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OSCILLATION OF SUBLINEAR DIFFERENCE EQUATIONS WITH POSITIVE NEUTRAL TERM

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Abstract. In this paper, we consider the oscillation of first order sublinear difference equation with positive neutral term

$$\triangle(x(n)+p(n)x(\tau(n)))+f(n,x(g_1(n)),\cdots,x(g_m(n)))=0.$$

We obtain necessary and sufficient conditions for the solutions of this equation to be oscillatory.

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1. Introduction

Recently, there has been an increasing interest in the study of the oscillatory behaviors to the solutions of nonlinear and linear difference equations. The reference [1-6] concern with the oscillation of nonlinear difference equations, while [7] concern with that of linear difference equations. The reference [8-10] studied respectively the oscillation of sublinear, superlinear and half-linear difference equations. In this paper, our objective is to give necessary and sufficient conditions for the oscillation of the following sublinear equation (1.1).

We mainly concern with oscillation of solutions for the following first order sublinear difference equation with positive neutral term

$$\triangle(x(n)+p(n)x(\tau(n)))+f(n,x(g_1(n)),\cdots,x(g_m(n)))=0 \tag{1.1}$$

where $\triangle$ is the forward difference operator: $\triangle x(n)=x(n+1)-x(n)$; $p(n)$ is a sequence of real numbers, $1<p_1\le p(n)\le p_2$, p_1 and p_2 are constants; $\tau(n)$ is a sequence of strictly increasing integers with $\tau(n)<n$, $\lim_{n\to\infty}\tau(n)=\infty$; $g_i(n)$ is sequence of positive integers with $\lim_{n\to\infty}g_i(n)=\infty$; $f(n,x_1,\cdots,x_m)$ is

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continuous in each x_i; $g(n) = \max\limits_{1\le i\le m}\{g_i(n)\} \le \tau(n)$.

A solution $x(n)$ of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative, otherwise, it is nonoscillatory. The equation is called oscillatory if and only if all its solutions are oscillatory.

$f(n, y_1, \cdots, y_m)$ is said to be strongly sublinear if there exists constants $\beta \in (0,1)$, β is a quotient of odd positive integers and $d > 0$ such that

$$|y|^{-\beta}|f(n, y, \cdots, y)|$$

is nonincreasing in $|y|$ for $0 < |y| \le d$.

2. Related lemmas

To obtain our main results, we need the following lemmas.

Lemma 2.1. *Assume*

$$z(n) = x(n) + p(n)x(\tau(n)) \tag{2.1}$$

where $\tau(n)$ is strictly increasing with $\tau(n) < n$, $\triangle z(n) < 0$, $x(n) > 0$, $1 < p_1 \le p(n) \le p_2$. Then

$$x(n) \ge \frac{p_1 - 1}{p_1 p_2} z(\tau^{-1}(n)).$$

Proof. Let τ^{-1} be the inverse function of τ and $\tau^{-2}(n) \equiv \tau^{-1}(\tau^{-1}(n))$. From(2.1), we have

$$\begin{aligned} x(n) &= \frac{z(\tau^{-1}(n)) - x(\tau^{-1}(n))}{p(\tau^{-1}(n))} \\ &= \frac{z(\tau^{-1}(n))}{p(\tau^{-1}(n))} - \frac{1}{p(\tau^{-1}(n))}\left[\frac{z(\tau^{-2}(n)) - x(\tau^{-2}(n))}{p(\tau^{-2}(n))}\right] \\ &\ge \frac{z(\tau^{-1}(n))}{p(\tau^{-1}(n))} - \frac{z(\tau^{-2}(n))}{p(\tau^{-1}(n))p(\tau^{-2}(n))} \\ &\ge \left[\frac{1}{p(\tau^{-1}(n))} - \frac{1}{p(\tau^{-1}(n))p(\tau^{-2}(n))}\right] z(\tau^{-1}(n)) \\ &\ge \frac{p_1 - 1}{p_1 p_2} z(\tau^{-1}(n)) \end{aligned}$$

for all large n. The proof is completed. □

Lemma 2.2. *Assume the difference inequality*

$$\triangle y(n) + q(n)y^{\beta}(\sigma(n)) \le 0 \tag{2.2}$$

has an eventually positive solution, where $q(n) \geq 0$, $q(n) \not\equiv 0$, $\beta > 0$, β *is a quotient of odd positive integers. Then the difference equation*

$$\triangle y(n) + q(n)y^{\beta}(\sigma(n)) = 0 \tag{2.3}$$

also has an eventually positive solution.

