y} \in L^*$ the following expressions are known.

- $\diamond$ $\tilde{x} \leq_{L^*} \tilde{y} \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$;
- $\diamond$ $\tilde{x} \preceq_{L^*} \tilde{y} \Leftrightarrow x_1 \leq y_1$ and $x_2 \leq y_2$;
- $\diamond$ $\tilde{x} \sqsubseteq_{L^*} \tilde{y} \Leftrightarrow 1 - x_1 \leq y_1$;
- $\diamond$ $\tilde{x} \vee_{L^*} \tilde{y} = ((x_1 \vee y_1), (x_2 \wedge y_2))$;
- $\diamond$ $\tilde{x} \wedge_{L^*} \tilde{y} = ((x_1 \wedge y_1), (x_2 \vee y_2))$;
- $\diamond$ $\tilde{x}^c = (x_2, x_1)$;

$\diamond$ $1_{L^*} = (1,0)$; $0_{L^*} = (0,1)$.

Evidently, the intuitionistic fuzzy index of $\tilde{x} \in L^*$ is defined as $\pi_{\tilde{x}} = 1 - x_1 - x_2$. Obviously, $0 \leq \pi_{\tilde{x}} \leq 1$.

There are several notations will be used in this paper in order to construct intuitionistic fuzzy implication operators.

Definition 2.1. A strong negation is any strictly decreasing and continuous $[0,1] \to [0,1]$ mapping n satisfying $n(0) = 1, n(1) = 0$ and $n(n(x)) = x$ for all x in X.

A lot has been written in the fuzzy literature about aggregation operators. Normally these operators are demanded the boundary conditions and monotonicity. H.Bustince gave a definition in [3] shown below, in addition to the properties above, symmetry is also demanded.

Definition 2.2. A function $M : [0,1]^2 \to [0,1]$ is called *aggregation operator* if it satisfies the following conditions,

$A1$. $M(0,0) = 0$;
$A2$. $M(1,1) = 1$;
$A3$. M is nondecreasing in both places;
$A4$. $M(x,y) = M(y,x)$ for all $(x,y) \in [0,1]^2$.

An aggregation operator is called idempotent if it satisfies

$$M(x,x) = x, \quad \text{for all} \quad x \in [0,1].$$

It is significant that any aggregation operator M that satisfies idempotency and $A3$ also satisfies the inequalities $\wedge(x,y) \leq M(x,y) \leq \vee(x,y)$.

3. Fuzzy coimplication Operators

In fuzzy theory a conditional rule in expert systems has the form

$$\text{If} \quad x \quad \text{is} \quad A \quad \text{then} \quad y \quad \text{is} \quad B \tag{3.1}$$

where x is a variable taking values in A and y is a variable taking values in B, A and B are fuzzy sets in X and Y, respectively.

In the framework of Zadeh's calculus of fuzzy restrictions, the fuzzy conditional rule is interpreted as a fuzzy relation restricting the possible values of the ordered pair (x,y). This fuzzy relation can be defined using a generalized implication operator I as

$$\begin{aligned} X \times Y &\rightarrow [0,1] \\ (x,y) &\mapsto I(\mu_A(x), \mu_B(y)) \quad \text{for all} \quad (x,y) \in X \times Y \end{aligned}$$

where I is a fuzzy implication operator (FIO), that is, the function $I : [0,1]^2 \to [0,1]$ satisfies the properties:

p_1. If $x \leq z$ then $I(x,y) \geq I(z,y)$ for all $y \in [0,1]$;
p_2. If $y \leq t$ then $I(x,y) \leq I(x,t)$ for all $x \in [0,1]$;

p_3. $I(0, y) = 1$ for all $y \in [0, 1]$;
p_4. $I(x, 1) = 1$ for all $x \in [0, 1]$;
p_5. $I(1, 0) = 0$.

Other properties (always called D-P properties) usually demanded of I are the following.

p_6. $I(1, y) = y$ for all $y \in [0, 1]$;
p_7. $I(x, I(y, z)) = I(y, I(x, z))$;
p_8. $x \geq y$ if and only if $I(x, y) = 1$;
p_9. $I(x, 0) = 1 - x$ for all $x \in [0, 1]$;
p_{10}. $I(x, y) \geq y$;
p_{11}. $I(x, x) = 1$;
p_{12}. $I(x, y) = I(1 - y, 1 - x)$;
p_{13}. I is continuous on $[0, 1] \times [0, 1]$;
p_{14}. If $x > 0$, then $I(x, 0) < 1$; If $y < 1$, then $I(1, y) < 1$.

It is known that $I(\mu_A(x), \mu_B(y))$ is always interpreted as the truth degree of the conditional rule (3.1). By the definition of $\mathcal{IFS}$, in order to get an intuitionistic fuzzy implication operator, we have to know the non-truth degree of the conditional rule (3.1).

In [3], the definition of fuzzy coimplication which is dual to fuzzy implication is given for interpreting the non-truth degree of the conditional rule (3.1).

