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IMPLICIT DIFFERENCE APPROXIMATION FOR THE TIME FRACTIONAL DIFFUSION EQUATION

P. ZHUANG AND F. LIU*

ABSTRACT. In this paper, we consider a time fractional diffusion equation on a finite domain. The equation is obtained from the standard diffusion equation by replacing the first-order time derivative by a fractional derivative (of order $0 < \alpha < 1$). We propose a computationally effective implicit difference approximation to solve the time fractional diffusion equation. Stability and convergence of the method are discussed. We prove that the implicit difference approximation (IDA) is unconditionally stable, and the IDA is convergent with $O(\tau + h^2)$, where τ and h are time and space steps, respectively. Some numerical examples are presented to show the application of the present technique.

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1. Introduction

Time-fractional diffusion equations (TFDE), obtained from the standard diffusion equation by replacing the first-order time derivative by a fractional derivative of order $0 < \alpha < 1$, (in Riemann-Liouville or Caputo sense), have been treated in different contexts by a number of authors. Mainardi [14] first discussed the fractional diffusion-wave equation. Using the method of the Laplace transform, it is shown the fundamental solutions of the basic Cauchy and Signalling problems can be expressed in the terms of an auxiliary function $M(z;\beta)$, where $z = |x|/t^\beta$ is the similarity variable. Wyss [19] considered the time fractional diffusion equation and the solution is given in closed form in terms of Fox functions. Schneider and Wyss [18] considered the time fractional diffusion and wave equations. The corresponding Green functions are obtained in closed form for arbitrary space dimensions in terms of Fox functions and their properties are exhibited. Gorenflo et al. [6] used the similarity method and the method

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of Laplace transform to obtain the scale-invariant solution of time-fractional diffusion-wave equation in terms of the wright function. Liu et al. [10] considered time-fractional advection-dispersion equation and derived the complete solution. Huang and Liu [8] considered the time-fractional diffusion equations in a n-dimensional whole-space and half-space. They investigated the explicit relationships between the problems in whole-space with the corresponding problems in half-space by the Fourier-Laplace transforms. Time fractional diffusion and wave equations are derived by considering continuous time random walk problems, which are in general non-Markovian processes. Space fractional diffusion equations with the fractional spatial derivative are used for studying Markovian processes. The physical interpretation of the fractional derivative both cases is that it represents a degree of memory in the diffusing material [7] . Many authors considered more general equations. Anh and Leonenko [2] presented a spectral representation of the mean-square solution of the fractional diffusion equation with random initial condition. Gorenflo et al. [5] gave a mapping between solutions of fractional diffusion-wave equations. Enzo et al. [3] and Luisa et al. [13] considered and proved the solutions to the Cauchy problem of the fractional telegraph equation can be expressed as the distribution of a suitable composition of different processes. However, published papers on the numerical solution of fractional partial differential equations are sparse. This motivates us to consider their effective numerical methods. Liu et al. [9, 11] used fractional Method of Lines to solve the space fractional diffusion equation, they transform this partial differential equation into a system of ordinary differential equations. Fix and Roop [4] developed a finite element method for a two-point boundary value problem. Meerschaert et al. proposed finite difference approximations for two-sided space-fractional partial differential equations [15] and fractional advection-dispersion flow equations [16]. Liu et al. [12] considered a discrete non-Markovian random walk approximation for the time fractional diffusion equation and discussed the stability and convergence of the approximation.

In this paper, we consider the time fractional diffusion equation. This paper is organized as follows: The analytical solution of the fractional diffusion equation in a bounded domain is given in section 2. The fractional implicit difference approximation is proposed in section 3. In sections 4 and 5, the stability and convergence of the implicit difference approximation are analyzed respectively. In section 6, the numerical examples are given.

2. The analytical solution of the TFDE in a bounded domain

In this section, we consider the fractional diffusion equation (FDE) of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \le x \le L, \ 0 < t \le T \tag{1}$$

with initial and boundary conditions:

$$u(0,t) = u(L,t) = 0, \quad 0 \le t \le T, \tag{2}$$

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{3}$$

where $0 < \alpha < 1$.

The fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ in (1) is the Caputo fractional derivatives of order α defined [17] by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{d\xi}{(t-\xi)^\alpha}, \quad 0 < \alpha < 1. \tag{4}$$

When $\alpha = 1$, we recover in the limit the well-known diffusion equation (Markovian process)

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \le x \le L, \ 0 < t \le T. \tag{5}$$

In the case $\alpha < 1$, we have to consider the previous all time levels (non-Markovian process).

By taking the finite *sine* transform and Laplace transform, the analytical solution for the equation (1) with the boundary conditions as above is obtained [1] as:

$$u(x,t) = \frac{2}{L} \sum_{k=0}^{\infty} E_\alpha(-a^2 n^2 t^\alpha) \sin(anx) \int_0^L f(r) \sin(anr) dr, \tag{6}$$

where $a = \frac{\pi}{L}$ and

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{7}$$

is the Mittag-Leffler function.

Some special Mittag-Leffler type functions are listed as follow:

$$E_1(-z) = e^{-z}, \ E_2(-z^2) = \cos(z), \ E_{\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{k}{2}+1)} = e^{z^2} \mathtt{erfc}(-z), \tag{8}$$

where erfc(z) is the error function complement defined by

$$\mathtt{erfc}(z) = \frac{1}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt. \tag{9}$$

3. Implicit difference approximation for the TFDE

In this section the following time-fractional diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \le x \le L, \ 0 < t \le T, \tag{10}$$

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{11}$$

$$u(0,t) = u(L,t) = 0 \tag{12}$$

is considered. A fractional implicit difference approximation is proposed.

Define $t_k = k\tau, k = 0, 1, 2, \cdots, n, x_i = ih, i = 0, 1, 2, \cdots, m$, where $\tau = \frac{T}{n}$ and $h = \frac{L}{m}$ are space and time steps, respectively. Let u_i^k be the numerical approximation to $u(x_i, t_k)$.

In the differential equation (10) we have adopted a symmetric second difference quotient in space at level $t = t_{k+1}$ for approximating the second-order space derivative. The time fractional derivative term can be approximated by the following scheme:

$$\begin{aligned}
\frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{j\tau}^{(j+1)\tau} \frac{\partial u(x,\xi)}{\partial \xi} \frac{d\xi}{(t_{k+1}-\xi)^\alpha} \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{d\xi}{(t_{k+1}-\xi)^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{(k-j)\tau}^{(k-j+1)\tau} \frac{d\eta}{\eta^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{d\eta}{\eta^\alpha} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} \frac{u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})}{\tau} \Big[(j+1)^{1-\alpha} - j^{1-\alpha}\Big] \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \Big[u(x_i, t_{k+1}) - u(x_i, t_k)\Big] \\
&\quad + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} \Big[u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})\Big]\Big[(j+1)^{1-\alpha} - j^{1-\alpha}\Big]
\end{aligned}$$

Now, let $b_j = (j+1)^{1-\alpha} - j^{1-\alpha},\ j = 0, 1, 2, \cdots, n$, and define

$$L_{h,\tau}^\alpha u(x_i, t_k) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} b_j \Big[u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})\Big]. \qquad (13)$$

Then we have

$$\begin{aligned}
&\left| \frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} - L_{h,\tau}^\alpha u(x_i, t_{k+1}) \right| \\
&\le \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left| \frac{\partial u(x_i,\xi)}{\partial \xi} - \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \right| \frac{d\xi}{(t_{k+1}-\xi)^\alpha}
\end{aligned}$$

$$\le \frac{C_1}{\Gamma(1-\alpha)}\tau\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\frac{d\xi}{(t_{k+1}-\xi)^{\alpha}}$$
$$\le \frac{C_1}{\Gamma(1-\alpha)}\tau\int_{0}^{t_{k+1}}\frac{d\xi}{(t_{k+1}-\xi)^{\alpha}}$$
$$\le \hat{C}_1\tau \tag{14}$$

where C_1, are $\hat{C}_1$ are constants.