Proof. Let $y(n)$ is an eventually positive solution of (2.2). Define a set $\Omega = \{\ x(n) \mid 0 \leq x(n) \leq y(n), n \geq N\}$. Then define a mapping Γ on Ω as follows:

$$(\Gamma x)(n) = \begin{cases} \sum\limits_{s=n}^{\infty} q(s)x^{\beta}(\sigma(s)), & n \geq N \\ (\Gamma x)(N) + y(n) - y(N), & n_0 \leq n \leq N. \end{cases}$$

Define a sequence $x_k(n),\ k = 1, 2, \cdots$ as follows:

$$\begin{aligned} &x_1(n) = y(n), \\ &x_{k+1}(n) = (\Gamma x_k)(n) \quad k = 1, 2, \cdots . \end{aligned}$$

Since $y(n)$ is a solution of (2.2), we get

$$\triangle y(n) + q(n)y^{\beta}(\sigma(n)) \leq 0.$$

Summing the above inequality from n to N and letting $N \to \infty$, we obtain

$$-y(n) + \sum_{s=n}^{\infty} q(s)y^{\beta}(\sigma(s)) \leq 0,$$

or

$$x_2(n) = (\Gamma y)(n) = \sum_{s=n}^{\infty} q(s)y^{\beta}(\sigma(s)) \leq y(n) = x_1(n).$$

By induction, we see that

$$0 \leq x_k(n) \leq x_{k-1}(n) \leq \cdots \leq x_1(n) = y(n),\ n \geq n_0.$$

Hence $\lim\limits_{k\to\infty} x_k(n) = x(n)$ exists with $0 \leq x(n) \leq y(n)$. Then we can apply the Lebesgue's dominated convergence theorem to show that $x = \Gamma x$, i.e.,

$$x(n) = \sum_{s=n}^{\infty} q(s)x^{\beta}(\sigma(s)),\ n \geq n_0.$$

Obviously, $x(n)$ is an eventually positive sollution of (2.3). Since $x(n) > 0$ for $n \in [n_0, N]$, it follows from (2.3) that $x(n) > 0$ for all $n \geq n_0$.
The proof is completed. □

Lemma 2.3([1]). *Consider the difference equation*

$$\triangle x(n) + q(n)x^{\alpha}(n - k) = 0 \tag{2.4}$$

where $q(n)$ is a sequence of nonnegative real numbers and α is a quotient of odd positive integers. Assume $0 < \alpha < 1$. Then every solution of (2.4) oscillates if and only if

$$\sum_{n=0}^{\infty} q(n) = \infty.$$

3. Main results

Theorem 3.1. *Assume that*

(i) $f \in C\ ([n_0, \infty) \times R^m, R\ )$ *is nondecreasing in each* y_i *and* $y_1 f(n, y_1, \cdots, y_m) > 0$ *for* $y_1 y_i > 0,\ 1 \le i \le m$;

(ii) $f \in C\ ([n_0, \infty) \times R^m, R\ )$ *is strongly sublinear;*

(iii) $$\sum_{s=n}^{\infty} f(s, a, \cdots, a) = \infty \text{ for every constant } a > 0. \tag{3.1}$$

Then every solution of Eq.(1.1) is oscillatory.

