IL NUOVO CIMENTO VOL. 74 A, N. 1 1 Marzo 1983

A Mathematical Approach to the Nuclear-Matter Problem.

C. VILLI

Istituto di Fisica dell'Università - Padova
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 26 Luglio 1982)

Summary. — The aim of this paper is to outline a mathematical scheme for the description of the overall behaviour of infinite nuclear matter: it has been conceived with the pragmatic purpose of providing simple tools for carrying out realistic numerical calculations. To this end we shall revive some insufficiently explored aspects of vintage theories, which seem to have been by-passed by recent theoretical developments.

1. – Introduction.

We begin by briefly summarizing the problem. Let $p \leqslant \varkappa$ be the momentum of a nucleon inside a sphere having in momentum space radius

$$\varkappa = (9\pi/8)^{\frac{1}{3}}(A^{\frac{1}{3}}/R)\,, \tag{1.1}$$

where A is the mass number and R the corresponding radius of the sphere in co-ordinate space. The kinetic energy of a single nucleon is

$$t(p) = p^2/2M; \tag{1.2}$$

then, the internal kinetic energy of nuclear matter is

$$T(\varkappa) = (3A/4\pi\varkappa^3)\int t(p)\,\mathrm{d}\boldsymbol{p} = (3A/10M)\varkappa^2\,. \tag{1.3}$$

If $V(\varkappa)$ indicates the potential energy of the system, the total energy is

$$W(\varkappa) = T(\varkappa) + V(\varkappa) = (3A/\varkappa^3)\int_0^{\varkappa}\{t(p) + \tfrac{1}{2}v(p, \varkappa)\}p^2\,\mathrm{d}p\,, \tag{1.4}$$

where $v(p, \varkappa)$ is the potential energy of a single nucleon in the Fermi sea; the factor $\frac{1}{2}$ in the last term underlies the assumption that the whole potential energy is due to interactions between pairs of nucleons. The saturation properties of nuclear forces are expressed by the conditions

$$\{\mathrm{d}W(\varkappa)/\mathrm{d}\varkappa\}_{\varkappa=\varkappa_F} = 0\,, \qquad W(\varkappa_F) = b_v A\,, \tag{1.5}$$

where $b_v = -(15.0 \pm 1.0)$ MeV is the observed volume energy *per* nucleon and $\varkappa_F = (9\pi)^{\frac{1}{3}}/2r_0$ is the Fermi momentum obtained from (1.1) by putting $R = r_0 A^{\frac{1}{3}}$ at the minimum of $W(\varkappa)$. In the following we shall explore the interval

$$0.9\cdot 10^{-13}\ \mathrm{cm} \leqslant r_0 \leqslant 1.4\cdot 10^{-13}\ \mathrm{cm}\,. \tag{1.6}$$

The compressibility C of infinite nuclear matter is defined at the equilibrium density by the relation

$$C = \varrho^2\{\mathrm{d}^2W(\varrho)/\mathrm{d}\varrho^2\}\,, \tag{1.7}$$

and the compressibility modulus is defined as $K = 9C$. Taking into account that $\varrho = A/\Omega = 2\varkappa_F^3/3\pi^2$, from eq. (1.7) one obtains

$$K = (\varkappa_F^2/A)\{\mathrm{d}^2W(\varkappa)/\mathrm{d}\varkappa^2\}_{\varkappa=\varkappa_F}\,. \tag{1.8}$$

At present there is no direct empirical evidence of the value of K and the theoretical estimates are controversial. The value of K is crucially dependent on the behaviour of $v(p, \varkappa)$ as a function of the limiting momentum $\varkappa$; in fact, taking into account that the necessary condition for nuclear saturation implies that

$$\frac{1}{\varkappa_F^2}\int_0^{\varkappa_F}\left[\frac{\partial v(p, \varkappa)}{\partial \varkappa}\right]_{\varkappa=\varkappa_F} p^2\,\mathrm{d}p = 2(b_v - \varepsilon_F) - v(\varkappa_F, \varkappa_F)\,, \tag{1.9}$$

where $\varepsilon_F = t(\varkappa_F)$ is the Fermi energy, it is found that

$$K = 6\varepsilon_F + \frac{3}{2\varkappa_F}\int_0^{\varkappa_F}\left[\left(\frac{\partial^2}{\partial\varkappa^2} - \frac{2}{\varkappa}\frac{\partial}{\partial\varkappa}\right)v(p, \varkappa)\right]_{\varkappa=\varkappa_F} p^2\,\mathrm{d}p + \tag{1.10}$$

$$+ \frac{3\varkappa_F}{2}\left[\frac{\partial v(p, \varkappa)}{\partial\varkappa}\right]_{p=\varkappa=\varkappa_F} + \frac{3\varkappa_F}{2}\left[\frac{\mathrm{d}v(\varkappa, \varkappa)}{\mathrm{d}\varkappa}\right]_{\varkappa=\varkappa_F}.$$

The energy of a particle on the Fermi surface is the extra energy which would be obtained by adding one particle to the nuclear-matter system without changing its volume Ω, while the average energy is the extra energy which would be obtained by adding one particle to the system without changing its density ϱ: these two quantities must be equal if the system is in equilibrium at zero pressure. The proof is straightforward. Let us define the single-particle energy w_F at the Fermi surface to be the lowest-energy state of a single particle in interaction with the system, namely

$$w_F = (\partial W/\partial A)_\Omega = (\partial W/\partial A)_\varrho , \tag{1.11}$$

where account has been taken of the fact that the system is in equilibrium with its surrounding, *i.e.* $(\partial W/\partial \varrho)_A = 0$. Since the saturation implies that

$$(\partial W/\partial A)_\varrho = W/A , \tag{1.12}$$

it follows that

$$w_F = W(\varkappa_F)/A = b_v ; \tag{1.13}$$

relation (1.13) is trivial and its physical content rather poor. A conceptually remarkable result is achieved by defining the total energy of a single nucleon in nuclear matter as

$$w(p, \varkappa) = t(p) + v(p, \varkappa); \tag{1.14}$$

indeed, the separation energy theorem due to HUGENHOLTZ and VAN HOVE [1] expresses the remarkable property

$$w_F \equiv w(\varkappa_F, \varkappa_F) = b_v , \tag{1.15}$$

namely, *at the density minimizing the total energy of nuclear matter, the average volume energy is equal to the total energy of the most energetic nucleon.* From eqs. (1.14) and (1.15) one has

$$v(\varkappa_F, \varkappa_F) = b_v - \varepsilon_F ; \tag{1.16}$$

then, from eqs. (1.9) and (1.16) one can infer that the dependence of $v(p, \varkappa)$ on the momenta p and $\varkappa$ must fulfil the condition

$$\int_0^{\varkappa_F} \left[\frac{\partial v(p, \varkappa)}{\partial \varkappa}\right]_{\varkappa=\varkappa_F} p^2 \, \mathrm{d}p = \varkappa_F^2 v(\varkappa_F, \varkappa_F) . \tag{1.17}$$

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[1] N. H. HUGENHOLTZ and L. VAN HOVE: *Physica (The Hague)*, **24**, 363 (1958).

A convincing theory of nuclear matter must simultaneously account for eqs. (1.5) and (1.15) consistently with the phenomenological information on the energy dependence of the optical potential ([2]) extracted from the experimental data concerning nucleons scattered from heavy nuclei. The optical potential of a nucleon in motion in the field of a nucleus with energy $E = k^2/2M$ is taken of the form

$$\mathfrak{V}(E, \varkappa_F) = \mathfrak{V}_R(E, \varkappa_F) - i\mathfrak{V}_I(E, \varkappa_F) , \tag{1.18}$$

where $\mathfrak{V}_R(E, \varkappa_F)$ and $\mathfrak{V}_I(E, \varkappa_F)$ are the real and the imaginary parts, respectively. We define the real optical potential as the optical transform of the potential energy of a single nucleon embedded in the Fermi sea, *i.e.*

$$\mathfrak{V}_R(E, \varkappa_F) = V\{p(E), \varkappa_F\} = V_0(\varkappa_F) f(E, \varkappa_F) , \tag{1.19a}$$

where momentum $p(E)$ of the incident nucleon is

$$p(E) = k[1 - \{\mathfrak{V}_R(E, \varkappa_F)/E\}]^{\frac{1}{2}}; \tag{1.19b}$$

we shall assume that the target nucleus is infinitely heavy, so that E is the kinetic energy in the laboratory system of reference. A test of the validity of the optical description is provided by the comparison of the measured nucleon-nucleus total and elastic-scattering differential cross-sections with those predicted using $\mathfrak{V}_R(E, \varkappa_F)$ and the imaginary optical potential

$$\mathfrak{V}_I(E, \varkappa_F) = C_I\{E - \mathfrak{V}_R(E, \varkappa_F)\} g(E, \varkappa_F) . \tag{1.20}$$

Our purpose is to discuss all these matters according to a unified theoretical scheme. In such a scheme there is no room for many of the dissertations which so far have but contributed to shattering the nuclear-matter problem into innumerable aspects generally contradicting one another.

2. – Remarks on Brueckner's theory of infinitely extended nuclear matter.

2·1. Assessing the validity of the picture of nuclear matter as it evolved in the past is rather difficult. This is due to the practical impossibility of disentangling the validity or failures of the adopted physical assumptions from the merits or inadequacies of the mathematical techniques used in handling them.

([2]) P. E. HODGSON: *The Optical Model of Elastic Scattering* (Oxford, 1963); *Nuclear Reactions and Nuclear Structure* (Oxford, 1971).

