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Mathematical Model of the Identical Slope Surface

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Abstract: **The formation of the identical slope surface and the method of construction are discussed. On the basement of building the parameter equation of variable-radius circle family envelope, the frequently used parameter equation of the identical slope surface of the top of taper moving along column helix, horizental arc and line is built. The equation can be used to construct the identical slope surface's contours, gradient lines and three dimensional figures correctly.**

Key words: **identical slope surface; family of circles; mathematical model; envelope; elevation projection; contours**

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0 Introduction

In the design of civil engineering and water conservancy project, the identical slope surface represented as an envelope of a cone with its vertex moving along the cylindrical helix, level arc or line etc is frequently used. For example, the earth dams in the hydraulic engineering systems or the corner on both sides of the road in the highway construction often need digging or backfilling to form such slope, determining the intersecting lines and excavation lines, where slope intersects slope or slope intersects land surfaces. For computer aided design and plotting the identical slope surface, the formulation of its mathematical model presents a critical problem.

The identical slope surface can be viewed as the envelope surface of taper family of which the axis of taper is vertical to level and the top of taper moves along spatial curve ABC. When the horizontal plane π of different height ($z=h$) and the envelope surface of taper family cross, the crossed line with taper family becomes variable-radius circle family L_1, the crossed line with the envelope surface of taper family becomes identical slope surface's contour L_2, L_1 and L_2 are tangent everywhere. So the article mainly discusses how to use variable-radius circle family envelope to build the mathematical model of the contour of the identical slope surface.

1 Equation for the Family of Circles Envelope Curve

Assuming that horizontal projection of the space curve $\widehat{ABC}$ is curve $\widehat{abc}$. The parametric equation for the curve may be expressesed as Fig. 1.

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$$\boldsymbol{r}(s) = \{x(s), y(s)\}$$

s expresses the length parameter of arc $\widehat{abc}$. $R(s)$ expresses the variable radius of circles of family of circles. $\alpha(s)$ expresses the unit tangent vector of the curve $\widehat{abc}$. $\beta(s)$ expresses the normal vector of the curve $\widehat{abc}$. The equation for the envelope curve of circles may be expressesed as

$$\begin{cases} [\rho - \boldsymbol{r}(s)]^2 = R^2(s) \\ [\rho - \boldsymbol{r}(s) + R(s) \cdot \dfrac{\mathrm{d}R}{\mathrm{d}s} \\ \quad \cdot \alpha(s)] \cdot \alpha(s) = 0 \end{cases} \tag{1}$$

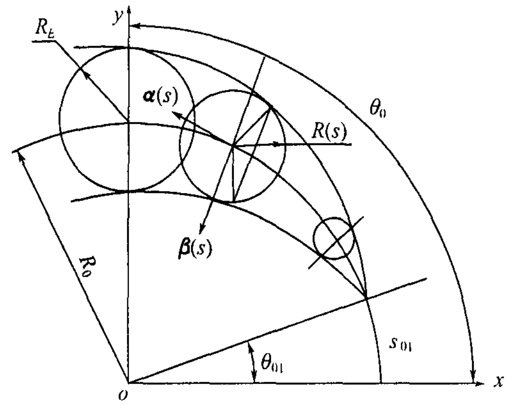

Fig. 1 The principle of contours

From Eq. (1), it is obtained that vector $\rho - \boldsymbol{r}(s) + R(s) \cdot \dfrac{\mathrm{d}R}{\mathrm{d}s} \cdot \alpha(s)$ is perpendicular to the unit tangent vector $\alpha(s)$, thus, the unit normal vector $\beta(s)$ must be collinear with the vector $\rho - \boldsymbol{r}(s) + R(s) \cdot \dfrac{\mathrm{d}R}{\mathrm{d}s} \cdot \alpha(s)$ on the deferent $\boldsymbol{r}(s)$, and they are in linear correlation, so

$$\rho - \boldsymbol{r}(s) + R(s) \cdot \frac{\mathrm{d}R}{\mathrm{d}s} \cdot \alpha(s) = \lambda\beta(s)$$

or

$$\rho - \boldsymbol{r}(s) = \lambda\beta(s) - R(s) \cdot \frac{\mathrm{d}R}{\mathrm{d}s} \cdot \alpha(s) \tag{2}$$

where λ as the undetermined coefficient.