Definition 3.1. A function $I_c : [0, 1]^2 \to [0, 1]$ is called *coimplication* if it satisfies the properties
p_{c1}. If $x \leq z$ then $I_c(x, y) \geq I_c(z, y)$ for all $y \in [0, 1]$;
p_{c2}. If $y \leq t$ then $I_c(x, y) \leq I_c(x, t)$ for all $x \in [0, 1]$;
p_{c3}. $I_c(x, 0) = 0$ for all $x \in [0, 1]$;
p_{c4}. $I_c(1, y) = 0$ for all $y \in [0, 1]$;
p_{c5}. $I_c(0, 1) = 1$.

The proposition below states the condition under which an implication gives rise to a coimplication. With the proposition, we can construct fuzzy coimplication operators (FCOs) by their dual FIOs.

Proposition 3.1. *A function $I_c : [0, 1]^2 \to [0, 1]$ is a coimplication if and only if the function $I(x, y) = n(I_c(n(x), n(y))$ is an implication for any strong negation n.*

In this paper, the strong negation n will be considered as standard $\mathcal{N}$, that is, $n(x) = \mathcal{N}(x) = 1 - x$ for all $x \in [0, 1]$. In this sense, $I_c(1 - \mu_A(x), 1 - \mu_B(y))$ can be interpreted as the degree of non-truth of the fuzzy conditional rule (3.1). Obviously, $I(\mu_A(x), \mu_B(y)) + I_c(1 - \mu_A(x), 1 - \mu_B(y)) = 1$.

From the definition of intuitionistic fuzzy implication operator in [3], we know that a class of intuitionistic fuzzy implication operators can be constructed by using of FIOs, FCOs and aggregation operators. In this reason we will discuss the 32 FIOs in common use concluded in [5-7] and their dual FCOs. The expressions are to be shown below.

R_0: $I_0(x,y)=\begin{cases}1 & x\le y\\ (1-x)\vee y & x>y,\end{cases}$ $\quad I_{0c}(x,y)=\begin{cases}0 & x\ge y\\ (1-x)\wedge y & x<y.\end{cases}$

Kleene-Dienes: $I_1(x,y)=(1-x)\vee y,\quad I_{1c}(x,y)=(1-x)\wedge y.$

Reichenbach: $I_2(x,y)=1-x+xy,\quad I_{2c}(x,y)=y-xy$

Lukasiewicz: $I_3(x,y)=\begin{cases}1 & x\le y\\ 1-x+y & x>y,\end{cases}$ $\quad I_{3c}(x,y)=\begin{cases}0 & x\ge y\\ y-x & x<y.\end{cases}$

Goguen: $I_4(x,y)=\begin{cases}1 & x=0\\ (\frac{y}{x})\wedge 1 & x>0\end{cases}$ $\quad I_{4c}(x,y)=\begin{cases}0 & x=1\\ (\frac{y-x}{1-x})\vee 0 & x<1.\end{cases}$

Gödel: $I_5(x,y)=\begin{cases}1 & x\le y\\ y & x>y,\end{cases}$ $\quad I_{5c}(x,y)=\begin{cases}0 & x\ge y\\ y & x<y.\end{cases}$

Dubois-Prade: $I_6(x,y)=\begin{cases}(1-x)\vee y & (1-x)\wedge y=0\\ 1 & \text{otherwise},\end{cases}$

$I_{6c}(x,y)=\begin{cases}(1-x)\wedge y & x\wedge(1-y)=0\\ 0 & \text{otherwise}.\end{cases}$

Zadeh: $I_7(x,y)=(1-x)\vee(x\wedge y),\quad I_{7c}(x,y)=(1-x)\wedge(x\vee y)$

Gaines-Rescher: $I_8(x,y)=\begin{cases}1 & x\le y\\ 0 & x>y,\end{cases}$ $\quad I_{8c}(x,y)=\begin{cases}(0 & x\ge y\\ 1 & x<y.\end{cases}$

Yager: $I_9(x,y)=y^x,\quad I_{9c}(x,y)=1-(1-y)^{1-x}.$

Mamdani: $I_{10}(x,y)=x\wedge y,\quad I_{10c}(x,y)=x\vee y.$

P.C.: $I_{11}(x,y)=xy,\quad I_{11c}(x,y)=x+y-xy.$

B.C.: $I_{12}(x,y)=0\vee(x+y-1),\quad I_{12c}(x,y)=1\wedge(x+y).$

E.C.: $I_{13}(x,y)=\begin{cases}x\wedge y & x\vee y=1\\ 0 & x\vee y<1,\end{cases}$ $\quad I_{13c}(x,y)=\begin{cases}(x\vee y & x\wedge y=0\\ 1 & x\wedge y>0.\end{cases}$