Proof. Assume the contrary and let $x(n)$ be a nonoscillatory solution of Eq.(1.1). Without loss of generality we may assume that $x(n)$ is eventually positive. From (2.1), we get $z(n) > 0$, for all large n. From Eq.(1.1) and condition (i), we have $\triangle z(n) < 0$. Since $\tau(n)$ is strictly increasing , τ^{-1} is the inverse function of τ and $\tau^{-2}(n) \equiv \tau^{-1}(\tau^{-1}(n))$. Therefore, by Lemma 2.1, we have

$$x(n) \ge \frac{p_1 - 1}{p_1 p_2} z(\tau^{-1}(n)). \tag{3.2}$$

Setting

$$y(n) = \frac{p_1 - 1}{p_1 p_2} z(n),$$

From Eq.(1.1), we have

$$\triangle y(n) + \frac{p_1 - 1}{p_1 p_2}\ f(n, x(g_1(n)), \cdots, x(g_m(n))) = 0,\ n \ge N.$$

From (3.2) and (i), we obtain

$$\triangle y(n) + \frac{p_1 - 1}{p_1 p_2}\ f(n, y(\tau^{-1}(g_1(n))), \cdots, y(\tau^{-1}(g_m(n)))) \le 0,\ n \ge N. \tag{3.3}$$

So

$$\triangle y(n) \le 0.$$

We claim that

$$\lim_{n\to\infty} y(n) = 0.$$

Assume the contrary and let $\lim\limits_{n\to\infty} y(n) = l > 0$. Thus $y(n) \ge l$. From $g(n) \le \tau(n)$ and the fact that $f(n, y_1, \cdots, y_m)$ is nondecreasing in each y_i, we obtain

$$f(n, y(\tau^{-1}(g_1(n))), \cdots, y(\tau^{-1}(g_m(n)))) \ge f(n, l, \cdots, l). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\triangle y(n)+\frac{p_1-1}{p_1p_2}f(n,l,\cdots,l)\leq 0.$$

Summing the above inequality from n to N and letting $N\to\infty$, we get

$$l-y(n)+\frac{p_1-1}{p_1p_2}\sum_{s=n}^{\infty}f(s,l,\cdots,l)\leq 0.$$

That is,

$$\frac{p_1-1}{p_1p_2}\sum_{s=n}^{\infty}f(s,l,\cdots,l)\leq y(n)<\infty,$$

which contradicts (iii).
So

$$\lim_{n\to\infty}y(n)=0.$$

Because $f(n,y_1,\cdots,y_m)$ is strongly sublinear, there exists $d>0$ and $0<\beta<1$ such that

$$\frac{|f(n,y(\tau^{-1}(g(n))),\cdots,y(\tau^{-1}(g(n))))|}{|y(\tau^{-1}(g(n)))|^{\beta}}$$

is nonincreasing in $|y|$ for $0<|y|\leq d$.
There exists $N_1\geq n_0$ such that $0<y(\tau^{-1}(g(n)))\leq d$ for $n\geq N_1$.
From (i) (ii) and above statements, we have

$$\begin{aligned}&f\Big(n,y(\tau^{-1}(g_1(n))),\cdots,y(\tau^{-1}(g_m(n)))\Big)\\&\geq f\Big(n,y(\tau^{-1}(g(n))),\cdots,y(\tau^{-1}(g(n)))\Big)\\&\geq d^{-\beta}\ (y(\tau^{-1}(g(n))))^{\beta}\ f(n,d,\cdots,d)\quad n\geq N_1.\end{aligned}\tag{3.5}$$

In view of (3.3) and (3.5), we get

$$\triangle y(n)+\frac{p_1-1}{p_1p_2d^{\beta}}\ \big(y(\tau^{-1}(g(n)))\big)^{\beta}f(n,d,\cdots,d)\leq 0.$$

From Lemma 2.2, the difference equation

$$\triangle y(n)+\frac{p_1-1}{p_1p_2d^{\beta}}\ \big(y(\tau^{-1}(g(n)))\big)^{\beta}f(n,d,\cdots,d)=0.\tag{3.6}$$

also has an eventually positive solution.
However, from (3.1) and Lemma 2.3 ,we obtain every solution of Eq.(3.6) is oscillatory, which is a contradiction. The proof is completed. □

Theorem 3.2. *Assume that (i) in Theorem 3.1 hold. Then Eq.(1.1) has a bounded nonoscillatory solution which is bounded away from zero if and only if*

$$\sum_{s=n}^{\infty}|f(s,d,\cdots,d)|<\infty\tag{3.7}$$

for some constant $d\neq 0$.