From Eq. (1) and Eq. (2), λ is computed by from $\lambda = \pm R(s) \cdot \sqrt{1 - (\frac{\mathrm{d}R}{\mathrm{d}s})^2}$. Substituing λ into Eq. (2), the equation for the family of circles envelope curve is obtained as follows:

$$\rho = \boldsymbol{r}(s) - R(s) \cdot \frac{\mathrm{d}R}{\mathrm{d}s} \cdot \alpha(s) \pm R(s) \cdot \sqrt{1 - (\frac{\mathrm{d}R}{\mathrm{d}s})^2} \cdot \beta(s) \tag{3}$$

This is a parametric equation, s is arc length parameter.

2 Mathematical Models for the Identical Slope Surface in Common Use

2.1 Equation of the Identical Slope Surface for the Vertex of a Cone Moving Along a Cylindrical Helix

Now let the vertex of a cone moves along a cyindrical helix. The cylindrical helix horizontal projection on the plane ($z=h$) is a circular arc[1] with radius R_0 and it's equation is $\boldsymbol{r}(s)$. $R(s)$ is variable radius of the family circles which length increases uniformly from 0 to R_E[1-3], when the cone goes down. Each parameter shows as Fig. 1, θ_{01} = angle of the starting point, s_{01} = arc length of the starting point, θ=angle of the end point, s_0=arc length of the end point, taking θ as parameter of the curve $\boldsymbol{r}(s)$, then $R(s) = \dfrac{R_E(s - s_{01})}{s_0 - s_{01}}, \dfrac{\mathrm{d}R}{\mathrm{d}s} = \dfrac{R_E}{s_0 - s_{01}}, R(s)\dfrac{\mathrm{d}R}{\mathrm{d}s} = \dfrac{R_E^2(s - s_{01})}{(s_0 - s_{01})^2}$, $\boldsymbol{r}(s) = \boldsymbol{r}(s(\theta)) = \boldsymbol{r}(\theta) = \{R_0 \cos\theta, R_0 \sin\theta\}$.

Substituting each parameter above into Eq. (3), the equation for the family of circles envelope curve with θ as the parameter may be expressesed as

$$\rho(\theta) = \boldsymbol{r}(\theta) - \frac{R_E^2(\theta - \theta_{01})}{(\theta_0 - \theta_{01})^2 R_0} \cdot \alpha(\theta) \pm \frac{R_E(\theta - \theta_{01})}{\theta_0 - \theta_{01}} \cdot \sqrt{1 - \left[\frac{R_E}{(\theta_0 - \theta_{01})R_0}\right]^2} \cdot \beta(\theta) \tag{4}$$

The coordinate equation is

$$\begin{cases} X(\theta) = \left[R_0 \mp \frac{R_E(\theta-\theta_{01})}{(\theta_0-\theta_{01})} \cdot \sqrt{1-\left[\frac{R_E}{(\theta_0-\theta_{01})R_0}\right]^2}\right]\cos\theta + \frac{R_E^2(\theta-\theta_{01})}{(\theta_0-\theta_{01})^2R_0}\sin\theta \\ Y(\theta) = \left[R_0 \mp \frac{R_E(\theta-\theta_{01})}{(\theta_0-\theta_{01})} \cdot \sqrt{1-\left[\frac{R_E}{(\theta_0-\theta_{01})R_0}\right]^2}\right]\sin\theta - \frac{R_E^2(\theta-\theta_{01})}{(\theta_0-\theta_{01})^2R_0}\cos\theta \end{cases} \quad (5)$$