P.D.: $I_{14}(x,y)=x+y-xy,\quad I_{14c}(x,y)=xy.$

B.D.: $I_{15}(x,y)=1\wedge(x+y),\quad I_{15c}(x,y)=0\vee(x+y-1).$

E.D.: $I_{16}(x,y)=\begin{cases}(x\vee y & x\wedge y=0\\ 1 & x\wedge y>0,\end{cases}$ $\quad I_{16c}(x,y)=\begin{cases}x\wedge y & x\vee y=1\\ 0 & x\vee y<1.\end{cases}$

Einstein C.: $I_{17}(x,y)=\frac{y}{1+(1-x)(1-y)},\quad I_{17c}(x,y)=\frac{x+y}{1+xy}.$

Einstein D.* $I_{18}(x,y)=\frac{x+y}{1+xy},\quad I_{18c}(x,y)=\frac{y}{1+(1-x)(1-y)}.$

$I_{19}(x,y)=\begin{cases}1 & x<1\\ y & x=1,\end{cases}$ $\quad I_{19c}(x,y)=\begin{cases}(0 & x>0\\ y & x=0.\end{cases}$

*For saving room, P. denotes Probability, B. denotes Bounded, E. denotes Extreme, D. denotes Disjunction and C. denotes Conjunction.

$$I_{20}(x,y)=(1-x+2xy-x^2y)\wedge 1, \qquad I_{20c}(x,y)=(x^2+y-x-x^2y)\vee 0.$$

$$I_{21}(x,y)=\begin{cases}1 & x<1 \text{ or } y=1\\ 0 & \text{otherwise,}\end{cases} \qquad I_{21c}(x,y)=\begin{cases}(0 & x>0 \text{ or } y=0\\ 1 & \text{otherwise.}\end{cases}$$

$$I_{22}(x,y)=x\vee y, \qquad I_{22c}(x,y)=x\wedge y.$$

$$I_{23}(x,y)=\begin{cases}1 & x\le y\\ \frac{1-x}{1-y} & x>y,\end{cases} \qquad I_{23c}(x,y)=\begin{cases}0 & x\ge y\\ \frac{y-x}{y} & x<y.\end{cases}$$

$$I_{24}(x,y)=\begin{cases}1 & x=0\\ y & x>0,\end{cases} \qquad I_{24c}(x,y)=\begin{cases}0 & x=1\\ y & x<1.\end{cases}$$

$$I_{25}(x,y)=\begin{cases}1 & x\le y\\ \dfrac{\log x}{\log y} & x>y>0\\ 0 & x>y=0,\end{cases} \qquad I_{25c}(x,y)=\begin{cases}0 & x\ge y\\ \dfrac{\log\dfrac{1-y}{1-x}}{\log(1-y)} & x<y<1\\ 1 & x<y=1.\end{cases}$$

$$I_{26}(x,y)=(1-x^p+y^p)\wedge 1.$$

$$I_{26c}(x,y)=(1-(1-(1-x)^p+(1-y)^p)^{\frac{1}{p}})\vee 0.$$

$$I_{27}(x,y)=xy+\frac{1-x}{2}, \qquad I_{27c}(x,y)=y-xy+\tfrac{x}{2}.$$

$$I_{28}(x,y)=\frac{x+y}{2}, \qquad I_{28c}(x,y)=\frac{x+y}{2}.$$

$$I_{29}(x,y)=\begin{cases}1 & x\le y\\ 1-x+xy & x>y,\end{cases} \qquad I_{29c}(x,y)=\begin{cases}0 & x\ge y\\ y-xy & x<y.\end{cases}$$

$$I_{30}(x,y)=\begin{cases}1 & x\le y\\ (1-x)\vee y\vee\frac{1}{2} & 0<y<x<1\\ (1-x)\vee y & \text{otherwise,}\end{cases}$$

$$I_{30c}(x,y)=\begin{cases}0 & x\ge y\\ (1-x)\wedge y\wedge\frac{1}{2} & 0<x<y<1\\ (1-x)\wedge y & \text{otherwise.}\end{cases}$$

$$I_{31}(x,y)=\begin{cases}1 & x\le y\\ 1-x & x>y,\end{cases} \qquad I_{31c}(x,y)=\begin{cases}0 & x\ge y\\ 1-x & x<y.\end{cases}$$

Just as we know, an FIO should satisfy the properties given above. Based on the definition(3.1), a fuzzy coimplication operator (FCO) can be demanded to satisfy other properties in addition to $I_{c1}-I_{c5}$, and we will call them Co-D-P properties:

p_{c6}. $I_c(0,y)=y$;
p_{c7}. $I_c(x,I_c(y,z))=I_c(y,I_c(x,z))$;
p_{c8}. $I_c(x,y)=0$ if and only if $x\ge y$;
p_{c9}. $I_c(x,1)=1-x$;
p_{c10}. $I_c(x,y)\le y$;

p_{c11}. $I_c(x,x)=0$;
p_{c12}. $I_c(x,y)=I_c(1-y,1-x)$;
p_{c13}. I_c is continuous on $[0,1]\times[0,1]$;
p_{c14}. If $x<1$, then $I_c(x,1)>0$; If $y>0$, then $I_c(0,y)>0$.