The capability of Brueckner's theory of infinitely extended nuclear matter to explain the observed value of b_v according to the prescriptions in (1.5) has not been proved with the limpidity that such a remarkable theory deserves. It is not clear whether the unsatisfactory results so far obtained are due to intrinsic faults concealed in the theory or whether they are brought about by the approximations adopted with the purpose of obtaining numerical results. The fundamental assumption of the theory is that the saturation properties of nuclear forces can be explained in terms of the parameters characterizing the nucleon-nucleon elastic-scattering collisions, provided the effects of the other nucleons on the elastic scattering of two nucleons in nuclear matter are properly evaluated. A clear-cut proof that this assumption is correct (or wrong) has not yet been reached: in general, two-nucleon potentials sufficiently strong to give the correct volume energy cause the nucleons to collapse to a much smaller radius than that observed, whereas potentials which account for stability at approximately the observed radius do not give the correct volume energy.

The use of phenomenological potentials has led to the discrepancy noted by BRUECKNER and MASTERSON [3] and by RAZAVY [4]. Many quodlibetic suggestions have been put forward in order to clarify the origin of such a discrepancy (for example, there has been speculation about the possibility that the two-body forces between nucleons should be changed in such a way as to preserve the agreement with the nucleon-nucleon scattering data, but increase the volume energy of nuclear matter, etc.). A blow to the whole problem has been given by the doubts concerning the validity of the description of the forces existing within nuclear matter through nucleon-nucleon potentials derived from experiments on the energy shell. This conceptual impasse becomes even worse if one takes into account eq. (1.15). Brueckner's theory has been made formally consistent with the Hungeholtz and Van Hove theorem by BRUECKNER and GOLDMAN [5]: the implications of the assumption that the K-matrix for the ground-state energy has to be expressed as a function of the occupation numbers of the Fermi gas have not been investigated in detail. In the Brueckner and Goldman treatment the consistency of the separation energy theorem with the first of equations (1.5) is still an open problem.

2·2. The conjectured role of the nucleon-nucleon interactions in determining the properties of infinite nuclear matter can be simply understood by resorting to a modification of the Hartree-Fock theory which consists in expressing the

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(3) K. A. BRUECKNER and K. S. MASTERSON: *Phys. Rev.*, **128**, 2267 (1963); see also R. RAJARAMAN and H. A. BETHE: *Rev. Mod. Phys.*, **39**, 249, 745 (1967).

(4) M. RAZAVY: *Phys. Rev.*, **130**, 1091 (1963).

(5) K. A. BRUECKNER and T. D. GOLDMAN: *Phys. Rev.*, **117**, 207 (1960); see also K. A. BRUECKNER, J. L. GAMMEL and J. T. KUBIS: *Phys. Rev.* **118**, 1438 (1960).

ordinary and exchange K-matrix elements in terms of the nucleon-nucleon elastic-scattering amplitudes [6]; the potential energy of a single nucleon is found to be

$$(2.1a) \qquad v(p,\varkappa) = -\frac{4}{\pi M}\left[\int_0^{\frac{1}{2}(\varkappa-p)} F(q)q^2\,\mathrm{d}q + \frac{1}{8p}\int_{\frac{1}{2}(\varkappa-p)}^{\frac{1}{2}(\varkappa+p)} F(q)\{\varkappa^2-(2q-p)^2\}q\,\mathrm{d}q\right], \quad p\leqslant\varkappa,$$

$$(2.1b) \qquad v(p,\varkappa) = -\frac{1}{2\pi M\varkappa}\int_{\frac{1}{2}(p-\varkappa)}^{\frac{1}{2}(p+\varkappa)} F(q)\{\varkappa^2-(2q-p)^2\}q\,\mathrm{d}q\,, \qquad p\geqslant\varkappa,$$

where $F(q)$ is a complicated function of the nucleon momentum. Since in nuclear matter elastic collisions can occur only in the forward and backward directions, and the scattering amplitudes related to the latter situation can be simply expressed in terms of those related to the former one, it follows that $F(q)$ is constructed as a sum of forward scattering amplitudes, classified according to spin and isobaric spin substates for the two-nucleon system:

$$(2.2) \qquad F(q) = F_{ss}(q) + 3F_{st}(q) + 3f_{ts}(q) + 9F_{tt}(q);$$

function $F(q)$ can then be evaluated using the asymptotic phase shifts determined from the analyses of the nucleon-nucleon elastic-scattering data. The total potential energy is obtained from eq. (2.1a), *i.e.*

$$(2.3) \qquad V(\varkappa) = (3A/8\pi\varkappa^3)\int v(p,\varkappa)\,\mathrm{d}\boldsymbol{p};$$

from the saturation prescriptions in (1.5) one determines the quantities b_v and r_c: the separation energy theorem requires that at the minimum of the total energy the following relation be valid:

$$(2.4) \qquad b_v = \varepsilon_F - (2/\pi M\varkappa_F)\int_0^{\varkappa_F} qF(q)(\varkappa_F - q)\,\mathrm{d}q\,.$$

We shall not go into numerical details, but rather use the description (2.1) as a heuristic paradigm.

In first-order perturbation theory the potential energy of nuclear matter is

$$(2.5) \qquad V(\varkappa) = \frac{9A(A-1)}{32\pi^2\varkappa^6}\int G(2P)\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,,$$

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(6) K. A. BRUECKNER, C. A. LEVINSON and H. M. MAHMOUD: *Phys. Rev.*, **95**, 219 (1954); K. A. BRUECKNER: *Phys. Rev.*, **96**, 1558 (1956); N. FUKUDA and R. G. NEWTON: *Phys. Rev.*, **103**, 1558 (1956).

where $\boldsymbol{P} = \frac{1}{2}(\boldsymbol{p} - \boldsymbol{q})$ is the relative momentum of a pair of nucleons and $G(2P)$ is the three-dimensional Fourier transform of the two-body potential: this function is independent of the limiting momentum $\varkappa$. Equation (2.5) becomes conceptually more significant by inverting the order of integrations; performing very carefully the required trivial manipulations, it is found that

$$V(\varkappa)/A = \int_0^{\varkappa} G(2P)\Pi(P, \varkappa)\,\mathrm{d}P\,, \tag{2.6a}$$

$$\Pi(P, \varkappa) = \frac{24P^2}{\varkappa^3}\left[1 - \frac{3}{2}\left(\frac{P}{\varkappa}\right) + \frac{1}{2}\left(\frac{P}{\varkappa}\right)^3\right], \tag{2.6b}$$

$$\Pi(0, \varkappa) = \Pi(\varkappa, \varkappa) = 0\,, \qquad \int_0^{\varkappa} \Pi(P, \varkappa)\,\mathrm{d}P = 1\,. \tag{2.6c}$$

It is readily established that function $\Pi(P, \varkappa)$ plays the role of the distribution of the probabilities that a pair of nucleons have relative momentum P; thus, the meaning of eq. (2.6) is: *in first-order perturbation theory the $\varkappa$-dependence of the average potential energy of nuclear matter is given by the three-dimensional Fourier transform of the two-body potential averaged over the relative-momentum distribution of pairs of noninteracting nucleons.*

This result corresponds to the absolute zero of temperature of the degenerate Fermi gas of nucleons. The distribution of the momenta of the single nucleons is simply described by saying that all the lower states are occupied up to the limiting momentum $\varkappa$, fixed by the total number of nucleons confined in the nuclear sphere of radius R, according to the saturation prescriptions (1.15). The effect of the nucleon-nucleon interactions, by causing virtual transitions from occupied states into unoccupied ones, is to spread the momentum distribution of the single nucleons in the neighbourhood of the limiting momentum $\varkappa$, lowering the density in momentum space below $\varkappa$ and giving rise to a tail in the distribution above this value. The modification of the momentum distribution of single nucleons also alters $\Pi(P, \varkappa)$: these effects can be conceived qualitatively as being of the same kind as those due to thermal agitation, and are responsible for the discontinuity of the momentum distribution at $p = \varkappa_{\mathrm{F}}$, which—in turn—also implies a discontinuity of the energy spectrum $w(p, \varkappa_{\mathrm{F}})$ at the Fermi surface (incidentally the latter discontinuity is in no way related to the energy gap of superconductivity). The physical content of the nuclear-matter description based on eqs. (2.1) now becomes clear: the experimental momentum dependence of function $F(q)$ implicitly accounts for the unknown two-body potential and for many of the theoretical intricacies due to the nonuniform momentum distribution of the single nucleons in the Fermi sea. The potential energy (2.3) calculated with the single-particle potential energy (2.1a) is far more realistic than that predicted by eq. (2.6). The func-

tion $F(q)$ also accounts for the effects due to tensor interactions, which in perturbation theory contribute to the potential energy of nuclear matter only in second-order and in higher-order approximations. From a conceptual point of view the theoretical scheme according to which eqs. (2.1) have been deduced is intermediate between perturbation theory and Brueckner's theory.