Proof. (I) Necessity: Let $x(n)$ is a bounded positive solution satisfying $x(g_i(n)) \geq b > 0,\ x(g_i(n)) \leq c$ for $n \geq n_1 \geq n_0, 1 \leq i \leq m$.
From Eq.(2.1), (i) of Theorem 3.1 and Eq.(1.1), we have

$$\triangle z(n) + f(n, b, \cdots, b) \leq 0.$$

Summing the above inequality from n to N and letting $N \to \infty$, we get

$$-z(n) + \sum_{s=n}^{\infty} f(s, b, \cdots, b) \leq 0,$$

i.e.,

$$\sum_{s=n}^{\infty} f(s, b, \cdots, b) \leq z(n).$$

Since $x(n)$ and $p(n)$ are all bounded, $z(n)$ defined by (2.1) is bounded.
So

$$\sum_{s=n}^{\infty} f(s, b, \cdots, b) < \infty.$$

That is to say, (3.7) hold. For the case that $x(n)$ is eventually negative, the proof is similar.

(II) Sufficiency: Assume there exists $d > 0$ such that (3.7) hold. Let $c > 0$ be such that $p_1 c \leq d$. In view of (3.7), there exists $N \geq n_0$ and $N_0 = \min\{\tau(N), \inf_{n \geq N}\{g_1(n)\}, \cdots, \inf_{n \geq N}\{g_m(n)\}\}$ such that for $n \geq N$

$$\sum_{s=n}^{\infty} f(s, d, \cdots, d) \leq (p_1 - 1)c. \tag{3.8}$$

We denote with B, the set of all sequences with the topology of uniform convergence on $[N_0, N]$. Define a set $X \subset B$ as follows:

$$X = \Big\{x(n) \in B:\ c \leq x(n) \leq p_1 c,\ \triangle x(n) \leq 0,\ n \geq N;\ x(n) = x(N), \\ N_0 \leq n \leq N\Big\}$$

Since $\tau(n)$ is strictly increasing, the inverse function of τ exists. We denote it with τ^{-1}, and $\tau^{-i}(n) \equiv \tau^{-1}(\tau^{-(i-1)}(n))$
For every $x \in X$ we define:

$$\overline{x}(n) = \begin{cases} \displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{i-1} x(\tau^{-i}(n))}{Q_i(\tau^{-i}(n))}, & n \geq N \\ \overline{x}(N), & N_0 \leq n \leq N, \end{cases} \tag{3.9}$$

where

$$\tau_0(n) \equiv n,\ Q_0(n) \equiv 1,\ Q_i(n) = \prod_{j=0}^{i-1} p(\tau^j(n)),\ i = 1, 2, \cdots \tag{3.10}$$

Since (3.9) is Leibniz's series for $n \geq N$, it is convergent and the sign of $\overline{x}(n)$ is the same as that of the first term.
So

$$0 < \overline{x}(n) \leq \frac{x(\tau^{-1}(n))}{Q_1(\tau^{-1}(n))} = \frac{x(\tau^{-1}(n))}{p(\tau^{-1}(n))} \leq p_1 c.$$

Define a mapping Γ on X as follows:

$$(\Gamma x)(n) = \begin{cases} c + \sum_{s=n}^{\infty} f(s, \overline{x}(g_1(s)), \cdots, \overline{x}(g_m(s))), & n \geq N \\ (\Gamma x)(N), & N_0 \leq n \leq N. \end{cases} \quad (3.11)$$

From (3.8), it is easy to see that $\Gamma X \subset X$. By definition, we get Γ is continuous and $\Gamma(X)$ is uniformly bounded. For every $x(n) \in X$ and $\varepsilon > 0$, there exists $N_1 \geq N$ such that

$$\begin{aligned} |(\Gamma x)(n_1) - (\Gamma x)(n_2)| &= \left| \sum_{s=n_1}^{n_2} f(s, \overline{x}(g_1(s)), \cdots, \overline{x}(g_m(s))) \right| \\ &\leq \left| \sum_{s=n_1}^{n_2} f(s, d, \cdots, d) \right| \\ &< \varepsilon \end{aligned}$$

for $n_1, n_2 \geq N_1$. Thus $\Gamma(X)$ is relatively compact in the topology of B. By Schauder Tychonoff fixed point theorem, there exists a $x \in X$ such that $x = \Gamma x$. i.e.,

$$x(n) = c + \sum_{s=n}^{\infty} f(s, \overline{x}(g_1(s)), \cdots, \overline{x}(g_m(s))). \quad (3.12)$$