Theorem 3.1: *These properties are not independent.*
(i) *If p_{c12} holds, then p_{c1} and p_{c2} are equivalent.*
(ii) *p_{c10} can be inferred from p_{c4} and p_{c6}.*
(iii) *p_{c4} can be inferred from p_{c10} and p_{c12}.*
(iv) *Sufficient condition of p_{c8} can be inferred from p_{c1} and p_{c11}.*

Easier method to know which one of 32 FCOs shown above is t-norm, t-conorm or which one should be selected during designing an intuitionistic fuzzy control system is to know which property an FCO satisfy. So next we will discuss whether 32 FCOs $I_{0c}-I_{31c}$ satisfy the properties $p_{c1}-p_{c14}$. The results will be represented in Table 1 (Y stands for "satisfy" and N stands for "don't satisfy").

Compared with the Table 1 in [5], we can make a conclusion that the FCOs hold the corresponding properties to their dual FIOs'.

4. Intuitionistic fuzzy implication operators

Atanassov and Gargov [4] and later Cornelis and Deschrijver[8] gave the definition of intuitionistic fuzzy implication operator.

An intuitionistic fuzzy implication operator (IFIO) is any $\mathcal{I}: L^{*2}\to L^*$, mapping satisfying the border conditions:

$$\mathcal{I}((0,1),(0,1))=(1,0);\mathcal{I}((0,1),(1,0))=(1,0);$$
$$\mathcal{I}((1,0),(1,0))=(1,0);\mathcal{I}((1,0),(0,1))=(0,1),$$

and the two following conditions:

1) If $\tilde{x}\le\tilde{y}$, then $\forall\tilde{z}\in L^*,\mathcal{I}(\tilde{x},\tilde{z})\ge\mathcal{I}(\tilde{y},\tilde{z})$.
2) If $\tilde{y}\le\tilde{z}$, then $\forall\tilde{x}\in L^*,\mathcal{I}(\tilde{x},\tilde{y})\le\mathcal{I}(\tilde{x},\tilde{z})$.

In [3], Bustince, Barrenechea and Mohedano gave another definition of IFIO. It satisfied the conditions above and recovered J Fodor's definition of FIO when the sets are fuzzy.

Definition 4.1: An intuitionistic fuzzy implication operator is a function: $\mathcal{I}: L^{*2}\to L^*$ with the properties below.

P_0. If $\tilde{x},\tilde{y}\in L^*$ are such that $x_1+x_2=1$ and $y_1+y_2=1$, then $\pi_{\mathcal{I}(x,y)}=0$;
P_1. $\forall\tilde{z}\in L^*$, if $\tilde{x}\le\tilde{y}$, then $\mathcal{I}(\tilde{x},\tilde{z})\ge\mathcal{I}(\tilde{y},\tilde{z})$;
P_2. $\forall\tilde{x}\in L^*$, if $\tilde{y}\le\tilde{z}$, then $\mathcal{I}(\tilde{x},\tilde{y})\le\mathcal{I}(\tilde{x},\tilde{z})$;
P_3. $\forall\tilde{z}\in L^*$,$\mathcal{I}(0_{L^*},\tilde{z})=1_{L^*}$;
P_4. $\forall\tilde{x}\in L^*$,$\mathcal{I}(\tilde{x},1_{L^*})=1_{L^*}$;
P_5. $\mathcal{I}(1_{L^*},0_{L^*})=0_{L^*}$.

	p_{c1}	p_{c2}	p_{c3}	p_{c4}	p_{c5}	p_{c6}	p_{c7}	p_{c8}	p_{c9}	p_{c10}	p_{c11}	p_{c12}	p_{c13}	p_{c14}
I_{0c}	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y
I_{1c}	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	N	Y	Y	Y
I_{2c}	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	N	Y	Y	Y
I_{3c}	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
I_{4c}	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	N	Y	Y
I_{5c}	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	N	N	Y
I_{6c}	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	N	Y
I_{7c}	N	Y	N	Y	Y	Y	N	N	Y	N	N	N	Y	Y
I_{8c}	Y	Y	Y	Y	Y	N	N	Y	N	N	Y	Y	N	Y
I_{9c}	Y	Y	Y	Y	Y	Y	Y	N	N	Y	N	N	Y	Y
I_{10c}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	Y	Y
I_{11c}	N	Y	N	N	Y	Y	Y	N	N	N	Y	N	Y	Y
I_{12c}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	Y	Y
I_{13c}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	N	Y
I_{14c}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	Y	N
I_{15c}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	Y	N
I_{16c}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	N	N
I_{17c}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	Y	Y
I_{18c}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	Y	N
I_{19c}	Y	Y	Y	Y	Y	Y	Y	N	N	Y	Y	N	N	N
I_{20c}	N	Y	Y	Y	Y	Y	N	N	Y	Y	N	N	N	Y
I_{21c}	Y	Y	Y	Y	Y	N	N	N	N	N	Y	N	N	N
I_{22c}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	Y	N
I_{23c}	Y	Y	Y	Y	Y	N	N	Y	Y	N	Y	N	Y	Y
I_{24c}	Y	Y	Y	Y	Y	Y	Y	N	N	Y	N	N	N	Y
I_{25c}	Y	Y	Y	Y	Y	N	N	Y	N	N	Y	N	Y	Y
I_{26c}	Y	Y	N	Y	Y	Y	Y	Y	N	Y	Y	N	Y	Y
I_{27c}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	Y	Y
I_{28c}	N	Y	N	N	N	N	N	N	N	N	N	N	Y	Y
I_{29c}	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y
I_{30c}	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	N	Y
I_{31c}	Y	Y	Y	Y	Y	N	N	Y	Y	N	Y	N	N	Y