2·3. Brueckner's theory is a mathematical model invented either for meeting the requirements with the theoretical scenery of infinite nuclear matter, mentioned in subsect. **2·2**, or for circumventing the drawbacks of the conventional perturbation theory. Its fundamental goal is to determine elements $K_{ij,kl}$ of the antisymmetrized K-matrix, calculated on the basis of a preconceived choice of the nucleon-nucleon potential. By using the standard notation, the basic aspect of Brueckner's theory is expressed by the equations

$$K_{ij,kl} = v_{ij,kl} + \sum_{m,n} \frac{v_{ij,mn} K_{mn,kl}}{e_k + e_l - e_m - e_n}, \tag{2.7a}$$

where the self-consistent single-particle energies are given by

$$e_i = t(p_i) + \sum_j (K_{ij,ij} - K_{ij,ji}). \tag{2.7b}$$

The exact solutions of the nonlinear system (2.7) are unknown and the approximations used to obtain numerical results give rise to complications and ambiguities which outweigh the heuristic value of the theory. For this reason, the attitude of laying all the blames for the unsatisfactory results thus obtained on the choice of the two-body potential seems objectionable. This opinion is also supported by the remark of Brueckner and Masterson that two-body potentials which give supposedly equally good fit to the scattering data do not lead necessarily to identical nuclear properties. Unfortunately, a stringent critical analysis of the approximations, corrections and modifications of Brueckner's theory cannot be carried out because the major part of the available results has been obtained by means of calculations which are neither transparent nor controllable on numerical ground. Several results, usually quoted in the literature, can be proved to be wrong. It is indeed discouraging to realize that misleading conclusions have often been taken for granted out of faith or mental indolence. Substantiating this statement properly would be too pedantic. Suffice it to point out, as an example, that the modification of Brueckner's theory [7], developed for removing the discrepancy discovered by BRUECKNER and MASTERSON, implies a strong violation of the separation energy theorem and conflicts with the first of eqs. (1.5): the uncritical acceptance of the conjec-

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[7] H. A. BETHE: *Phys. Rev. B*, **4**, 804 (1965).

tured role of three-body correlations in nuclear matter has led to conclusions which are completely ungrounded [8].

All those who feel disenchanted enough in front of theories adjusted *ad hoc* by sudden flashes of wit (the therapeutical role ascribed to the three-body correlations in order to make up for the failure of Brueckner's theory is but an example) may also feel the lack of a reliable, analytically simple and numerically manageable tool which allows them to deal with the nuclear-matter problem. For this reason we have to lay aside the ambitious aim of developing a formally rigorous treatment (systematically spoilt by approximations whose validity cannot be assessed *a priori* and are often found *a posteriori* to be disastrous) and try instead to approach the problem along a different pathway. Our program is to extract the maximum of theoretical information from the saturation prescriptions (1.5) and from the Hugenholtz and Van Hove theorem (1.15) with a minimum of physical assumptions. More specifically, our primary aim is the explicit determination of the dependence of function $v(p, \varkappa)$ on p and $\varkappa$ without resorting to preconceived choices of the nucleon-nucleon potential or to sophisticated nuclear many-body procedures. This note intends to show that even in the realm of theoretical physics one can try to be faithful to Newton's statement « *hypotheses non fingo* ».

3. – The nucleon effective mass.

3·1. The concept of nucleon effective mass is not so trivial as it appears from many nuclear-matter calculations: a deeper insight into its meaning can be achieved with the help of elementary quantum mechanics. Resorting to the correspondence principle and requiring that the following classical relation

$$\dot{w}(\boldsymbol{p}, \varkappa_{\mathrm{F}}) = \boldsymbol{F}\cdot\boldsymbol{v} \tag{3.1}$$

between the force $\boldsymbol{F}$ acting on a nucleon having velocity $\boldsymbol{v}$ in the Fermi sea and its total energy $w(\boldsymbol{p}, \varkappa_{\mathrm{F}})$ remains valid for the mean values of the quantum theory is consistent with the idealized conception of infinite nuclear matter. The group velocity $\boldsymbol{v}(\boldsymbol{p})$ of the packet and the total energy $w(\boldsymbol{p}, \varkappa_{\mathrm{F}})$ satisfy the relation

$$\boldsymbol{v}(\boldsymbol{p}) = \boldsymbol{\nabla}_{\boldsymbol{p}} w(\boldsymbol{p}, \varkappa_{\mathrm{F}}); \tag{3.2}$$

consequently, one has

$$\langle \dot{\boldsymbol{v}} \rangle = \langle \boldsymbol{\nabla}_{\boldsymbol{p}} \boldsymbol{F}\cdot\boldsymbol{v} \rangle = \langle \boldsymbol{F}\cdot\boldsymbol{T} w(\boldsymbol{p}, \varkappa_{\mathrm{F}}) \rangle \,, \tag{3.3}$$

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[8] D. W. L. Sprung, P. C. Bhargava and T. K. Dahlblom: *Phys. Lett.*, **21**, 538 (1966); the quoted results are wrong also because based on an incorrect application of Bethe's three-body theory (see D. B. Day: *Rev. Mod. Phys.*, **39**, 719, 743 (1967)).

where $\boldsymbol{T} = \boldsymbol{\nabla}_p \boldsymbol{\nabla}_p$ is a differential tensor operator of rank 2. Equation (3.3) corresponds to the classical relation $\dot{\boldsymbol{v}} = \boldsymbol{F}/\boldsymbol{M}$ provided one introduces the mass tensor

$$1/\boldsymbol{M}^* = \boldsymbol{T}w(\boldsymbol{p}, \varkappa_{\mathrm{F}}) . \tag{3.4}$$

Relation (3.3) had previously been obtained by SEITZ [9] in a different context. Since nuclear matter is conceived as a spherically symmetric system, the mass tensor (3.4) reduces to the scalar nucleon effective mass

$$M/M^*(p) = M\{\mathrm{d}^2 w(p, \varkappa_{\mathrm{F}})/\mathrm{d}p^2\} = 1 + M\{\mathrm{d}^2 v(p, \varkappa_{\mathrm{F}})/\mathrm{d}p^2\} . \tag{3.5}$$

It is indeed astonishing that eq. (3.5), which is undoubtedly well known to various individuals, should not have yet achieved the dignity of a specific mention in the literature concerning the nuclear-matter problem. It is customary [10] to define a parameter M^{**}, homogeneous to a mass, from the slope of $v(p, \varkappa_{\mathrm{F}})$ at $p = \varkappa_{\mathrm{F}}$ through the relation

$$M/M^{**} = M\big[(1/p)\{\mathrm{d}w(p, \varkappa_{\mathrm{F}})\,\mathrm{d}p\}\big]_{p=\varkappa_{\mathrm{F}}} = 1 + M\big[(1/p)\{\mathrm{d}v(p, \varkappa_{\mathrm{F}})/\mathrm{d}p\}\big]_{p=\varkappa_{\mathrm{F}}}; \tag{3.6}$$

definitions (3.5) and (3.6) lead to identical results only if $v(p, \varkappa_{\mathrm{F}})$ depends quadratically on the momentum p. Although definition (3.6) is completely unrelated to the concept of mass tensor, it is nevertheless used to express the nuclear effective mass in infinite nuclear matter also when the momentum dependence of $v(p, \varkappa_{\mathrm{F}})$ is not quadratic in p: this is the cause of several misleading conclusions drawn from nuclear-matter calculations.

3'2. – One of the most important goals of a nuclear-matter theory is to provide reliable values of the compressibility parameter K, consistently with eqs. (1.5) and (1.15) and with the phenomenological information on the energy dependence of the optical potential (1.18). Definition (3.5) of the nucleon effective mass, reached independently of any detail concerning the nuclear matter, stimulates curiosity about the possible existence of a general relation between the compressibility modulus and the effective mass. To satisfy such a curiosity, we shall resort to our imagination and prove, as a purely intellec-

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[9] F. SEITZ: *The Modern Theory of Solids* (New York, N. Y., 1940).

[10] K. A. BRUECKNER: *The Many Body Problem*, Vol. I (Paris, 1959), p. 169; M. A. PRESTON: *Physics of the Nucleus* (Reading, Mass., 1962), p. 202. A rather queer criterion for calculating M^{**} has been suggested by L. C. GOMES, J. D. WALECKA and V. F. WEISSKOPF: *Ann. Phys.* (*N. Y.*), **3**, 241, 252 (1958). Recently, definition (3.6) has been adopted also by J. P. BLAIZOT in his review paper on nuclear compressibilities (*Phys. Rep.*, **64**, 171 (1980), formula (7.1), p. 234). See also K. A. BRUECKNER and J. L. GAMMEL: *Phys. Rev.*, **109**, 1840 (1958), formula (21).

tual *divertissement*, the following theorem: *if the infinitely extended nuclear matter is conceived as a nonviscous and incompressible fluid and the relation* $R = r_0 A^{\frac{1}{3}}$ *is assumed to be valid, then the effective mass of a nucleon plunged into it is*

$$M^*/M = \tfrac{1}{2} . \tag{3.7}$$

It is a matter of common experience that the apparent inertia of a body of mass M, moving in a fluid, is greatly increased by the fluid around it. The increased inertia is called the effective mass M^* of the body, and the difference between the effective mass and the real mass is called the induced mass M_i ([11]). According to classical hydrodynamics, the motion of a body in a fluid is dynamically equivalent to the motion of a *heavier* body in vacuo, whose effective mass is

$$M^* = M + M_i . \tag{3.8}$$

It is easily realized that this result, which was first given an exact mathematical formulation by GREEN and STOKES well over a century ago ([12]), has to be changed if one attempts to describe, along the same line, the motion of a nucleon in nuclear matter, because of the peculiar property of the nuclear fluid of exerting attractive forces upon the moving particle. Since in this case the increase of the kinetic energy of the fluid must be thought of as due to nuclear interactions, an elementary energy balance shows that the motion of a nucleon in nuclear matter is dynamically equivalent to the motion of a free nucleon having a *lighter* mass

$$M^* = M - M_i , \tag{3.9}$$

i.e. the effect of the attractive interactions between the nuclear body and the nuclear fluid is dynamically equivalent to a negative induced mass, contrary to what is to be expected in connection with the motion of an ordinary body in an ordinary fluid.