Thus, we obtain the following form

$$u_i^{k+1}-u_i^k+\sum_{j=1}^{k}b_j\Big(u_i^{k+1-j}-u_i^{k-j}\Big)=\mu\Gamma(2-\alpha)\Big(u_{i+1}^{k+1}-2u_i^{k+1}+u_{i-1}^{k+1}\Big), \tag{15}$$

for $i=1,2,\cdots,m-1,\ k=0,1,2,\cdots,n-1$, where $\mu=\dfrac{\tau^{\alpha}}{h^2}$. Let $r=\mu\Gamma(2-\alpha)$, the above equation can be rewritten as the following form

$$-ru_{i+1}^{k+1}+(1+2r)u_i^{k+1}-ru_{i-1}^{k+1}=u_i^k-\sum_{j=1}^{k}b_ju_i^{k+1-j}+\sum_{j=1}^{k}b_ju_i^{k-j}.$$

Hence, for $k=0$:

$$-ru_{i+1}^1+(1+2r)u_i^1-ru_{i-1}^1=u_i^0, \tag{16}$$

for $k>0$:

$$-ru_{i+1}^{k+1}+(1+2r)u_i^{k+1}-ru_{i-1}^{k+1}=(1-b_1)u_i^k+\sum_{j=1}^{k-1}u_i^{k-j}(b_j-b_{j+1})+b_ku_i^0, \tag{17}$$

where $i=1,2,\cdots,m$. Eqs. (16) and (17) can be written as

$$\begin{cases} A\mathbf{u}^1=\mathbf{u}^0,\\ A\mathbf{u}^{k+1}=c_1\mathbf{u}^k+c_2\mathbf{u}^{k-1}+\cdots+c_k\mathbf{u}^1+b_k\mathbf{u}^0,\ k>0,\\ \mathbf{u}^0=\mathbf{f},\end{cases} \tag{18}$$

where

$$A=\begin{bmatrix} 1+2r & -r & 0 & \cdots & 0 & 0\\ -r & 1+2r & -r & \cdots & 0 & 0\\ 0 & -r & 1+2r & \cdots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1+2r & -r\\ 0 & 0 & 0 & \dots & -r & 1+2r \end{bmatrix}, \tag{19}$$

$$\mathbf{u}^k=\begin{bmatrix} u_1^k\\ u_2^k\\ \vdots\\ u_{m-1}^k\end{bmatrix},\ \mathbf{f}=\begin{bmatrix} f(x_1)\\ f(x_2)\\ \vdots\\ f(x_{m-1})\end{bmatrix} \tag{20}$$

and $c_j=b_{j-1}-b_j,\ j=1,2,\cdots,n.$

Using the properties of the function $g(x)=x^{1-\alpha}(x\geq 1)$, the following results can be obtained:

$$\begin{cases} 1=b_0>b_1>b_2>\cdots\to 0, \\ \sum_{j=1}^{k} c_j = 1-b_k, \\ \sum_{j=1}^{\infty} c_j = 1, 1>2-2^{1-\alpha}=c_1>c_2>c_3>\cdots\to 0. \end{cases} \tag{21}$$

Hence, we have

Remark 1. In (18) we see that A is strictly diagonally dominant with positive diagonal terms and nonpositive offdiagonal terms. The equation (18) can be solved.

Remark 2. Matrix A is an M-matrix. The solution $u_i^k, (i=0,1,2,\cdots,m; k=0,1,\cdots,n)$ preserves non-negativity, if $u_i^0,\ i=0,1,2,\cdots,m$ are non-negative.

Using inductive approach, we can obtain the following conclusion from (16) and (17).