Table 1.FCOs' situation of satisfying the properties

IFIOs can be demanded to satisfy other properties in addition to $P_{I0} - P_{I5}$.

P_6. $\pi_{\mathcal{I}(\tilde{x},\tilde{y})} \leq \vee(1-x_1, x_2, 1-y_1, y_2) \leq \vee(1-x_1, 1-y_1)$;

P_7. If $\tilde{x} = \tilde{y}$, then $\pi_{\mathcal{I}(\tilde{x},\tilde{y})} = \pi_{\tilde{x}}$;

P_8. If $\pi_{\tilde{x}} = \pi_{\tilde{y}}$, then $\pi_{\mathcal{I}(\tilde{x},\tilde{y})} = \pi_{\tilde{x}}$;

P_9. $\mathcal{I}(1_{L^*}, \tilde{y}) = \tilde{y}$;

P_{10}. $\mathcal{I}(\tilde{x}, \mathcal{I}(\tilde{y}, \tilde{z})) = \mathcal{I}(\tilde{y}, \mathcal{I}(\tilde{x}, \tilde{z}))$;

P_{11}. $\mathcal{I}(\tilde{x}, \tilde{y}) = 1_{L^*}$ if and only if $\tilde{x} \sqsubseteq \tilde{y}$;

P_{12}. $\mathcal{I}(\tilde{x}, 0_{L^*}) = \tilde{x}^c$;

P_{13}. $\mathcal{I}(\tilde{x}, \tilde{y}) \geq \tilde{y}$;

P_{14}. $\mathcal{I}(\tilde{x}, \tilde{x}) = 1_{L^*}$;

P_{15}. $\mathcal{I}(\tilde{y}^c, \tilde{x}^c) = \mathcal{I}(\tilde{x}, \tilde{y})$;

P_{16}. If $\tilde{x} = 0_{L^*}$ or $\tilde{y} = 1_{L^*}$, then $\mathcal{I}(\tilde{x}, \tilde{y}) = 1_{L^*}$;
P_{17}. $\mathcal{I}(\tilde{x}, \tilde{y}) = 1_{L^*}$ if and only if $\tilde{x} = 1_{L^*}$ and $\tilde{y} = 0_{L^*}$;
P_{18}. $\mathcal{I}(\tilde{x}, \tilde{x}^c) = \tilde{x}^c$;
P_{19}. $\mathcal{I}(\tilde{x}, \mathcal{I}(\tilde{y}, \tilde{x})) = 1_{L^*}$.

Where $P_6 - P_8$ are properties of the truly intuitionistic fuzzy and the others are the properties inherited when the fuzzy implications are generalized to the intuitionistic fuzzy case.

In intuitionistic fuzzy theory the intuitionistic fuzzy conditional rule has the form

$$\text{If } x \text{ is } A \text{ then } y \text{ is } B, \tag{4.1}$$

where x is a variable taking values in A and y is a variable taking values in B, A and B are $\mathcal{IFS}$ in X and Y, respectively.

With FIOs, FCOs and aggregation operators, the truth degree of the intuitionistic fuzzy conditional rule (4.1) is given by

$$I(M_1(\mu_A(x), 1 - \nu_A(x)), M_2(\mu_B(y), 1 - \nu_B(y))),$$

and the non-truth degree is given by

$$I_c(M_3(\nu_A(x), 1 - \mu_A(x)), M_4(\nu_B(y), 1 - \mu_B(y))),$$

where I is any FIO, I_c is any FCO and M_1, M_2, M_3, M_4 are any idempotent aggregation operators.

Evidently, the sum of these two values must be less than or equal to one, that is,

$$\begin{aligned} &I\Big(M_1(\mu_A(x), 1 - \nu_A(x)), M_2(\mu_B(y), 1 - \nu_B(y))\Big) \\ &+I_c\Big(M_3(\nu_A(x), 1 - \mu_A(x)), M_4(\nu_B(y), 1 - \mu_B(y))\Big) \leq 1. \end{aligned}$$

Obviously, the inequality above is not verified with all aggregations M_1, M_2, M_3, M_4, and just any I and I_c. Based on the following proposition, we can construct some IFIOs.