According to hydrodynamics, the induced mass M_i can be regarded as the mass of a virtual particle having a kinetic energy equal to the additional kinetic energy $\delta T(\varkappa_F)$ gained by the fluid. In order to evaluate $\delta T(\varkappa_F)$ we consider a single body of mass M and « virtual » radius r_0 in motion with velocity v through a nonviscous and incompressible fluid, extended over the whole space: the theoretical amusement lies in the identification of the single body with a nucleon and the nuclear matter with a perfect fluid. We shall assume Dirichlet

([11]) The idea of induced mass was originally expressed in a very crude way by DUBUAT in 1786 (see L. G. DUBUAT: *Principles d'hydraulic*, Vol. II (Paris, 1816), p. 222).

([12]) G. GREEN: *Mathematical Papers*, Vol. I (London, 1833), p. 315; G. STOKES: *Mathematical and Philosophical Papers*, Vol. I (London, 1834), p. 17.

flow and neglect, also in the adopted classical scheme, the effects of the intrinsic angular momentum of the nucleon on the surrounding nuclear fluid. Without loss of generality, we may choose the axis of spherical co-ordinates as the direction of motion. With respect to the fluid at infinity, the interaction body-fluid is expressed by the dipole-velocity potential ([13])

$$U = - r_0^3 v \cos\theta / 2r^2 . \tag{3.10}$$

At any point of the fluid the radial and angular components of the velocity are

$$v_r = \partial U/\partial r = r_0^3 v \cos\theta / r^2 , \tag{3.11a}$$

$$v_\theta = (1/r)(\partial U/\partial\theta) = r_0^3 v \sin\theta / 2r^3 . \tag{3.11b}$$

The additional kinetic energy is

$$\delta T(\varkappa_F) = \tfrac{1}{2}\int_{r \geqslant r_0} \varrho(\boldsymbol{r})(v_r^2 + v_\theta^2)\,\mathrm{d}\boldsymbol{r}; \tag{3.12}$$

assuming for $r \leqslant R$ a uniform density distribution of nuclear matter, from eq. (3.12) it is found that

$$\delta T(\varkappa_F) = (\pi \varrho_0 r_0^3/2) v^2 \equiv (M_1/2) v^2 . \tag{3.13}$$

Since the packing of A nucleons satisfies the relation $R = r_0 A^{\frac{1}{3}}$, one has

$$\varrho_0 = 3M/4\pi r_0^3 \quad \text{for } r \leqslant R , \qquad \varrho_0 = 0 \quad \text{for } r > R; \tag{3.14}$$

then, from eq. (3.13) it is found that $M_1 = M/2$ and from eq. (3.9) $M^* = 0.5M$, which proves the theorem.

4. – A mathematical property of the single-particle potential energy in the Fermi sea.

Conditions (1.5) are necessary, but not sufficient, for testing the validity of a nuclear-matter theory. The $\varkappa$-dependence of the single-particle potential energy has never been investigated in detail and, therefore, the calculation of $V(\varkappa)$ by means of eq. (2.3) makes also the p-dependence of $v(p, \varkappa)$ rather elusive: the separation energy theorem (1.15) provides crucial information on the value

([13]) G. BIRKHOFF: *Hydrodynamics* (Princeton, N. J., 1950); see also H. LAMB: *Hydrodynamics* (New York, N. Y., 1945), p. 123.

of $v(p, \varkappa)$ at $p = \varkappa = \varkappa_F$. Additional information on $v(p, \varkappa)$ can be obtained by direct or indirect comparison with the data of the predicted values of the compressibility modulus (dependent on the second-order derivatives of $V(\varkappa)$ evaluated at the minimum of $W(\varkappa)$) and with those concerning the real part of the optical potential (deduced as optical transform of $v(p, \varkappa_F)$ for $p > \varkappa_F$): from this information one can derive the required sufficient conditions. For these reasons we shall concentrate primarily on the two-variable function $v(p, \varkappa)$.

The necessary condition for nuclear saturation, expressed by the first of eqs. (1.5), means physically that there exists a value $\varkappa_F$ of the limiting momentum $\varkappa$ at which the pressure $P(\varkappa)$ of the Fermi gas is zero, *i.e.*

$$P(\varkappa_F) = (2\varkappa_F^4/9\pi^2 A)\{\mathrm{d}W(\varkappa)/\mathrm{d}\varkappa\}_{\varkappa=\varkappa_F} = 0 . \tag{4.1}$$

Let us work out eq. (4.1) using the integral form of $W(\varkappa)$ given in eq. (1.4); taking into account definition (1.14), one obtains

$$b_v = W(\varkappa_F)/A = w(\varkappa_F, \varkappa_F) - \{Q(\varkappa_F)/2\varkappa_F^2\} , \tag{4.2a}$$

$$Q(\varkappa_F) = \varkappa_F^2 v(\varkappa_F, \varkappa_F) - \int_0^{\varkappa_F} p^2 \left\{\frac{\partial v(p, \varkappa)}{\partial \varkappa}\right\}_{\varkappa=\varkappa_F} \mathrm{d}p . \tag{4.2b}$$

Clearly, the separation energy theorem (1.15) follows from eq. (4.2*a*) provided that

$$Q(\varkappa_F) = 0 ; \tag{4.3}$$

note that conditions (4.3) and (1.17) are identical. It is interesting to compare relations (4.2) with those obtained by BETHE [14] and then simplified by assuming that a quantity playing the role of our $Q(\varkappa_F)/2\varkappa_F^2$ is « negligible ».

Let us introduce the function

$$Q(\varkappa) = \varkappa^2 v(\varkappa, \varkappa) - \int_0^{\varkappa} p^2 \{\partial v(p, \varkappa)/\partial \varkappa\} \, \mathrm{d}p . \tag{4.4}$$

A lengthy calculation shows that $v(p, \varkappa)$, calculated in first-order perturbation theory, possesses the property

$$Q(\varkappa) \equiv 0 ; \tag{4.5}$$

the proof will be given in subsect. **5**·2. It is readily verified that also the single-particle potential energy (2.1*a*) is characterized by property (4.5). In fact,

[14] H. A. BETHE: *Phys. Rev.*, **103**, 1353 (1956); see formulae from (9.17) to (9.25).

one has

$$\varkappa^2 v(\varkappa, \varkappa) = -(2\varkappa/\pi M)\int_0^{\varkappa} F(q)(\varkappa - q) q^2 \,\mathrm{d}q \equiv I(\varkappa)\,, \tag{4.6a}$$

$$\frac{\partial v(p, \varkappa)}{\partial \varkappa} = -\frac{\varkappa}{\pi M p}\int_{\frac{1}{2}(\varkappa - p)}^{\frac{1}{2}(\varkappa + p)} F(q) q \,\mathrm{d}q\,, \tag{4.6b}$$

$$\int_0^{\varkappa} p^2 \left\{\frac{\partial v(p, \varkappa)}{\partial \varkappa}\right\} \mathrm{d}p = -\frac{1}{\pi M}\int_0^{\varkappa} p \,\mathrm{d}p \int_{\frac{1}{2}(\varkappa - p)}^{\frac{1}{2}(\varkappa + p)} F(q) q \,\mathrm{d}q \equiv J(\varkappa)\,; \tag{4.6c}$$

inverting the order of integration, eq. (4.6c) becomes

$$-\pi M J(\varkappa) = \int_0^{\varkappa/2} F(q) q \,\mathrm{d}q \int_{\varkappa - 2q}^{\varkappa} p \,\mathrm{d}p + \int_{\varkappa/2}^{\varkappa} F(q) q \,\mathrm{d}q \int_{2q - \varkappa}^{\varkappa} p \,\mathrm{d}p\,. \tag{4.6d}$$

In conclusion, one finds $Q(\varkappa) = I(\varkappa) - J(\varkappa) \equiv 0$. Since $Q(\varkappa) \equiv 0$ holds for any value of $\varkappa$, it is necessarily valid also for the value $\varkappa = \varkappa_{\mathrm{F}}$ corresponding to the minimum of the total energy $W(\varkappa)$: we have thus verified with striking simplicity the validity of the Hugenholtz and Van Hove theorem.

In nuclear-matter calculations the Hugenholtz and Van Hove theorem constitutes a dark corner around which there is much confusion [15]. Using our notation, such a theorem can be expressed by the exact relation, which has a well-defined meaning for large systems,

$$w(\varrho, \varrho) = \frac{\mathrm{d}}{\mathrm{d}\varrho}\left\{\frac{W(\varrho)}{\Omega}\right\}, \tag{4.7}$$

where $\varrho = 2\varkappa_{\mathrm{F}}^3/3\pi^2$; eq. (4.7) is by no means a trivial one: in fact, $w(\varrho, \varrho)$ is defined in terms of diagrams with an external line at both ends, whereas $W(\varrho)$ is obtained from ground-state diagrams. In Brueckner's theory the difference

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[15] The disconcert created by the suspicion that equality (1.15) could be true was well expressed by BETHE about two years before the discovery of the separation energy theorem by HUGENHOLTZ and VAN HOVE (H. A. BETHE: *Phys. Rev.*, **103**, 1353, 1372 (1956)): « It is perhaps somewhat surprising that one calculate in one case an average energy and in the other one a maximum energy, and that these two should be equal. This is made possible by the factor $\frac{1}{2}$ in the potential energy when the average W is calculated in eq. (9.17); this should just compensate for the difference between average and maximum ». The theoretical background of this problem will be examined in sect. **5**: we shall see that the « magic » role attributed by BETHE to the factor $\frac{1}{2}$ does not work because conceptually irrelevant.

between $w(\varkappa_F, \varkappa_F)$ and $W(\varkappa_F)/A$ is called the re-arrangement energy ([16]): this energy can be identified with quantity $Q(\varkappa_F^2)/2\varkappa^2$. Thus, we have previously given a general argument for concluding that *the validity of the Hugenholtz and Van Hove theorem implies that the re-arrangement energy is equal to zero because it must be* $Q(\varkappa_F) = 0$.