Remark 3. The solution u_i^j is conservative, i.e.,

$$\sum_{i=-\infty}^{\infty} |u_i^0| < \infty \Rightarrow \sum_{i=-\infty}^{\infty} |u_i^j| = \sum_{i=-\infty}^{\infty} |u_i^0|, j\in N. \tag{22}$$

4. Stability of implicit difference approximation

We suppose that $\widetilde{u}_i^j, (i=0,1,2,\cdots,m; j=0,1,2,\cdots,n)$ is the approximate solution of (16) and (17), the error $\varepsilon_i^j=\widetilde{u}_i^j-u_i^j, (i=0,1,2,\cdots,m; j=0,1,2,\cdots,n)$ satisfies

$$\begin{aligned} & -r\varepsilon_{i+1}^1+(1+2r)\varepsilon_i^1-r\varepsilon_{i-1}^1=\varepsilon_i^0, \\ & -r\varepsilon_{i+1}^{k+1}+(1+2r)\varepsilon_i^{k+1}-r\varepsilon_{i-1}^{k+1}=c_1\varepsilon_i^k+\sum_{j=1}^{k-1} c_{j+1}\varepsilon_i^{k-j}+b_k\varepsilon_i^0, k>0, \\ & (i=1,2,\cdots,m-1), \end{aligned} \tag{23}$$

which can be written as

$$\begin{cases} A\mathbf{E}^{k+1}=c_1\mathbf{E}^k+c_2\mathbf{E}^{k-1}+\cdots+c_k\mathbf{E}^1+b_k\mathbf{E}^0, \\ \mathbf{E}^0, \end{cases} \tag{24}$$

where $\mathbf{E}^k=\begin{bmatrix} \varepsilon_1^k \\ \varepsilon_2^k \\ \vdots \\ \varepsilon_{m-1}^k \end{bmatrix}$. Hence, the following result can be proved using mathematical induction.

Proposition 1. $\|\mathbf{E}^k\|_\infty \le \|\mathbf{E}^0\|_\infty,\ k = 1, 2, 3, \cdots$.

Proof. For $k = 1$, $-r\varepsilon_{i+1}^1 + (1+2r)\varepsilon_i^1 - r\varepsilon_{i-1}^1 = \varepsilon_i^0$. Let $|\varepsilon_l^1| = \max\limits_{1\le i\le m-1} |\varepsilon_i^1|$. Then we have

$$\begin{aligned} |\varepsilon_l^1| &\le -r|\varepsilon_{l+1}^1| + (1+2r)|\varepsilon_l^1| - r|\varepsilon_{l-1}^1| \\ &\le \Big| - r\varepsilon_{l+1}^1 + (1+2r)\varepsilon_l^1 - r\varepsilon_{l-1}^1 \Big| \\ &= |\varepsilon_l^0| \le \|E^0\|_\infty. \end{aligned} \tag{25}$$

Thus, $\|\mathbf{E}^1\|_\infty \le \|\mathbf{E}^0\|_\infty$. Suppose that $\|\mathbf{E}^j\|_\infty \le \|\mathbf{E}^0\|_\infty, j = 1, 2, \cdots, k$. Let $|\varepsilon_l^{k+1}| = \max\limits_{1\le i\le m-1} |\varepsilon_i^{k+1}|$.

Then we also have

$$\begin{aligned} |\varepsilon_l^{k+1}| &\le -r|\varepsilon_{l+1}^{k+1}| + (1+2r)|\varepsilon_l^{k+1}| - r|\varepsilon_{l-1}^{k+1}| \\ &\le \Big| - r\varepsilon_{l+1}^{k+1} + (1+2r)\varepsilon_l^{k+1} - r\varepsilon_{l-1}^{k+1} \Big| \\ &= \left| c_1\varepsilon_l^k + \sum_{j=1}^{k-1} c_{j+1}\varepsilon_l^{k-j} + b_k\varepsilon_l^0 \right| \\ &\le c_1|\varepsilon_l^k| + \sum_{j=1}^{k-1} c_{j+1}|\varepsilon_l^{k-j}| + b_k|\varepsilon_l^0| \qquad (26) \\ &\le c_1\|\mathbf{E}^k\|_\infty + \sum_{j=1}^{k-1} c_{j+1}\|\mathbf{E}^{k-j}\|_\infty + b_k\|\mathbf{E}^0\|_\infty \\ &\le \left\{ c_1 + \sum_{j=1}^{k-1} c_{j+1} + b_k \right\} \|\mathbf{E}^0\|_\infty \\ &= \|\mathbf{E}^0\|_\infty, \end{aligned}$$