Proposition 4.1[3]. *Let I be a fuzzy implication operator and I_c be the coimplication associated to I. Let M_1, M_2, M_3, M_4 be four idempotent aggregation operators such that*

$$\begin{aligned} &M_1(x, y) + M_3(1 - x, 1 - y) \geq 1 \\ \text{and} \quad &M_2(x, y) + M_4(1 - x, 1 - y) \leq 1, \quad \text{for all} \quad x, y \in [0, 1]. \end{aligned} \tag{4.2}$$

Then $\mathcal{I} : L^{*2} \to L^*$ given by

$$\mathcal{I}(\tilde{x}, \tilde{y}) = (I(M_1(x_1, 1 - x_2), M_2(y_1, 1 - y_2)), I_c(M_3(x_2, 1 - x_1), M_4(y_2, 1 - y_1)))$$

is an IFIO.

From proposition 4.1, we can conclude that M_1, M_2, M_3, M_4 satisfy the inequality (4.2), then IFIO can be constructed. In this paper, taking $M_1 = \vee = M_3$ and $M_2 = \wedge = M_4$, then we get 32 IFIOs by 32 FIOs and their dual FCOs shown in part 3. The expressions will be shown in the following.

R_0: $$\mathcal{I}_1(\tilde{x},\tilde{y}) = \begin{cases} (1,0) & 1-x_2 \le y_1 \\ (x_2 \vee y_1, x_1 \wedge y_2) & 1-x_1 < y_2 \\ (x_2 \vee y_1, 0) & \text{otherwise} \end{cases}$$

Kleene-Dienes: $\mathcal{I}_2(\tilde{x},\tilde{y}) = (x_2 \vee y_1, x_1 \wedge y_2)$

Reichenbach $\mathcal{I}_3(\tilde{x},\tilde{y}) = (x_2 + y_1 - x_2 y_1, x_1 y_2)$

Lukasiewicz: $\mathcal{I}_4(\tilde{x},\tilde{y}) = (1 \wedge (x_2 + y_1), 0 \vee (x_1 + y_2 - 1))$

Goguen: $$\mathcal{I}_5(\tilde{x},\tilde{y}) = \begin{cases} (1,0) & x_1 = 0 \text{ and } x_2 = 1 \\ (1 \wedge \frac{y_1}{1-x_2}, 0 \vee \frac{x_1+y_2-1}{x_1}) & \text{otherwise} \end{cases}$$

Gödel: $$\mathcal{I}_6(\tilde{x},\tilde{y}) = \begin{cases} (1,0) & 1-x_2 \le y_1 \\ (y_1, y_2) & 1-x_1 < y_2 \\ (y_1, 0) & \text{otherwise} \end{cases}$$

Dubois-Prade: $$\mathcal{I}_7(\tilde{x},\tilde{y}) = \begin{cases} (x_2 \vee y_1, x_1 \wedge y_2) & x_2 \wedge y_1 = 0 \text{ and } x_1 \vee y_2 = 1 \\ (1,0) & \text{otherwise} \end{cases}$$

Zadeh: $\mathcal{I}_8(\tilde{x},\tilde{y}) = (x_2 \vee ((1-x_2) \wedge y_1), x_1 \wedge ((1-x_1) \vee y_2))$

Gaines-Rescher: $$\mathcal{I}_9(\tilde{x},\tilde{y}) = \begin{cases} (1,0) & 1-x_2 \le y_1 \\ (0,1) & 1-x_1 < y_2 \\ (0,0) & \text{otherwise} \end{cases}$$

Yager: $\mathcal{I}_{10}(\tilde{x},\tilde{y}) = (y_1^{1-x_2}, 1-(1-y_2)^{x_1})$

Mamdani: $\mathcal{I}_{11}(\tilde{x},\tilde{y}) = ((1-x_2) \wedge y_1, (1-x_1) \vee y_2)$

P.C.: $\mathcal{I}_{12}(\tilde{x},\tilde{y}) = (y_1 - y_1 x_2, 1 - x_1 + x_1 y_2)$

B.C.: $\mathcal{I}_{13}(\tilde{x},\tilde{y}) = (0 \vee (y_1 - x_2), 1 \wedge (1 - x_1 + y_2))$

E.C. $$\mathcal{I}_{14}(\tilde{x},\tilde{y}) = \begin{cases} ((1-x_2) \wedge y_1, (1-x_1) \vee y_2) & (1-x_2) \vee y_1 = 1 \\ & \text{and } (1-x_1) \wedge y_2 = 0 \\ (0,1) & \text{otherwise} \end{cases}$$

P.D.: $\mathcal{I}_{15}(\tilde{x},\tilde{y}) = (1 - x_2 + x_2 y_1, y_2 - y_2 x_1)$

B.D.: $\mathcal{I}_{16}(\tilde{x},\tilde{y}) = (1 \wedge (1 - x_2 + y_1), 0 \vee (y_2 - x_1))$

E.D. $$\mathcal{I}_{17}(\tilde{x},\tilde{y}) = \begin{cases} ((1-x_2) \vee y_1, (1-x_1) \wedge y_2)) & (1-x_2) \wedge y_1 = 0 \\ & \text{and } (1-x_1) \vee y_2 = 1 \\ (1,0) & \text{otherwise} \end{cases}$$