Supposing the slope of the identical slope surface is $1 : L_P$, where, L_P is horizon range. From $\tan\alpha = \frac{1}{L_P} = \frac{z(\theta)-h}{R(\theta)}$, $R(\theta) = L_P(z(\theta)-h) = \frac{R_E(\theta-\theta_{01})}{\theta_0-\theta_{01}}$, where h is the parameter of elevation, $z(\theta)$ is height of the conic vertex. Substituting them into Eq. (5), so the coordinate equation of the identical slope surface for the conic vertex moving along the cylindrical helix is obtained as follows:

$$\begin{cases} X(\theta,h) = \left[R_0 \mp L_P(z(\theta)-h) \cdot \sqrt{1-\left[\frac{L_P(z(\theta)-h)}{(\theta-\theta_{01})R_0}\right]^2}\right]\cos\theta + \frac{L_P^2(z(\theta)-h)^2}{(\theta-\theta_{01})R_0}\sin\theta \\ Y(\theta,h) = \left[R_0 \mp L_P(z(\theta)-h) \cdot \sqrt{1-\left[\frac{L_P(z(\theta)-h)}{(\theta-\theta_{01})R_0}\right]^2}\right]\sin\theta - \frac{L_P^2(z(\theta)-h)^2}{(\theta-\theta_{01})R_0}\cos\theta \\ Z(\theta,h) = h, \ (\theta \neq \theta_{01}) \end{cases} \quad (6)$$

where θ is in circular measure, and $\theta \neq \theta_{01}$. From $\boldsymbol{r}\{x(\theta), y(\theta), z(\theta)\}$, $x(\theta) = R_0\cos\theta$, $y(\theta) = R_0\sin\theta$, $z(\theta)$ is the height of the cylindrical helix, which computed by the actual needs in the project.

From Eq. (6), it is derived that the identical slope surface can be considered as a parametric curvilinear mesh consisting of parameter θ and h. When fixing parameter h, we can obtain the parametric curve for θ, i.e., the contour line. When the parameter θ, we can obtain the parametric curve for h, i.e. the grade line, it is a straight line.

When $\theta - \theta_{01} = 0$, Eq. (6) can not be used to compute. At this time, the starting point for the contour line is located on the curve $\boldsymbol{r}_0(\theta)$, and can be computed by

$$\begin{cases} X(\theta,h) = R_0\cos\theta_{01} \\ Y(\theta,h) = R_0\sin\theta_{01} \\ Z(\theta,h) = h \end{cases} \quad (7)$$

where θ_{01} is computed by $z(\theta)$.

For example: on the indetical slope surface, find the contour curves which elevation are 15, 16, 17. Fig. 2 illustrates the elevation projection of the identical slope surface for the conic vertex moving along a levorotatory cylindrical helix. Fig. 2(a) needs backfilling, Fig. 2(b) needs excavating, the equation is expressesed as (unit: mm):

$$\begin{cases} x(\theta) = 15\,000\cos\theta \\ y(\theta) = 15\,000\sin\theta \\ z(\theta) = 18\,500 - 50\,\theta \end{cases}$$

the slope $i = \frac{1}{2}$, $L_P = 2$, successive elevation is 17、16、15、14, respectively.

2.2 Equation of the Identical Slope Surface for a Conic Vertex Moving Along a Level Circular Arc

When a conic vertex moves along a level circular arc the radius of circle of its family is constant. Eq. (3) becomes as follow:

$$\rho = r(s) \pm R \cdot \dot{\beta}(s) \quad (8)$$

We obtain correspondingly the equation of the identical slope surface as follow:

$$\begin{cases} X(\theta,h) = [R_0 \mp L_P(z(\theta)-h)]\cos\theta \\ Y(\theta,h) = [R_0 \mp L_P(z(\theta)-h)]\sin\theta \\ Z(\theta,h) = h \end{cases} \quad (9)$$

2.3 Equation of the Identical Slope Surface for a Conic Vertex Moving Along a Space Straight Line

When the vertex of a cone moving along space line the regular circular conic envelope surface becomes a plane.