i.e., $\|\mathbf{E}^{k+1}\|_\infty \le \|\mathbf{E}^0\|_\infty$. □

Hence, the following theorem is obtained.

Theorem 1. *The fractional implicit difference approximations defined by (16) and (17) are unconditionally stable.*

5. Convergence of implicit difference approximation

Let $u(x_i, t_k), (i = 1, 2, \cdots, m-1; k = 1, 2, \cdots, n)$ be the exact solution of the TFDE (10) - (12) at mesh point (x_i, t_k). Define $e_i^k = u(x_i, t_k) - u_i^k, i =$

$1, 2, \cdots, m-1; k = 1, 2, \cdots, n$ and $\mathbf{e}^k = (e_1^k, e_2^k, \cdots, e_{m-1}^k)^T$. Using $\mathbf{e}^0 = 0$, substitution into (16) and (17) leads to

$$
\begin{aligned}
&-re_{i+1}^1 + (1+2r)e_i^1 - re_{i+1}^1 = R_i^1, \\
&-re_{i+1}^{k+1} + (1+2r)e_i^{k+1} - re_{i+1}^{k+1} = c_1 e_i^k + \sum_{j=1}^{k-1} c_{j+1} e_i^{k-j} + R_i^{k+1}, k > 0
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
R_i^{k+1} &= u(x_i, t_{k+1}) - u(x_i, t_k) + \sum_{j=1}^{k-1} b_j \Big[u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})\Big] \\
&\quad -\mu\Gamma(2-\alpha)\Big[u(x_{i+1}, t_{k+1}) - 2u(x_i, t_{k+1}) + u(x_{i-1}, t_{k+1})\Big] \\
&= \sum_{j=0}^{k} b_j \Big[u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})\Big] \\
&\quad -\mu\Gamma(2-\alpha)\Big[u(x_{i+1}, t_{k+1}) - 2u(x_i, t_{k+1}) + u(x_{i-1}, t_{k+1})\Big].
\end{aligned} \tag{28}
$$

From (14), we have

$$
\frac{1}{\Gamma(2-\alpha)\tau^\alpha} \sum_{j=0}^{k} b_j \Big[u(x_i, t_{k+1-j}) - u(x_i, t_{k-j})\Big] = \frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} + \hat{C}_1 \tau
$$

and

$$
\frac{u(x_{i+1}, t_{k+1}) - 2u(x_i, t_{k+1}) + u(x_{i-1}, t_{k+1})}{h^2} = \frac{\partial^2 u(x_i, t_{k+1})}{\partial x^2} + C_2 h^2.
$$

Hence,

$$
R_i^{k+1} = \tau^\alpha \Gamma(2-\alpha) \left[\frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} - \frac{\partial^2 u(x_i, t_{k+1})}{\partial x^2}\right] + \hat{C}_1 \tau^{1+\alpha} + C_2 \tau^\alpha h^2.
$$

Also

$$
|R_i^{k+1}| \leq C(\tau^{1+\alpha} + \tau^\alpha h^2), \ i = 1, 2, \cdots, m-1; \ k = 0, 1, 2, \cdots, n
$$

where C is a constant.