The single-particle potential (2.1*a*) is intrinsically consistent with the Hugenholtz and Van Hove theorem, although the numerical results obtained from eq. (2.4) exhibit a discrepancy quantitatively similar to that noted by BRUECKNER and MASTERSON. Energy $Q(\varkappa)/2\varkappa^2$ has been evaluated according to Bethe's treatment of the nuclear many-body problem ([14]): it has been ascertained that Bethe's integral equation obeyed by the scattering matrix leads to results intrinsically inconsistent with the separation energy theorem. This also occurs in the older formulation of Brueckner's theory: for example, the results obtained by BRUECKNER and GAMMEL ([17]) disclose a large discrepancy between the values of $w(\varkappa_F, \varkappa_F)$ and b_v which can only be partially explained by taking into account the contribution of cluster terms underestimated by the authors. The re-arrangement energy is conceived as a contribution (not necessarily additive) to the single-particle potential energy due to the density dependence of the K-matrix, which also includes some of the effects of correlations between particles: when the density changes, the particles re-arrange themselves, and the K-matrix also changes. This justification is unsatisfactory because it ignores the fact that the appearance of the re-arrangement energies is strictly bound to the criterion adopted in the application of variational methods, which—in turn—influence the choice of the definition of the single-particle potential energy ([18]). We do not share the opinion according to which the nonzero magnitude of the re-arrangement energy is a characteristic many-body effect which manifests itself through high-order effects in the K-matrix. There are very good reasons to retain that the re-arrangement energy question is a false problem, originated by objectionable mathematical approximations used in handling the nuclear-matter problem: no re-arrangement energy would probably appear in an exact formulation of Brueckner's theory.

The most conceptually troublesome aspect of any nuclear-matter theory of Brueckner's type arises from the somewhat frantic theoretical justifications of the appearance of « theoretical energies », which (like the re-arrangement energy) are « created » by the adopted approximations and/or by the techniques used to handle the nuclear-matter problem (model energies, single-particle

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([16]) K. A. BRUECKNER: *The Many Body Problem*, Vol. I (Paris, 1959), p. 160.

([17]) K. A. BRUECKNER and J. L. GAMMEL: *Phys. Rev.*, **105**, 1679 (1957).

([18]) Suggestions concerning the application of variational methods in such a way that the re-arrangement energy should not occur have been given by R. J. EDEN V. J. EMERY and S. SAMPANTHAR: *Proc. R. Soc. London Ser. A*, **253**, 177, 186 (1959); see also D. J. THOULESS: *The Quantum Mechanics of Many-Body Systems* (New York, N. Y., 1961).

energies included in self-consistent energy denominators, etc.). All these ghost energies, which in some cases depend even on the particular arrangement of the terms of series expansions, are experimentally unobservable. We shall not waste time with unfruitful theoretical subtleties and try instead to outline a description of nuclear matter rigorously based on the property $Q(\varkappa) \equiv 0$. This heuristic frontier has never been explored before: we shall call it *the Hugenholtz and Van Hove approximation of infinitely extended nuclear matter.*

5. – The differential equation of infinite nuclear matter.

5·1. – The kinetic and potential energies of nuclear matter, and the potential energy of a single nucleon plunged into it, tend to zero at the limit of zero nuclear density. This trivial statement follows from the fact that for $\varkappa \to 0$ the average distance [19]

$$d(\varkappa) \simeq (\tfrac{5}{3})^{\frac{1}{2}} (1/\varkappa) \tag{5.1}$$

between the nearest neighbours of the A nucleons, uniformly distributed in the interior of the nuclear sphere, tends to infinity and, consequently, the nucleon-nucleon interactions vanish. It follows that it must be $V(0) = 0$ and

$$v(p, 0) = 0; \tag{5.2}$$

an obvious implication of property (5.1) is

$$v(0, 0) = 0 . \tag{5.3}$$

The following theorem holds: *the necessary and sufficient condition for the validity of the physically obvious property* $v(0, 0) = 0$ *is given by*

$$Q(\varkappa) = \varkappa^2 v(\varkappa, \varkappa) - \int_0^{\varkappa} p^2 \{\partial v(p, \varkappa)/\partial \varkappa\} \, \mathrm{d}p \equiv 0 , \tag{5.4}$$

provided the potential energy of a single nucleon in the Fermi sea obeys the partial differential equation

$$\left(\frac{\partial^2}{\partial p^2} + \frac{2}{p} \frac{\partial}{\partial p} - \frac{\partial^2}{\partial \varkappa^2} + \frac{2}{\varkappa} \frac{\partial}{\partial \varkappa} \right) v(p, \varkappa) = 0 . \tag{5.5}$$

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[19] J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, N. Y., 1952), p. 129.

The proof of the theorem is straightforward. Let us evaluate the first-order derivative of the function $Q(\varkappa)/\varkappa^2$ given by eq. (4.4). Taking into account that for $p = \varkappa$ the following relation exists:

$$\frac{\mathrm{d}v(\varkappa, \varkappa)}{\mathrm{d}\varkappa} = \left[\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial \varkappa}\right) v(p, \varkappa)\right]_{p=\varkappa}, \tag{5.6}$$

one obtains

$$\frac{\partial}{\partial \varkappa}\left[\frac{Q(\varkappa)}{\varkappa^2}\right] = \left[\frac{\partial v(p, \varkappa)}{\partial p}\right]_{p=\varkappa} - \int_0^{\varkappa} \frac{\partial}{\partial \varkappa}\left[\left(\frac{p}{\varkappa}\right)^2 \frac{\partial v(p, \varkappa)}{\partial \varkappa}\right] \mathrm{d}p\,. \tag{5.7}$$

The differential equation (5.5) can be written as

$$\frac{\partial}{\partial p}\left[\left(\frac{p}{\varkappa}\right)^2 \frac{\partial v(p, \varkappa)}{\partial p}\right[= \frac{\partial}{\partial \varkappa}\left[\left(\frac{p}{\varkappa}\right)^2 \frac{\partial v(p, \varkappa)}{\partial \varkappa}\right]; \tag{5.8}$$

substituting (5.8) into (5.7) it is found that

$$\frac{\mathrm{d}}{\mathrm{d}\varkappa}\left[\frac{Q(\varkappa)}{\varkappa^2}\right] = \left[\left(\frac{p}{\varkappa}\right)^2 \frac{\partial v(p, \varkappa)}{\partial p}\right]_{p=0} = 0\,, \tag{5.9}$$

where it has been taken into account that the invariance prescription of the potential under time reflection implies that for small momenta $\partial v(p, \varkappa)/\partial p$ varies linearly with p. From eq. (5.9) one gets

$$Q(\varkappa) = C\varkappa^2\,, \tag{5.10}$$

where C is an arbitrary constant of integration. Substituting eq. (4.4) into (5.10) and performing the transformation $x = p/\varkappa$, one has

$$v(\varkappa, \varkappa) - \varkappa \int_0^1 x^2 \{\partial v(\varkappa x, \varkappa)/\partial \varkappa\}\, \mathrm{d}x = C\,. \tag{5.11}$$

At the limit $\varkappa \to 0$ eq. (5.11) becomes $v(0, 0) = C$; it follows that property (5.4) is satisfied if $C = 0$, which proves the theorem. We shall denominate eq. (5.5) as *the differential equation of infinite nuclear matter*. Taking the implications of condition (4.3) into account, the theorem proved in this subsection can also be formulated in the following way: *solutions $v(p, \varkappa)$ of the differential equation of infinite nuclear matter automatically satisfy the Hugenholtz and Van Hove theorem*. The nuclear-matter description constructed on the basis of these solutions constitutes the Hugenholtz and Van Hove approximation of infinite nuclear matter, mentioned at the end of sect. 4.