Einstein C.: $$\mathcal{I}_{18}(\tilde{x},\tilde{y}) = \left(\frac{y_1 - y_1 x_2}{1 + x_2 - x_2 y_1}, \frac{1 - x_1 + y_2}{1 + y_2 - x_1 y_2} \right)$$

Einstein D.: $$\mathcal{I}_{19}(\tilde{x},\tilde{y}) = \left(\frac{1 - x_2 + y_1}{1 + y_1 - x_2 y_1}, \frac{y_2 - y_2 x_1}{1 + x_1 - y_2 x_1} \right)$$

$$\mathcal{I}_{20}(\tilde{x},\tilde{y}) = \begin{cases} (y_1, y_2) & x_1 = 1 \text{ and } x_2 = 0 \\ (1,0) & \text{otherwise} \end{cases}$$

$$\mathcal{I}_{21}(\tilde{x},\tilde{y}) = (1 \wedge (x_2 + y_1 - x_2^2 y_1), 0 \vee (x_1^2 - x_1 + 2x_1 y_2 - x_1^2 y_2))$$

$$\mathcal{I}_{22}(\tilde{x},\tilde{y}) = \begin{cases} (1,0) & y_1 = 1 \text{ and } x_1 < 1 \text{ and } x_2 > 0 \\ (0,1) & \text{otherwise} \end{cases}$$

$$\mathcal{I}_{23}(\tilde{x},\tilde{y}) = ((1-x_2) \vee y_1, (1-x_1) \wedge y_2)$$

$$\mathcal{I}_{24}(\tilde{x},\tilde{y})=\begin{cases}(1,0), & 1-x_2\leq y_1\\ \left(\dfrac{x_2}{1-y_1},\dfrac{y_2+x_1-1}{y_2}\right), & 1-x_1<y_2\\ \left(\dfrac{x_2}{1-y_1},0\right), & 1-x_2\geq y_1 \text{ and } 1-x_1\geq y_2\end{cases}$$

$$\mathcal{I}_{25}(\tilde{x},\tilde{y})=\begin{cases}(1,0) & x_1=0 \text{ and } x_2=1\\ (y_1,y_2) & \text{otherwise}\end{cases}$$

$$\mathcal{I}_{26}(\tilde{x},\tilde{y})=\begin{cases}(1,0), & 1-x_2\leq y_1\\ \left(\dfrac{\log(1-x_2)}{\log y_1},\dfrac{\log\frac{1-y_2}{x_1}}{\log(1-y_2)}\right), & 1-x_2>y_1>0 \text{ and}\\ & 1-x_1<y_2<1\\ \left(\dfrac{\log(1-x_2)}{\log y_1},0\right), & 1-x_2>y_1>0 \text{ and } 1-x_1\geq y_2\\ \left(0,\dfrac{\log\dfrac{1-y_2}{x_1}}{\log(1-y_2)}\right), & 1-x_2>y_1=0 \text{ and } 1-x_1<y_2<1\\ (0,0), & 1-x_2>y_1=0 \text{ and } 1-x_1\geq y_2\\ (0,1), & \text{otherwise.}\end{cases}$$

$$\mathcal{I}_{27}(\tilde{x},\tilde{y})=\left(1\wedge(1-(1-x_2)^p+y_1^p)^{\frac{1}{p}}\right),0\wedge\left(1-(1-x_1^p+(1-y_2)^p)^{\frac{1}{p}})\right)$$

$$\mathcal{I}_{28}(\tilde{x},\tilde{y})=\left(y_1-y_1x_2+\frac{y}{2},x_1y_2+\frac{1-x_1}{2}\right)$$

$$\mathcal{I}_{29}(\tilde{x},\tilde{y})=\left(\frac{1-x_2+y_1}{2},\frac{1-x_1+y_2}{2}\right)$$

$$\mathcal{I}_{30}(\tilde{x},\tilde{y})=\begin{cases}(1,0) & 1-x_2\leq y_1\\ (x_2+y_1-x_2y_1,x_1y_2) & 1-x_1<y_2\\ (x_2+y_1-x_2y_1,0) & \text{otherwise}\end{cases}$$

$$\mathcal{I}_{31}(\tilde{x},\tilde{y})=\begin{cases}(1,0) & 1-x_2\leq y_1\\ (x_2\vee y_1\vee\frac{1}{2},x_1\wedge y_2\wedge\frac{1}{2}) & 0<y_1<1-x_2<1\\ & \text{and } 0<1-x_1<y_2<1\\ (x_2\vee y_1\vee\frac{1}{2},0) & 0<y_1<1-x_2<1\\ & \text{and } 1-x_1\geq y_2\\ (x_2\vee y_1,x_1\wedge y_2\wedge\frac{1}{2}) & y_1=0 \text{ or } x_2=0\\ & \text{and } 0<1-x_1<y_2<1\\ (x_2\vee y_1,x_1\wedge y_2)) & \text{otherwise}\end{cases}$$

$$\mathcal{I}_{32}(\tilde{x},\tilde{y})=\begin{cases}(1,0) & 1-x_2\leq y_1\\ (x_2,x_1) & 1-x_1<y_2\\ (x_2,0) & \text{otherwise}\end{cases}$$

For convenience of making use of these IFIOs in reality, in the following table, we will discuss whether the 32 IFIOs satisfy the properties P_0-P_{19}(Y stands for "Yes" and N stands for "No").