Proposition 2. $\|\mathbf{e}^k\|_\infty \leq C b_{k-1}^{-1}(\tau^{1+\alpha} + \tau^\alpha h^2), \ k = 1, 2, \cdots, n,$ *where* $\|\mathbf{e}^k\|_\infty = \max\limits_{1 \leq i \leq m-1} |e_i^k|$ *and* C *is a constant.*

Proof. Using mathematical induction method. For $k = 1$, let $\|\mathbf{e}^1\|_\infty = |e_l^1| = \max\limits_{1 \leq i \leq m-1} |e_i^1|$, we have

$$
\begin{aligned}
|e_l^1| &\leq -r|e_{l+1}^1| + (1+2r)|e_l^1| - r|e_{l-1}^1| \\
&\leq |-re_{l+1}^1 + (1+2r)e_l^1 - re_{l-1}^1| \\
&= \ |R_i^1| \\
&\leq Cb_0^{-1}(\tau^{1+\alpha} + \tau^\alpha h^2)
\end{aligned} \tag{29}
$$

Suppose that

$$\|\mathbf{e}^{j-1}\|_\infty \le C b_j^{-1}(\tau^{1+\alpha}+\tau^\alpha h^2), j=1,2,\cdots,k \quad \text{and} \quad |e_l^{k+1}| = \max_{1\le i\le m-1} |e_i^{k+1}|.$$

Note that $b_j^{-1} \le b_k^{-1},\ j=0,1,\cdots,k$. We have

$$\begin{aligned}
|e_l^{k+1}| &\le -r|e_{l+1}^{k+1}| + (1+2r)|e_l^{k+1}| - r|e_{l-1}^{k+1}| \\
&\le |-re_{l+1}^{k+1} + (1+2r)e_l^{k+1} - re_{l-1}^{k+1}| \\
&= \left| c_1 e_l^k + \sum_{j=1}^{k-1} c_{j+1} e_l^{k-j} + R_i^{k+1} \right| \\
&\le \left| c_1 e_l^k + \sum_{j=1}^{k-1} c_{j+1} e_l^{k-j} + R_i^{k+1} \right| \\
&\le c_1 |e_l^k| + \sum_{j=1}^{k-1} c_{j+1} |e_l^{k-j}| + C(\tau^{1+\alpha}+\tau^\alpha h^2) \\
&\le c_1 \|\mathbf{e}^k\|_\infty + \sum_{j=1}^{k-1} c_{j+1} \|\mathbf{e}^{k-j}\|_\infty + C(\tau^{1+\alpha}+\tau^\alpha h^2) \\
&\le \left[c_1 + \sum_{j=1}^{k-1} c_{j+1} + b_k \right] b_k^{-1} C(\tau^{1+\alpha}+\tau^\alpha h^2) \\
&= b_k^{-1} C(\tau^{1+\alpha}+\tau^\alpha h^2),
\end{aligned} \tag{30}$$

Because

$$\begin{aligned}
\lim_{k\to\infty} \frac{b_k^{-1}}{k^\alpha} &= \lim_{k\to\infty} \frac{k^{-\alpha}}{(k+1)^{1-\alpha} - k^{1-\alpha}} \\
&= \lim_{k\to\infty} \frac{k^{-1}}{(1+\frac{1}{k})^{1-\alpha} - 1} \\
&= \lim_{k\to\infty} \frac{k^{-1}}{(1-\alpha)k^{-1}} \\
&= \frac{1}{1-\alpha},
\end{aligned} \tag{31}$$

hence, there is a constant $\bar{C}$,

$$\|\mathbf{e}^k\|_\infty \le \bar{C} k^\alpha (\tau^{1+\alpha}+\tau^\alpha h^2).$$

□

Because $k\tau \le T$ is finite, we obtain the following result.