Equation (5.5) is a special case of the general second-order partial differential equation in two variables

$$\{\mathscr{A}\partial_p^2 + 2\mathscr{B}\partial_{p\varkappa}^2 + \mathscr{C}\partial_\varkappa^2 + \mathscr{D}\partial_p + \mathscr{E}\partial_\varkappa + \mathscr{F}\}v(p,\varkappa) + \mathscr{G} = 0\,, \tag{5.12}$$

where $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}$ and $\mathscr{G}$ are real functions of p and $\varkappa$ (or possibly real constants), and $\partial_p \equiv \partial/\partial p$, $\partial_{p\varkappa}^2 \equiv \partial^2/\partial p\,\partial\varkappa$, etc. The characteristic curves of eq. (5.12) are given by the equation

$$\frac{\mathrm{d}\varkappa}{\mathrm{d}p} = \frac{\mathscr{B} \pm \sqrt{\mathscr{B}^2 - \mathscr{A}\mathscr{C}}}{\mathscr{A}}\,. \tag{5.13}$$

Since in our case $\mathscr{A} = -\mathscr{C} = 1$ and $\mathscr{B} = 0$, eq. (5.5) turns out to be of hyperbolic type ($\mathscr{B}^2 - \mathscr{A}\mathscr{C} > 0$): its hyperbolic nature can also be exhibited in normal form by using the transformation

$$\lambda = (\varkappa + p)/2\,, \qquad \mu = (\varkappa - p)/2\,. \tag{5.14}$$

The characteristic curves are given by the straight lines

$$\varkappa = c \pm p\,, \tag{5.15}$$

where c is an arbitrary integration constant. The physical meaning of the variables p and $\varkappa$ implies $p \geqslant 0$ and $\varkappa \geqslant 0$; the Hugenholtz and Van Hove theorem requires $c = 0$, because $v(p,\varkappa)$ must obey prescription (1.16) at point $p = \varkappa = \varkappa_F$ lying on the characteristic line passing through the origin of the reference system chosen in the positive region of the $(\varkappa, p)$-plane. The fundamental role played by eq. (5.5) arises from the fact that it provides a remarkable clue for disentangling the mathematical aspects of the nuclear-matter problem from the physical ones. In fact, the analytic structure of the equation is « universal » in the sense that it is entirely independent of any detail concerning the nucleon-nucleon interactions as well as the nuclear sphere, where the interactions occur [20]: this stimulating circumstance implies that all physical properties have a crucial role only in the specification of the Cauchy problem associated with the equation itself. These considerations are restricted to the saturation properties of nuclear matter: indeed, they open a new path also for a deeper insight into the optical model outlined in sect. 1. The optical transform (1.19) operates on the tail of the single-particle potential energy $v(p, \varkappa_F)$; it should be evident that the optical-model analyses do not give any information on $v(p, \varkappa_F)$ at the Fermi sphere ($p = \varkappa_F$). In fact, the closest approach (in momentum space) of the probe neutron to the Fermi

[20] This circumstance justifies the attitude expressed at the end of subsect. 2·3.

sphere occurs at the zero-energy limit; putting $E = 0$ in eq. (1.19*b*) one has

$$p(0) = [-2M\mathcal{V}_R(0, \varkappa_F)]^{\frac{1}{2}} = [-2Mv\{p(0), \varkappa_F\}]^{\frac{1}{2}} . \tag{5.16}$$

According to the Hugenholtz and Van Hove theorem it is

$$\varkappa_F^2/2M + v(\varkappa_F, \varkappa_F) = b_v , \tag{5.17a}$$

whereas from eq. (5.16) one has

$$\{p(0)\}^2/2M + v\{p(0), \varkappa_F\} = 0; \tag{5.17b}$$

it follows that the equality $p(0) = \varkappa_F$ would imply $b_v = 0$: since $v(p, \varkappa_F)$ is a decreasing function of p, eq. (5.17*b*) shows that it must be

$$p(0) > \varkappa_F . \tag{5.18}$$

The optical-model analyses determine, through transformation (1.19), the p-dependence of function $v(p, \varkappa_F)$ along the straight line $\varkappa = \varkappa_F$, parallel to the p-axis and for $p \geqslant p(0) > \varkappa_F$ crossing the characteristic curves of eq. (5.5): this situation is summarized in fig. 1. It is evident that the phenomenological

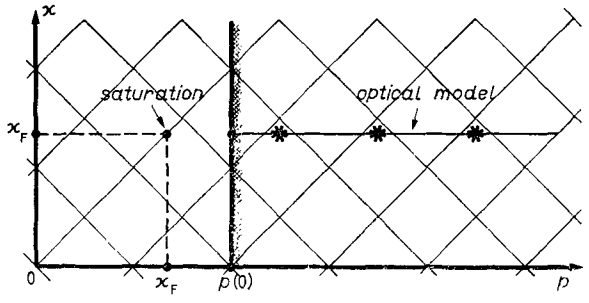

Fig. 1. – The saturation of nuclear matter and the optical model.

information extracted from the optical-model analyses contributes to the specification of the Cauchy problem associated with the differential equation of nuclear matter. This is the motivation of our efforts to link the description of the saturation properties of infinite nuclear matter with the gross properties characterizing the scattering of neutrons from heavy nuclei within the frame of a consistent theoretical scheme.

The hyperbolic partial differential equation (5.5) is indeed remarkable. One may speculate whether it expresses in a differential form the Pauli principle, whose role in the description of nuclear matter prevails upon the partic-

ular feature of any realistic two-body potential ([21]). In fact, it seems plausible to retain that the p-dependence of $v(p, \varkappa)$ arises primarily from the total antisymmetry of the nuclear wave function, although it can also include other effects (for instance, those brought about by the conjectured velocity dependence of the nucleon-nucleon interactions and/or by their nonlocal nature, etc.). Equation (5.5) seems to reconcile, in a surprisingly simple manner, the independent-particle and collective liquidlike aspects of nuclear matter, so that the properties of a Fermi liquid at not too high densities and those of a diluted Fermi gas are not mutually exclusive and must not be regarded as opposite extremes ([22]). These considerations are strictly based on the fact that the mathematical structure of the nuclear-matter equations is inextricably linked to the Hugenholtz and Van Hove theorem. A piece of corroborative evidence is given in the following. Let us express eq. (4.7) in terms of the energy *per* particle instead of the energy *per* unit volume; one has

$$w(\varrho, \varrho) = \frac{W(\varrho)}{A} + \varrho \frac{\mathrm{d}}{\mathrm{d}\varrho}\left\{\frac{W(\varrho)}{A}\right\} = \frac{W(\varrho)}{A} + \frac{1}{\varrho} P(\varrho) . \tag{5.19}$$

In a liquid the pressure is

$$P(\varrho) = \varrho^2 \frac{\mathrm{d}}{\mathrm{d}\varrho}\left\{\frac{W(\varrho)}{A}\right\} = 0 \tag{5.20a}$$

and, therefore, from eq. (5.19) one obtains

$$w(\varrho, \varrho) = W(\varrho)/A ; \tag{5.20b}$$

condition (5.20a) is the first of the saturation prescriptions (1.5) valid at $\varrho = 2\varkappa_{\mathrm{F}}^3/3\pi^2$, where eq. (5.20$b$) must be equal to the average volume energy b_v. Thus the differential equation of nuclear matter, which conceals within its mathematical structure the Hugenholtz and Van Hove theorem, governs the behaviour on p and $\varkappa$ of the potential energy of a single nucleon embedded in a very peculiar medium, the Fermi sea, which is described as a degenerate gas of nucleons, but nevertheless at equilibrium density exhibits properties which are typical of a liquid. The fall-out of this situation on the compressibility modulus is particularly interesting. To highlight this point we re-write eq. (5.5)

([21]) This idea has been suggested to the author by L. ROSENFELD; see also L. ROSENFELD: *Interactions nucléaires aux basses energies et structure des noyaux* (Paris, 1959), p. 330; C. VILLI: *Nucl. Phys.*, **9**, 306 (1959). I recall with gratitude M. BORN, W. HEISENBERG, R. E. PEIERLS, L. ROSENFELD and V. F. WEISSKOPF for the stimulating discussions I had with them long ago: they have influenced, directly or indirectly, whatever is good in this paper, but are not responsible for its shortcomings.

([22]) C. VILLI: *Atti dell'Istituto Veneto di Scienze, Lettere ed Arti*, Tomo CXXXVIII, **1** (1979-1980).

in the form

$$\left(\frac{\partial^2}{\partial\varkappa^2}-\frac{2}{\varkappa}\frac{\partial}{\partial\varkappa}\right)v(p,\varkappa)=\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\frac{\partial}{\partial p}\right)v(p,\varkappa); \tag{5.21}$$

replacing eq. (5.21) into eq. (1.10) one finds

$$K=6\varepsilon_{\mathrm{F}}+(3\varkappa_{\mathrm{F}}/2)\sum_{i=1}^{3}\varphi_i(\varkappa_{\mathrm{F}})\,, \tag{5.22a}$$

$$\begin{cases}\varphi_1(\varkappa_{\mathrm{F}})=\{\partial v(p,\varkappa)/\partial p\}_{p=\varkappa=\varkappa_{\mathrm{F}}}\,,\\ \varphi_2(\varkappa_{\mathrm{F}})=\{\partial v(p,\varkappa)/\partial\varkappa\}_{p=\varkappa=\varkappa_{\mathrm{F}}}\,,\\ \varphi_3(\varkappa_{\mathrm{F}})=\{\mathrm{d}v(\varkappa,\varkappa)/\mathrm{d}\varkappa\}_{\varkappa=\varkappa_{\mathrm{F}}}\,.\end{cases} \tag{5.22b}$$

If we take into account eqs. (1.16) and (5.6), the compressibility modulus becomes

$$K=6\varepsilon_{\mathrm{F}}+3(b_{\mathrm{v}}-\varkappa_{\mathrm{F}})\psi(\varkappa_{\mathrm{F}})\,, \tag{5.23a}$$

where (apart from factor $\varkappa_{\mathrm{F}}$) $\psi(\varkappa_{\mathrm{F}})$ is the first-order logarithmic derivative

$$\psi(\varkappa_{\mathrm{F}})=\varkappa_{\mathrm{F}}\{\mathrm{d}\ln v(\varkappa,\varkappa)/\mathrm{d}\varkappa\}_{\varkappa=\varkappa_{\mathrm{F}}}\,. \tag{5.23b}$$

The parameter $\psi(\varkappa_{\mathrm{F}})$ is a crucial one for testing the validity of a nuclear-matter theory. So far it has been ignored because of the lack of reliable information on K: such information can be extracted from the data only by means of a unified theoretical scheme suitable to match the saturation prescriptions of nuclear matter with the phenomenological information concerning the scattering of neutrons from heavy nuclei. Equation (5.5), combined with the optical transformation (1.19), opens the way towards the achievement of this goal.