In view of (3.9), we have

$$\overline{x}(n) + p(n)\overline{x}(\tau(n)) = x(n). \quad (3.13)$$

So

$$\overline{x}(n) + p(n)\overline{x}(\tau(n)) = c + \sum_{s=n}^{\infty} f(s, \overline{x}(g_1(s)), \cdots, \overline{x}(g_m(s))).$$

Now it is easy to see that $\overline{x}(n)$ is a solution of Eq.(1.1). From Lemma 2.1 and (3.13), we have

$$\frac{p_1 - 1}{p_1 p_2} c \leq \frac{p_1 - 1}{p_1 p_2} x(\tau^{-1}(n)) \leq \overline{x}(n) \leq x(n) \leq p_1 c.$$

Therefore, $\overline{x}(n)$ is a bounded nonoscillatory solution of Eq.(1.1) which is bounded away from zero. The proof is completed. □

Theorem 3.3. *Assume that (i)-(ii) in Theorem 3.1 hold, and $g(n) \leq \tau(n)$. Then all solutions of Eq.(1.1) are oscillatory if and only if*

$$\sum_{s=n}^{\infty} |f(s,a,\cdots,a)| = \infty$$

for every constant $a \neq 0$.

Proof. From Theorem 3.1, we get sufficiency immediatly. Necessity: If not, assume that

$$\sum_{s=n}^{\infty} |f(s,a,\cdots,a)| < \infty.$$

In view of Theorem 3.2, Eq.(1.1) has a nonoscillatory solution, which contradicts our assumption. The proof is completed. ☐

Remark. The condition (ii) is important for Theorem 3.1 and Theorem 3.3 to hold. In fact, Theorem 3.1 and Theorem 3.3 are not true probably for linear equations.

Example 1. Consider the difference equation

$$\triangle(x(n)+2x(n-1)) + \frac{5}{8}x(n-2) = 0.$$

In our notation, $p(n) = 2, \tau(n) = n-1, g(n) = n-2, f(n, x(g_1(n)), \cdots, x(g_m(n))) = \frac{5}{8}x(n-2)$. It is easy to show that the condition (i) and (ii) in Th. 3.1 and Th.3.3 are satisfied. But the above equation has a nonoscillatory solution. In fact, $x(n) = 2^{-n}$ is such a solution.

Example 2. Consider the difference equation

$$\triangle(x(n)+2x(n-1)) + 2x^{\frac{1}{3}}(n-3) = 0.$$

In our notation, $p(n) = 2, \tau(n) = n-1, g(n) = n-3, f(n, x(g_1(n)), \cdots, x(g_m(n))) = 2x^{\frac{1}{3}}(n-3)$. It is easy to see that all assumptions of Th.3.1 are satisfied. Therefore, every solution of equation is oscillatory.

Theorem 3.4 (Comparison Theorem). *Consider another difference equation*

$$\triangle(x(n)+p(n)x(\tau(n))) + q(n, x(g_1(n)), \cdots, x(g_m(n))) = 0. \tag{3.14}$$

Assume Eq.(3.14) satisfies (i) and (ii) of Th.3.1 with $g(n) < \tau(n)$ and

$$q(n, |y|, \cdots, |y|) \leq f(n, |y|, \cdots, |y|), \tag{3.15}$$

for every $|y| \neq 0$. If Eq.(3.14) is oscillatory, then Eq.(1.1) is oscillatory.

Proof. Assume the contrary and let $x(n)$ be an eventually positive solution of Eq.(1.1). In view of (3.15) we obtain $x(n)$ which is an eventually positive solution of the below inequality

$$\triangle(x(n)+p(n)x(\tau(n)))+q(n,x(g_1(n)),\cdots,x(g_m(n)))\leq 0. \tag{3.16}$$

Therefore, Eq.(3.14) also has an eventually positive solution, which contradicts the assumption of theorem. The proof is completed. □

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