Theorem 2. *Let u_i^k be the approximate value of $u(x_i,t_k)$ computed by use of the difference scheme (16) and (17). Then there is a positive constant C such that*

$$|u_i^k - u(x_i,t_k)| \le C(\tau+h^2),\ i=1,2,\cdots,m-1;\ k=1,2,\cdots,n \tag{32}$$

6. Numerical results

In this section, we present an example in a finite domain to demonstrate that the IDA is a computationally effective method, and the IDA can been applied to simulate the behavior of the solution of the fractional reaction-diffusion equation. We consider the following fractional diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2},\ 0 \le x \le 2,\ t > 0, \tag{33}$$

the boundary conditions $u(0,t) = u(2,t) = 0$ and initial condition

$$u(x,0) = f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ \dfrac{4-2x}{3}, & \frac{1}{2} \le x \le 2 \end{cases}. \tag{34}$$

The function $f(x)$ represents the temperature distribution in a bar generated by a point heat source kept in the point $x = \frac{1}{2}$ for long enough.

A comparison of the exact solution and the numerical solution using IDA with various time and space steps for TFDE at $t = 0.4$ and $\alpha = 0.5$ is shown in Figure 1. It is apparent from Figure 1 that the numerical solution is in complete agreement with exact solution. Computed errors between the exact solution and IDA for a various time and space steps for TFDE at $t = 0.4$ and $\alpha = 0.5$ are listed in Table 1 and show the effect of τ and h. From Table 1, it can be seen that our method (IDA) yields convergence with $O(\tau + h^2)$.

Figures 2 and 3 compare the response of the diffusion system for different real numbers $0 < \alpha < 1$ at $t = 0.4$ and different x, and at $x = 1.5$ and different t, respectively.

Table 1. The error $u_i^k - u(x_i, t_k)$ at $t = 0.4$

x_i	$h = \frac{1}{4}, \tau = \frac{1}{20}$	$h = \frac{1}{8}, \tau = \frac{1}{75}$	$h = \frac{1}{20}, \tau = \frac{1}{400}$
0.25	6.6909E-3	1.101E-3	-4.002E-4
0.50	1.49298E-2	4.2616E-3	1.3944E-3
0.75	1.30396E-2	3.035E-3	3.537E-4
1.00	7.6896E-3	-9.254E-4	-3.2303E-3
1.25	9.2103E-3	2.4512E-3	6.444E-4
1.50	6.2192E-3	1.593E-3	3.568E-4
1.75	2.8617E-3	6.264E-4	-1.093E-4

7. Conclusions

In this paper, an implicit finite difference approximation for the time fractional diffusion equation in a bounded domain have been described and demonstrated. We prove that the implicit difference approximation is unconditionally stable and convergent. The technique can be applied to solve fractional-order differential equation.

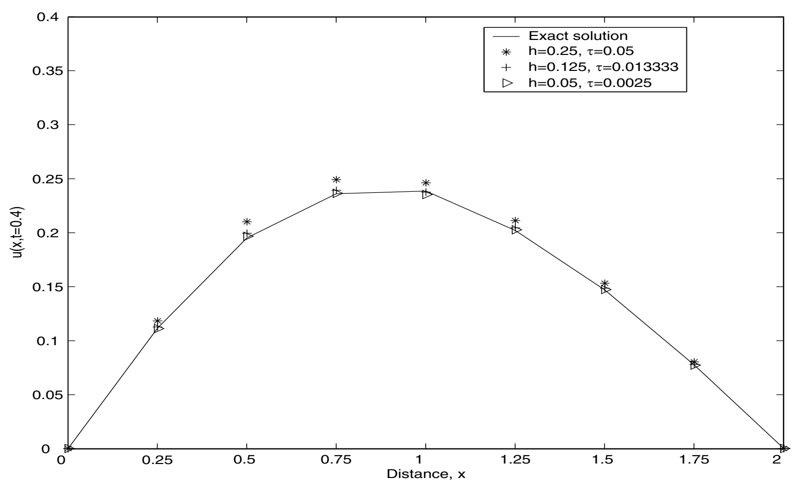

FIGURE 1. Comparison of the exact solution and numerical solution (IDA) at $t = 0.4$ and $\alpha = 0.5$.

