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The Optimal Solution of the Model with Physical and Human Capital Adjustment Costs

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Abstract: We prove that the model with physical and human capital adjustment costs has optimal solution when the production function is increasing return and the structure of vetor fields of the model changes substantially when the prodution function from decreasing return turns to increasing return. And it is shown that the economy is improved when the coefficients of adjustment costs become small.

Key words: optimal solution; nonzero equilibrium; adjustment costs

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0 Introduction

Recently, Petr Duczynski *et al*[1,2] set up an economic growth model with both physical capital and human capital adjustment costs which extends the model with single physical capital adjustment cost[3]. The production function is assumed decreasing return which derives the Jacobi matrix of the dynamical system at the nonzero equilibrium has two negative eigenvalues and two positive eigenvalues. This implies that the dynamical system has a two-dimension stable manifold near the nonzero equilibrium. So, the model has multiple feasible solutions and the existence of the optimal solution is not provided.

In this paper, we first strictly prove the model has unique nonzero equilibrium. Then we prove that the model has unique feasible solution so that the model has optimal solution when the production function is increasing return. We point out that the structure of vetor fields of the model changes substantially when the production funclion from decreasing return turns to increasing return, and we prove that the economy becomes better when coefficients the adjustment costs is smaller.

1 The Model

The original optimal problem which presented by Petr Duczynski[1] is given below:

$$\max\int_0^{\infty} e^{-(r-x-n)t}\{f(k,h)-i[1+\phi(\frac{i}{k})]-j[1+\psi(\frac{j}{h})]\}dt \tag{1}$$

$$\text{s.t } \dot{k} = i-(\delta_1+n+x)k, \dot{h} = j-(\delta_2+n+x)h,$$
$$k(0)=k_0, h(0)=h_0,$$

where $y=f(k,h)=Ak^{\alpha}h^{\eta}$[4] is the production function and r is the world real interest rate, x is the exogenously given growth rate of technology, n is the growth rate of raw labor.

In the model, k and h stand for physical and human capital per effective worker, i and j are investments in physical and human capital and δ_1,δ_2 are the depreciation rates for physical and human capital respectively. The unit adjustment cost functions are assumed to take the following forms:

$$\phi\left(\frac{i}{k}\right)=\frac{b_1}{\omega+1}\left(\frac{i}{k}\right)^{\omega}, \psi\left(\frac{i}{h}\right)=\frac{b_2}{\omega+1}\left(\frac{i}{h}\right)^{\omega}$$

where $\omega\geqslant 1, b_1>0, b_2>0$.

From the first order conditions to solve the optimal problem (1), the four-dimension dynamical system is obtained[5].

$$\dot{k}=[b_1^{-\frac{1}{\omega}}(\lambda-1)^{\frac{1}{\omega}}-(\delta_1+n+x)]k \tag{2}$$

$$\dot{h}=[b_1^{-\frac{1}{\omega}}(\mu-1)^{\frac{1}{\omega}}-(\delta_2+n+x)]h \tag{3}$$

$$\dot{\lambda}=(r+\delta_1)\lambda-[\alpha f(k,h)k^{-1}]+\frac{\omega b_1^{-\frac{1}{\omega}}}{\omega+1}(\lambda-1)^{1+\frac{1}{\omega}} \tag{4}$$

$$\dot{\mu}=(r+\delta_2)\mu-[\eta f(k,h)h^{-1}]+\frac{\omega b_2^{-\frac{1}{\omega}}}{\omega+1}(\mu-1)^{1+\frac{1}{\omega}} \tag{5}$$

2 The Existence and Uniqueness of the Nonzero Equilibrium

Let

$$\begin{cases} c_1=(r+\delta_1)[b_1(\delta_1+n+x)^{\omega}] \\ d_1=\dfrac{\omega b_1}{\omega+1}(\delta_1+n+x)^{\omega+1} \end{cases} \tag{6}$$

$$\begin{cases} c_2=(r+\delta_2)[b_1(\delta_1+n+x)^{\omega}] \\ d_2=\dfrac{\omega b_2}{\omega+1}(\delta_2+n+x)^{\omega+1} \end{cases} \tag{7}$$

then we have following theorem:

Theorem 1 If $c_1>d_1, c_2>d_2$, and the parameters α,η satisfy $\alpha+\eta\neq 1$, then the dynamical system (2)-(5) has unique nonzero equilibrium.