As already stressed, the importance of eq. (5.5) in the construction of a realistic description of nuclear matter is centred upon its intrinsic capability to account rigorously for the Hugenholtz and Van Hove theorem. A similar central role in solid-state physics is possessed by Koopmans' theorem [23], which is essential to the interpretation of solids in terms of energy band structures [24]. Both theorems, although proved along substantially different theoretical lines, lead to the same conclusion: this circumstance seems to reveal a common conceptual background. Serious mathematical complexities discourage any attempt to summarize the proof of the Hugenholtz and Van Hove theorem. We shall limit ourselves to recalling the most significant stages of the proof:

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[23] T. KOOPMANS: *Physica (The Hague)*, **1**, 104 (1933); see also W. JONES and N. H. MARCH: *Theoretical Solid State Physics* (London, 1973).

[24] W. A. HARRISON: *Solid State Theory* (New York, N. Y., 1970).

a) in a Fermi gas with interaction at the absolute zero of temperature, the single-particle total energy has a physical meaning only for particles of momentum p close to the Fermi momentum $\varkappa_F$; *b*) the single-particle energy states are metastable with a lifetime approaching to infinity at the limit $p \to \varkappa_F$; *c*) the limiting value w_F of the single-particle total energy is the pole on the real axis of a complicated analytic function, defined as the sum of contributions of all connected diagrams with one ingoing and one outgoing particle line: quantity w_F, consistent with eq. (1.13), becomes physically more significant by virtue of definition (1.14). We recall that Koopmans' theorem proves that the energy required to remove from a solid one electron in a Hartree-Fock state Ψ_s is the eigenvalue w_s of the Hartree-Fock equation for Ψ_s. More specifically, the difference in the total energies calculated by using Slater determinants with N and with $N-1$ wave functions (under the assumption that in both cases the individual one-electron wave functions are the same) is found to be equal to the Hartree-Fock parameter w_s corresponding to the state which has been omitted: at the top of the Fermi sphere this is just the conclusion reached by the Hugenholtz and Van Hove theorem, expressed for large systems by eq. (4.7) with $w(\varrho, \varrho) \equiv w_F$. Furthermore, under the assumption that the electron wave functions do not change as an electron is removed, the ionization energy of a crystal with respect to any given electron state is simply the Hartree-Fock parameter w_s. Since the removal of one electron changes the potential of only one part in N, one may neglect this change and conclude that the Hartree-Fock parameter in a solid is the negative of the ionization energy for the corresponding state in a crystal computed in the Hartree-Fock approximation. According to Koopmans' theorem the change in energy of the system when one electron is transferred from one state to another is simply the difference between the two Hartree-Fock parameters, because both the initial- and final-state energies may be directly related to the same ionization state: it follows that the calculated energy bands can be conceived as one-electron energy eigenvalues. Since the effects of the electron-electron interaction do not change much in going from the free atom to the solid, one may argue that Koopmans' theorem is valid for a crystal only if it is valid for the free atoms that constitute the crystal. Translating all this into nuclear-matter language is indeed intriguing. Two questions then arise: *a*) do the Hugenholtz and Van Hove theorem and Koopman's theorem provide the theoretical clue for describing multinucleon and, respectively, multielectron systems as «independent-particle systems»?; *b*) does the interpretation of a solid in terms of energy band structures correspond—*mutatis mutandis*—to the description of a nucleus in terms of energy shell structures? Finally, is eq. (5.5) also valid in solid-state physics? Alas, I have no exact mathematical answer to these questions.

5·2. – The single-particle potential energy $v(p, \varkappa)$, calculated in the framework of Brueckner's theory, is incompatible with eq. (5.5): this circumstance

makes it difficult to assess all the merits of Brueckner's work, whose importance can be judged only from its overall impact on our understanding of nuclear matter.

We have already verified in sect. 4 that $v(p, \varkappa)$, given by eq. (2.1*a*), possesses property (4.5): therefore, it also fulfils eq. (5.5). We shall now verify that also $v(p, \varkappa)$, expressed by eq. (2.1*b*) for $p \geqslant \varkappa$, is a particular solution of the differential equation of nuclear matter. One has

$$\partial_p v(p, \varkappa) = -(1/p) v(p, \varkappa) - \gamma_1(p, \varkappa) , \tag{5.24a}$$

$$\partial_p^2 v(p, \varkappa) = -(2/p) \partial_p v(p, \varkappa) + \gamma_0(p, \varkappa) - \Gamma(p, \varkappa) , \tag{5.24b}$$

$$\partial_\varkappa v(p, \varkappa) = -\varkappa \gamma_0(p, \varkappa) , \tag{5.24c}$$

$$\partial_\varkappa^2 v(p, \varkappa) = -(1/\varkappa) \partial_\varkappa v(p, \varkappa) - \Gamma(p, \varkappa) , \tag{5.24d}$$

where functions $\nu_\nu(p, \varkappa)$ $(\nu = 0, 1)$ and $\Gamma(p, \varkappa)$ are defined as follows:

$$\nu_\nu(p, \varkappa) = -\frac{1}{\pi M p} \int_{\frac{1}{2}(p-\varkappa)}^{\frac{1}{2}(p+\varkappa)} F(q)(2q - p)^\nu \mathrm{d}q , \tag{5.25a}$$

$$\Gamma(p, \varkappa) = (\varkappa/4\pi M p)[(p + \varkappa) F\{(p + \varkappa)/2\} + (p - \varkappa) F\{(p - \varkappa)/2\}]; \tag{5.25b}$$

using relations (5.24) it is trivially checked that eq. (5.5) is satisfied. It has to be stressed that the differential equation of nuclear matter governs the p-dependence of $v(p, \varkappa_F)$ also for $p \geqslant p(0) > \varkappa_F$ consistently with the separation energy theorem valid at $p = \varkappa_F$: this is the most remarkable feature of the Hugenholtz and Van Hove approximation of infinitely extended nuclear matter as far as the optical model is concerned.

We shall now prove that another particular solution of eq. (5.5) is provided by $v(p, \varkappa)$ calculated in first-order perturbation theory. To this end we construct the nuclear wave function, antisymmetric with the interchange of all the co-ordinates of any pair of nucleons, in Slater's determinantal form from different individual wave functions for single particles. To represent the ground state of the bound system idealized as «infinitely extended nuclear matter» composed by A interacting nucleons, one has to take into account that the $A/4$ spatial wave functions of lower momentum are associated, each in turn, with all four spins and isobaric spin wave functions. The potential energy of nuclear matter in first-order perturbation turns out to be

$$V(\varkappa) = \tfrac{1}{2} \mathbf{S}_{\alpha\beta} \{\langle \alpha\beta | V_C | \alpha\beta \rangle - \langle \alpha\beta | V_C | \beta\alpha \rangle\} , \tag{5.26}$$

where the symbol **S** indicates the operations of integration and summation over the nucleon states $|\alpha\rangle$ and $|\beta\rangle$ in co-ordinate, momentum spin and isobaric

spin spaces. We assume that the central nucleon-nucleon potential V_C is a regular function of r, so that we do not have to circumvent the well-known problems arising from the conjectured existence of a hard core. More specifically, we assume that $V_C \equiv V_C(r)$ is a second-order two-nucleon central potential deduced from the PS-PV Yukawa theory with an extended source, suitable for simulating core effects (suggested by the low-energy nucleon-nucleon elastic-scattering data and by the description of the deuteron S-state) without the necessity of resorting to unphysical discontinuities introduced *ad hoc* in the radial behaviour of $V_C(r)$. Using standard notation, the two-nucleon potential we have in mind reads

$$V_C(r) = V_0 \mathbf{O}_C v_C(r) , \tag{5.27a}$$

$$V_0 = (\mu c^2/3)(\mu/2M)^2 g^2 , \qquad \mathbf{O}_C = (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) , \tag{5.27b}$$

where the radial behaviour of the regular function $v_C(r)$ need not be specified in view of our further considerations; furthermore, we assume that $V_C(r)$ possesses the property [25]

$$\int V_C(r)\, \mathrm{d}\boldsymbol{r} = 0 . \tag{5.27c}$$

The ordinary and exchange contributions to (5.26) are

$$V(\varkappa) = V_{\mathrm{ord}}(\varkappa) - V_{\mathrm{exch}}(\varkappa) , \tag{5.28a}$$

$$V_{\mathrm{ord}}(\varkappa) = (V_0/2)\Big[\sum_{i,j=1}^{4} \langle ij|\mathbf{O}_C|ij\rangle\Big]\Big[\mathbf{I}_{ab}\langle ab|v_C|ab\rangle\Big] , \tag{5.28b}$$

$$V_{\mathrm{exch}}(\varkappa) = (V_0/2)\Big[\sum_{i,j=1}^{4} \langle ij|\mathbf{O}_C|ji\rangle\Big]\Big[\mathbf{I}_{ab}\langle ab|v_C|ba\rangle\Big] , \tag{5.28c}$$

where symbol $\mathbf{I}$ indicates the operations of integration over all the occupied states $|a\rangle \equiv |\boldsymbol{r}_1;\ \boldsymbol{p}_1 \equiv \boldsymbol{p}\rangle$ and $|b\rangle \equiv |\boldsymbol{r}_2;\ \boldsymbol{p}_2 \equiv \boldsymbol{q}\rangle$. Since

$$\mathbf{I}_{ab}\langle ab|v_C|ab = 0 , \qquad \sum_{i,j=1}^{4} \langle ij|\mathbf{O}_C|ji\rangle = 36 , \tag{5.29}$$