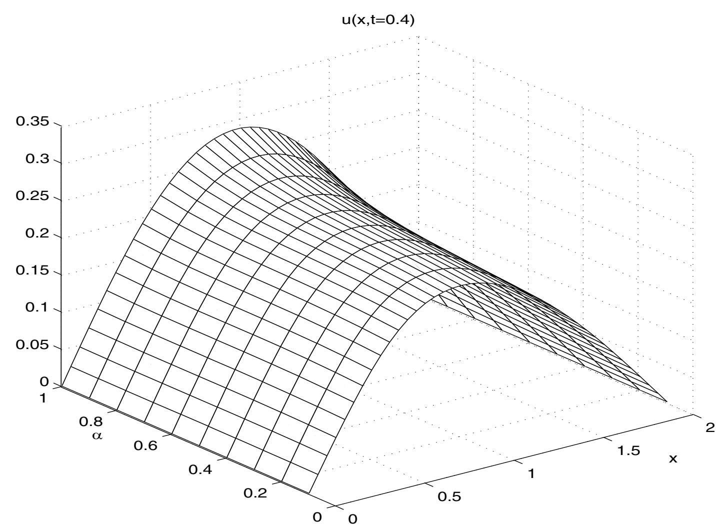

FIGURE 2. Displacement as a function of x at $t = 0.4$ for various α.

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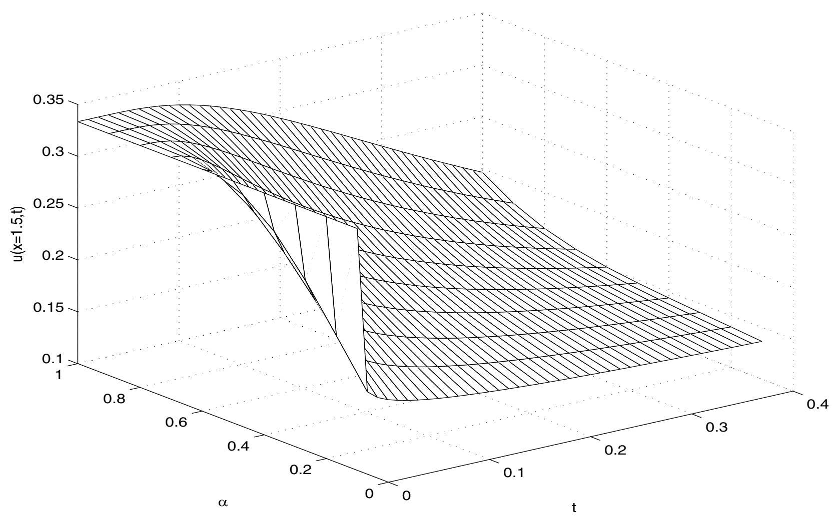

FIGURE 3. Displacement as a function of t at $x = 1.5$ for various α.

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Pinghui Zhuang received his MSc from Fuzhou University in 1988. Now he is an associate professor at Xiamen University. His research interest is numerical analysis and techniques for solving partial differential equations.

School of Mathematical Science, Xiamen University, Xiamen 361005, China
e-mail: zxy1104@xmu.edu.cn

Fawang Liu received his MSc from Fuzhou University in 1982 and PhD from Trinity College, Dublin, in 1991. Since graduation, he has been working in computational and applied mathematics at Fuzhou University, Trinity College Dublin and University College Dublin, University of Queensland, Queensland University of Technology and Xiamen University. Now he is a Professor at Xiamen University. His research interest is numerical analysis and techniques for solving a wide variety of problems in applicable mathematics, including semiconductor device equations, microwave heating problems, gas-solid reactions, singular perturbation problem, saltwater intrusion into aquifer systems and fractional differential equations.

(1) Department of Mathematics, Xiamen University, Xiamen 361005, China
(2) School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Qld. 4001, Australia
e-mail: fwliu@xmu.edu.cn; f.liu@qut.edu.au