Proof An point $P^*=(k^*,h^*,\lambda^*,\mu^*)$ is a nonzero equilibrium of dynamical system (2)-(5) if and only if it satisfies

$$b_1^{-\frac{1}{\omega}}(\lambda^*-1)^{\frac{1}{\omega}}-(\delta_1+n+x)]=0 \tag{8}$$

$$b_2^{-\frac{1}{\omega}}(\mu^*-1)^{\frac{1}{\omega}}-(\delta_2+n+x)]=0 \tag{9}$$

$$(r+\delta_1)\lambda^*-[\alpha f(k^*,h^*)(k^*)^{-1}]+\frac{\omega b_1^{-\frac{1}{\omega}}}{\omega+1}(\lambda^*-1)^{1+\frac{1}{\omega}}=0 \tag{10}$$

$$(r+\delta_2)\mu^*-[\eta f(k^*,h^*)(h^*)^{-1}]+\frac{\omega b_2^{-\frac{1}{\omega}}}{\omega+1}(\mu^*-1)^{1+\frac{1}{\omega}}=0 \tag{11}$$

that is, it satisfies

$$\lambda^*=1+b_1(\delta_1+n+x)^{\omega}$$
$$\mu^*=1+b_2(\delta_2+n+x)^{\omega}$$

and

$$(r+\delta_1)1+[1+b_1(\delta_1+n+x)^{\omega}]=\alpha f(k^*,h^*)(k^*)^{-1}+\frac{\omega b_1}{\omega+1}(\delta_1+n+x)^{\omega+1} \tag{12}$$

$$(r+\delta_2)1+[1+b_2(\delta_2+n+x)^{\omega}]=\eta f(k^*,h^*)(h^*)^{-1}+\frac{\omega b_2}{\omega+1}(\delta_2+n+x)^{\omega+1} \tag{13}$$

Eq. (12) and (13) imply that

$$\eta(c_1-d_1)k^*=\alpha\eta f(k^*,h^*),$$
$$\alpha(c_2-d_2)h^*=\alpha\eta f(k^*,h^*)$$

i.e. $h^*=\dfrac{\eta(c_1-d_1)}{\alpha(c_2-d_2)}k^*=mk^*$, where $m\overset{\text{def}}{=}\dfrac{\eta(c_1-d_1)}{\alpha(c_2-d_2)}$

So, we have

$$c_1=\alpha A(k^*)^{\alpha-1}(h^*)^{\eta}+d_1=\alpha Am^{\eta}(k^*)^{\alpha+\eta-1}+d_1,$$

that is,

$$k^*=\left(\frac{c_1-d_1}{\alpha Am^{\eta}}\right)^{\frac{1}{\alpha+\eta-1}}$$

provided $\alpha+\eta\neq 1$. Therefore, the equation (8)-(11) has unique solution.

Remark From Eqs. (6) and (7), we know that the conditions $c_1>d_1, c_2>d_2$ can be satisfied provided b_1 and b_2 are small enough.

3 The Optimal Solution of the Model

Denote the right hand of the equations of (2)-(5) by $F_i(k,h,\lambda,\mu)$, $i=1,2,3,4$ respectively and the Jacobi matrix at the equilibrium P^* by A, then

$$A=\begin{pmatrix} \frac{\partial F_1}{\partial k} & \frac{\partial F_1}{\partial h} & \frac{\partial F_1}{\partial \lambda} & \frac{\partial F_1}{\partial \mu} \\ \frac{\partial F_2}{\partial k} & \frac{\partial F_2}{\partial h} & \frac{\partial F_2}{\partial \lambda} & \frac{\partial F_2}{\partial \mu} \\ \frac{\partial F_3}{\partial k} & \frac{\partial F_3}{\partial h} & \frac{\partial F_3}{\partial \lambda} & \frac{\partial F_3}{\partial \mu} \\ \frac{\partial F_4}{\partial k} & \frac{\partial F_4}{\partial h} & \frac{\partial F_4}{\partial \lambda} & \frac{\partial F_4}{\partial \mu} \end{pmatrix}=\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ c & d & e & 0 \\ d & f & 0 & e \end{pmatrix} \tag{14}$$

where

$$\begin{cases} a = \dfrac{(\delta_1+n+x)^{1-\omega}}{b_1\omega}k^*, b = \dfrac{(\delta_2+n+x)^{1-\omega}}{b_2\omega}h^* \\ c = \dfrac{\alpha(1-\alpha)A(h^*)^\eta}{(k^*)^{2-\alpha}}, \quad d = -\dfrac{\alpha\eta A}{(k^*)^{1-\alpha}(h^*)^{1-\eta}} \\ e = r-n-x, \qquad f = \dfrac{\eta(1-\eta)A(k^*)^\alpha}{(h^*)^{2-\eta}} \end{cases} \tag{15}$$

Lemma 1 The matrix A has a couple positive and a couple negative eigenvalues when $\alpha+\eta<1$, and one eigenvalue with negative real part and three eigenvalue one with positive real parts when $\alpha+\eta>1$.