Showed as Fig. 3, point (X_E, Y_E, Z_E) and point (X_0, Y_0, Z_0) are extreme points of a space straight line. Horizontal projection of the space straight line is a straight line whose equation is

$$\boldsymbol{r}(s) = \boldsymbol{P}_{01} + s_x \cdot \boldsymbol{l}_0 \tag{10}$$

where $\boldsymbol{P}_{01}$ is the start point (X_{01}, Y_{01}), s_x is the length parameter of the line, $\boldsymbol{l}_0$ is the unit vector. The coordinate equation is

$$\begin{cases} x - X_{01} = \dfrac{X_E - X_{01}}{s_0 - s_{01}}(s - s_{01}) \\ y - Y_{01} = \dfrac{Y_E - Y_{01}}{s_0 - s_{01}}(s - s_{01}) \end{cases} \tag{11}$$

Substituting

$$\alpha(s)=\left\{\frac{X_E-X_{01}}{s_0-s_{01}},\frac{Y_E-Y_{01}}{s_0-s_{01}}\right\},\quad \beta(s)=\left\{-\frac{Y_E-Y_{01}}{s_0-s_{01}},\frac{X_E-X_{01}}{s_0-s_{01}}\right\},\quad R(s)=\frac{R_E(s-s_{01})}{s_0-s_{01}},\quad \frac{\mathrm{d}R}{\mathrm{d}s}=\frac{R_E}{s_0-s_{01}}$$

into Eq. (3), the coordinate equation is obtained as follows:

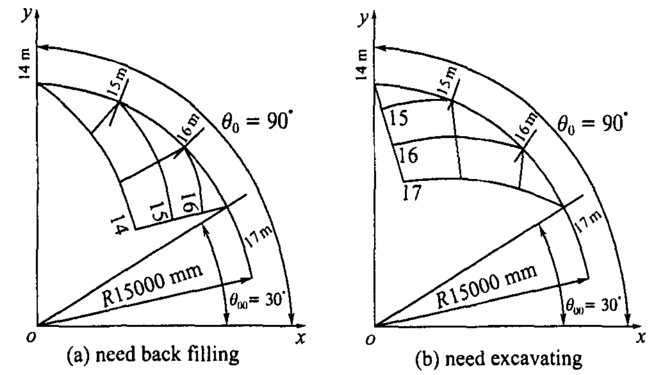

Fig. 2 Example

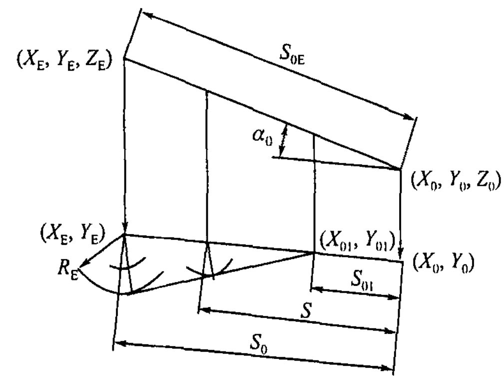

Fig. 3 The principle of contours

$$\begin{cases} X(s) = X_{01} + \dfrac{(X_E - X_{01})(s - s_{01})}{s_0 - s_{01}} - \dfrac{R_E^2(X_E - X_{01})(s - s_{01})}{(s_0 - s_{01})^3} \mp \dfrac{R_E(s - s_{01})}{(s_0 - s_{01})^2} \\ \quad \cdot \sqrt{1 - \left[\dfrac{R_E}{s_0 - s_{01}}\right]^2}(Y_E - Y_{01}) \\ Y(s) = Y_{01} + \dfrac{(Y_E - Y_{01})(s - s_{01})}{s_0 - s_{01}} - \dfrac{R_E^2(Y_E - Y_{01})(s - s_{01})}{(s_0 - s_{01})^3} \pm \dfrac{R_E(s - s_{01})}{(s_0 - s_{01})^2} \\ \quad \cdot \sqrt{1 - \left[\dfrac{R_E}{s_0 - s_{01}}\right]^2}(X_E - X_{01}) \end{cases} \tag{12}$$