	$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	$\mathcal{I}_4$	$\mathcal{I}_5$	$\mathcal{I}_6$	$\mathcal{I}_7$	$\mathcal{I}_8$	$\mathcal{I}_9$	$\mathcal{I}_{10}$	$\mathcal{I}_{11}$	$\mathcal{I}_{12}$	$\mathcal{I}_{13}$	$\mathcal{I}_{14}$	$\mathcal{I}_{15}$	$\mathcal{I}_{16}$	$\mathcal{I}_{17}$
P_0	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
P_1	Y	Y	Y	Y	Y	Y	N	Y	Y	N	N	N	N	N	N	N	N
P_2	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
P_3	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N	N
P_4	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	N	N	N	N	Y	Y	Y
P_5	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N
P_6	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	Y	Y
P_7	Y	Y	Y	N	N	Y	Y	Y	Y	N	N	N	N	N	N	N	N
P_8	N	Y	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
P_9	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	N	N	N
P_{10}	Y	Y	Y	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y	Y
P_{11}	Y	N	N	Y	Y	Y	N	N	Y	N	N	N	N	N	N	N	N
P_{12}	Y	Y	Y	Y	N	N	Y	Y	N	N	N	N	N	N	N	N	N
P_{13}	Y	Y	Y	Y	Y	Y	Y	N	N	Y	N	N	N	N	Y	Y	Y
P_{14}	Y	N	N	Y	Y	Y	Y	N	Y	N	N	N	N	N	N	N	N
P_{15}	Y	Y	Y	Y	N	N	Y	N	Y	N	N	N	N	N	N	N	N
P_{16}	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	Y	Y	Y
P_{17}	Y	Y	Y	Y	N	Y	N	Y	N	N	N	N	N	N	N	N	N
P_{18}	N	Y	N	N	N	N	N	Y	N	N	N	N	N	N	N	N	N
P_{19}	Y	N	N	Y	N	Y	Y	N	Y	N	N	N	N	N	N	N	N

Table 2. IFIOs' Situation of Satisfying the Properties(I)

	$\mathcal{I}_{18}$	$\mathcal{I}_{19}$	$\mathcal{I}_{20}$	$\mathcal{I}_{21}$	$\mathcal{I}_{22}$	$\mathcal{I}_{23}$	$\mathcal{I}_{24}$	$\mathcal{I}_{25}$	$\mathcal{I}_{26}$	$\mathcal{I}_{27}$	$\mathcal{I}_{28}$	$\mathcal{I}_{29}$	$\mathcal{I}_{30}$	$\mathcal{I}_{31}$	$\mathcal{I}_{32}$
P_0	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
P_1	Y	Y	Y	Y	Y	Y	N	Y	N	N	N	N	N	N	N
P_2	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
P_3	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N
P_4	Y	Y	Y	Y	Y	Y	N	Y	Y	N	N	N	N	Y	Y
P_5	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N
P_6	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	N	Y	Y
P_7	N	Y	N	Y	Y	Y	N	N	Y	N	N	N	Y	Y	N
P_8	Y	Y	Y	Y	Y	N	N	Y	N	N	Y	Y	N	Y	N
P_9	Y	Y	Y	Y	Y	Y	Y	N	N	Y	N	N	Y	Y	N
P_{10}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	Y	Y	Y
P_{11}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	Y	Y	N
P_{12}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	Y	Y	N
P_{13}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	N	Y	Y
P_{14}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	Y	N	N
P_{15}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	Y	N	N
P_{16}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	N	N	Y
P_{17}	N	Y	N	N	Y	Y	Y	N	N	N	N	N	Y	Y	N
P_{18}	N	Y	Y	N	N	N	Y	N	N	Y	N	N	Y	N	N
P_{19}	Y	Y	Y	Y	Y	Y	Y	N	N	Y	Y	N	N	N	N

Table 3.IFIOs' Situation of Satisfying the Properties(II)

5. Conclusion

In the paper, we have expressed 32 fuzzy implication operators $I_0 - I_{31}$ and their dual 32 fuzzy coimplication operators $I_{0c} - I_{31c}$. It is verified whether the coimplication operators satisfy the properties $p_{c1} - p_{c14}$. We have constructed 32 intuitionistic fuzzy implication operators $\mathcal{I}_1 - \mathcal{I}_{32}$ and have verified whether they satisfy the properties $P_0 - P_{19}$. The results obtained can be applied to design an intuitionistic fuzzy control system, which is one of our future work.

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