Proof From Eqs. (14)-(15), we have

$$|A| = ab(cf-d^2)$$
$$= \frac{\alpha\eta A^2(k^*)^{2\alpha-1}(h^*)^{2\eta-1}(1-\alpha-\eta)}{b_1b_2\omega^2(\delta_1+n+x)^{\omega-1}(\delta_2+n+x)^{\omega-1}} \tag{16}$$

and

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & a & 0 \\ 0 & \lambda & 0 & b \\ c & d & \lambda-e & 0 \\ d & f & 0 & \lambda-e \end{vmatrix}$$
$$= \lambda^4 - 2e\lambda^3 + (ac+bf+e^2)\lambda^2 - (ae+bf)e\lambda + ab(cf-d_2).$$

Let $\sigma_1 = 2ac+e^2+2bf$,

$\sigma_2 = a^2c^2+4abd^2-2abcf+b^2f^2$,

then it is easy to check that eigenvalues of matrix A are

$$\begin{cases} \lambda_1 = \dfrac{1}{2}\left\{e-\sqrt{\sigma_1-2\sqrt{\sigma_2}}\right\} \\ \lambda_2 = \dfrac{1}{2}\left\{e+\sqrt{\sigma_1-2\sqrt{\sigma_2}}\right\} \end{cases} \tag{17}$$

$$\begin{cases} \lambda_3 = \dfrac{1}{2}\left\{e-\sqrt{\sigma_1+2\sqrt{\sigma_2}}\right\} \\ \lambda_4 = \dfrac{1}{2}\left\{e+\sqrt{\sigma_1+2\sqrt{\sigma_2}}\right\} \end{cases} \tag{18}$$

Since $\sigma_2 = (ac+bf)^2+4ab(d^2-cf)$

$= (ac-bf)^2+4adb^2>0$,

the eigenvalues $\lambda_3<0, \lambda_4>0$, from Eq. (18) and

$$2\sqrt{\sigma_2}\begin{cases} > 2ac+2bf, \text{ if } d^2>cf, \\ < 2ac+2bf, \text{ if } d^2<cf \end{cases}$$

that is, by Eq. (17),

$$\sigma_1 - 2\sqrt{\sigma_2}\begin{cases} < e^2, \text{ if } d^2>cf \\ > e^2, \text{if } d^2<cf. \end{cases}$$

So, the eigenvalues λ_1, λ_2 has positive real parts if $d^2>cf$ and $\lambda_1<0, \lambda_2>0$ if $d^2<cf$.

From Eq. (16), $d^2>cf$ if and only if $\alpha+\eta>1$. Hence, the lemma holds.

From Lemma 1, we know that the dynamical system(2)-(5)has a two-dimension stable manifold near the equilibrium P^* when $\alpha+\eta<1$ and a one-dimension stable manifold near the equilibrium P^* when $\alpha+\eta>1$.

Remark The structure of vector fields of the dynamical system (2)-(5) changes substantially on the two sides of the line $\alpha+\eta=1$.

Theorem 2 When the production fuction is increasing return, that is, $\alpha+\eta>1$, then the model has optimal solution.

4 Summary

As we have proved that the model has unique optimal solution when $\alpha+\eta>1$. Near the nonzero equilibrium the optiml solution of the dynamical system (2)-(5) converges to the nonzero equilibrium at the approximal rate of λ_3. This implies the model includes the β convergence which is discussed by Barro[3].

On the other hand, that adjustment of human capital has much affection on the economy for the production process must be increasing return. This indicates that the economy uses increasing return's production method to offset the adjustment costs of physical and human capital.

If the condition $\delta+r-\dfrac{w}{w+1}(\delta+n+x)<0$ holds, then from $\dfrac{\partial(c_1-d_1)}{\partial b_1} = \dfrac{\partial(c_2-d_2)}{\partial b_2}<0$ and the representation of k^*, h^*, we have

$$\frac{\partial k^*}{\partial b_1}<0, \frac{\partial k^*}{\partial b_2}<0, \frac{\partial h^*}{\partial b_1}<0, \frac{\partial h^*}{\partial b_2}<0 \tag{19}$$

The inequality (19) implies that the economy will become worse if either the coefficient of adjustment cost of physical capital or the coefficient of the adjustment of human capital increases. This also implies that decrease the adjustment costs can improve the economy.

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