From $\dfrac{R_E(s-s_{01})}{s_0-s_{01}} = L_P(z(s)-h)$, hence $R_E = \dfrac{L_P(z(s)-h)(s_0-s_{01})}{s-s_{01}}$, substituting it into Eq. (12), Eq. (13) is the equation of the identical slope surface for conic vertex moving along space line,

$$\begin{cases} X(s,h) = X_{01} + \dfrac{(X_E - X_{01})(s - s_{01})}{s_0 - s_{01}} - \dfrac{L_P^2(z(s) - h)^2(X_E - X_{01})}{(s - s_{01})(s_0 - s_{01})} \\ \quad \mp \dfrac{L_P(z(s) - h)}{(s_0 - s_{01})} \cdot \sqrt{1 - \left[\dfrac{L_P(z(s) - h)}{s - s_{01}}\right]^2}(Y_E - Y_{01}) \\ Y(s,h) = Y_{01} + \dfrac{(Y_E - Y_{01})(s - s_{01})}{s_0 - s_{01}} - \dfrac{L_P^2(z(s) - h)^2(Y_E - Y_{01})}{(s - s_{01})(s_0 - s_{01})} \\ \quad \pm \dfrac{L_P(z(s) - h)}{(s_0 - s_{01})} \cdot \sqrt{1 - \left[\dfrac{L_P(z(s) - h)}{s - s_{01}}\right]^2}(X_E - X_{01}) \\ Z(s,h) = h \quad (s_0 \neq s_{01}, s \neq s_{01}) \end{cases} \tag{13}$$

where $X_{01}=X_0+\frac{X_E-X_0}{s_0}s_{01}, Y_{01}=Y_0+\frac{Y_E-Y_0}{s_0}s_{01}, s_{01}=\frac{(z(s_{01})-Z_0)s_{0E}}{Z_E-Z_0}\cos\alpha_0$.

When the conic vertex moves along space level straight beeline, the equation of identical slope surface may be expressesed as

$$\begin{cases} X(s,h) = X_0 + \dfrac{(X_E - X_0)s}{s_0} \mp \dfrac{Y_E - Y_0}{s_0}L_P(Z_0 - h) \\ X(s,h) = Y_0 + \dfrac{(Y_E - Y_0)s}{s_0} \pm \dfrac{X_E - X_0}{s_0}L_P(Z_0 - h) \\ Z(s,h) = h \end{cases} \tag{14}$$

3 Conclusions

The mathematical models established above for the identical slope surface are real three-dimensional model which can not only be applied to the elevation projection of all kinds of hydraulic structures, road slope and so on, but also is suitable for showing the three-dimensional view of such curved surface, providing computer aided design, computer assisted instruction for such projects with a practical, ideal mathematical model and providing accurate data of contours and grade lines.

References:

[1] Zhu Yu-wan. *Descriptive Geometry*. Beijing: Higher Education Press, 1997. 199-213.
[2] Hu Jian-guo. Application Research on Envelope Curved Surface of Family of Variable-Radius Spherical Surfaces. *Journal of Wuhan University of Hydraulic and Electric Engineering*, 1997, **6**: 91-94.
[3] Ai Yun-jun. *Analysis Introduction of Engineering Cartography*. Beijing: China Railway Publish House, 1987.

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Further Study on Responsibility of Si Photovoltaic Detector

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Abstract: The open circuit voltage of silicon photovoltaic detector (UV-110) was studied in this paper. An abnormal peak voltage was observed in some devices at definite illumination, then the open circuit voltage decreases with increasing light intensity. The results indicate that the rectification contact between metal and semiconductor is responsible for the abnormal phenomenon. It also strongly degrades the responsibility of these devices.

Key words: silicon photovoltaic detector; open circuit voltage; parallel resistance; series resistance; rectification contact