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[25] A one-pion nucleon-nucleon exchange potential of type (5.27) has recently been deduced in the framework of the static PS-PV theory of nuclear forces with an extended meson source density, which accounts for the presumptive spatial dimensions of the quark confinement region. For our purposes it is relevant to remark that such a potential possesses realistic physical features without resorting to the *ad hoc* introduction of the hard core and to cutting-off procedures at short distances (C. VILLI: *Nuovo Cimento A*, **67**, 178 (1982)).

where the first of (5.29) arises from property (5.27*c*) ([26]); the potential energy of nuclear matter becomes

$$V(\varkappa) = -\,18V_0 \sum_{ab} \langle ab|v_C|ba\rangle \,. \tag{5.30}$$

Let $\phi_p(\boldsymbol{r}_1)$ be the plane-wave functions describing the generic nucleon «1» of momentum $p_1 \equiv p \leqslant \varkappa$, etc.; one has

$$\sum_{ab} \langle ab|v_C|ba\rangle = \frac{9A(A-1)}{256\pi^2\varkappa^6} \int H(|\boldsymbol{p}-\boldsymbol{q}|)\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,, \tag{5.31}$$

$$H(|\boldsymbol{p}-\boldsymbol{q}|) = \frac{1}{\Omega^2}\int \phi_p^*(\boldsymbol{r}_1)\phi_q^*(\boldsymbol{r}_2)\,v_C(r_{12})\,\phi_q(\boldsymbol{r}_1)\phi_p(\boldsymbol{r}_2)\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2\,, \tag{5.32}$$

where $\Omega = (4\pi/3)R^3 = 3\pi^2 A/2\varkappa^3$. We now perform the co-ordinate transformation

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2\,, \qquad \boldsymbol{s} = \tfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)\,; \tag{5.33}$$

since the Jacobian is equal to 1, the integration over the variable $\boldsymbol{s}$ gives the volume factor Ω and, after two angular integrations, it is found that

$$V(\varkappa) = -\,\frac{27AV_0}{16\pi^3\varkappa^6}\int K(|\boldsymbol{p}-\boldsymbol{q}|)\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,, \tag{5.34a}$$

$$K(|\boldsymbol{p}-\boldsymbol{q}|) = \int_0^\infty r v_C(r)\left\{\sin\left(|\boldsymbol{p}-\boldsymbol{q}|r\right)/|\boldsymbol{p}-\boldsymbol{q}|\right\}\mathrm{d}r \equiv K(2P)\,. \tag{5.34b}$$

Finally, comparing eq. (5.34) with eq. (2.3), one has

$$v(p,\varkappa) = -\,(qV_0/2\pi^2)\int K(|\boldsymbol{p}-\boldsymbol{q}|)\,\mathrm{d}\boldsymbol{q}\,. \tag{5.35}$$

The $G(2P)$ function appearing in eq. (2.5) is

$$G(2P) = -\,\{6V_0/(A-1)\pi\}\,K(2P)\,. \tag{5.36}$$

([26]) This is also true for the central potential deduced from the old PS-PV theory with a pointlike source, provided one takes into account the contact interaction term expressed by the delta-singularity. In the older literature it was customary to omit this term, because for $r>0$ it does not influence the wave function of the two interacting nucleons. Also in this case, however, it was generally assumed, in first-order perturbation calculations, that the ordinary part of the total energy of nuclear matter be equal to zero (L. ROSENFELD: *Nuclear Forces*, Part III (Amsterdam, 1948); see also R. HUBY: *Proc. Phys. Soc. London Sect. A*, **62**, 62 (1949)).

Equation (5.35) can easily be worked out and one obtains

$$v(p, \varkappa) = -(18V_0/\pi)\varkappa^2 \int_0^\infty r v_C(r) j_0(pr) j_1(\varkappa r)\, \mathrm{d}r , \tag{5.37}$$

where $j_0(x)$ and $j_1(x)$ are spherical Bessel functions

$$j_0(x) = \sin x/x , \quad j_1(x) = \sin x/x^2 - \cos x/x . \tag{5.38}$$

Since $v_C(r)$ is supposed to be a continuous function of r, we can derive eq. (5.37) under the integral sign. It is found that

$$\{\partial_p^2 + (2/p)\,\partial_p - \partial_\varkappa^2 + (2/\varkappa)\,\partial_\varkappa\}\{\varkappa^2 j_0(pr) j_1(\varkappa r)\} = 0 ; \tag{5.39}$$

thus the single-particle potential energy (5.37) satisfies the differential equation of nuclear matter and, consequently, it possesses property (4.5) ([27]). In conclusion, the Hugenholtz and Van Hove theorem is formally fulfilled in first-order perturbation theory, but the observed value of b_v cannot be obtained for any reasonable value of the length r_0 and the PS-PV pion-nucleon coupling constant g^2. Function $v(p, \varkappa)$, calculated in first-order perturbation theory on the basis of a nucleon-nucleon potential characterized by a repulsive hard core at short distances, does not fulfil eq. (5.5); the conventional treatment of the hard core in a many-body system leads to results which violate the Hungenholtz and Van Hove theorem, thus simulating the existence of the so-called re-arrangement energies, already discussed in sect. **4**: this is generally true if the preceding calculations are carried out by taking into account the concept of « healing distance », introduced by GOMES, WALECKA and WEISSKOPF ([28]).

6. – Solutions of the differential equation of nuclear matter.

6'1. – We shall seek for solutions of eq. (5.5) by expanding $v(p, \varkappa)$ in power series of p, *i.e.*

$$v(p, \varkappa) = \sum_{n=0}^{\infty} A_n(\varkappa) p^n , \tag{6.1}$$

([27]) A. L. FETTER and J. D. WALECKA: *Quantum Theory of Many-Particle Systems* (New York, N. Y., 1971): it is readily verified that the single-particle potential energies given in formula (40.17), p. 355, in formula (40.18), p. 356, in formula (41.51), p. 369, etc. are particular solutions of the differential equation of nuclear matter.

([28]) L. C. GOMES, J. D. WALECKA and V. F. WEISSKOPF: *Ann. Phys.* (*N. Y.*), **3**, 241 (1958).

where $A_n(\varkappa)$ are unknown functions of the limiting momentum $\varkappa$. Substituting eq. (6.1) into eq. (5.5) it is found that

$$\sum_{n=2}^{\infty} n(n-1)\,A_n(\varkappa)\,p^{n-2} + 2\sum_{n=1}^{\infty} nA_n(\varkappa)\,p^{n-2} = \sum_{n=0}^{\infty} \{\mathfrak{D}A_n(\varkappa)\}\,p^n\,, \tag{6.2}$$

where we have introduced the differential operator ($\boldsymbol{D} \equiv \mathrm{d}/\mathrm{d}\varkappa$)

$$\mathfrak{D} = \boldsymbol{D}^2 - (2/\varkappa)\boldsymbol{D}\,. \tag{6.3}$$

Replacing $n-2$ with n in the summations on the left-hand side of eq. (6.2), one has

$$(2/p)\,A_1(\varkappa) + \sum_{n=0}^{\infty} [(n+2)(n+3)\,A_{n+2}(\varkappa) - \mathfrak{D}A_n(\varkappa)]p^n = 0\,; \tag{6.4}$$

consequently, it must be

$$A_1(\varkappa) = 0\,, \tag{6.5a}$$

$$(n+2)(n+3)\,A_{n+2}(\varkappa) = \mathfrak{D}A_n(\varkappa)\,. \tag{6.5b}$$

From condition (6.5*a*) it follows that *only even powers of the nucleon momentum p appear in expansion* (6.1) ([29,30]).

([29]) The expansion of $v(p, \varkappa)$ in even powers of p has been used previously, without any mathematical justification, with the purpose of describing the p-dependence only near $p = 0$. Originally it was privately suggested to K. A. BRUECKNER by J. A. WHEELER (see K. A. BRUECKNER: *Phys. Rev.*, **97**, 1353 (1955)). The drawbacks of various attempts to describe, even approximately, the nuclear-matter behaviour by using such an expansion were due to the fact that the $\varkappa$-dependence of $v(p, \varkappa)$ was completely unknown: we now know that the $\varkappa$-dependence of the single-particle potential energy is governed (to a very high degree of approximation) by eq. (5.5).

([30]) The ignorance of the existence of eq. (5.5) (and of its mathematical implications) has led—even on an approximate ground—to serious misunderstandings of the fundamental properties of infinitely extended nuclear matter. For example, the theoretical credibility of its claimed superfluid character is largely spoiled by such misunderstandings. A disconcerting proof in support of this opinion is given in a note by K. L. MILL, A. M. SESSLER, S. A. MOSKOWSKI and D. G. SHAUKLAND (*Phys. Rev. Lett.*, **3**, 383 (1959)): apart from several other critical remarks, it has to be pointed out that *a*) the single-body potential energy, given by their formula (9), does not fulfil eq. (5.5) and, therefore, conflicts with the saturation prescriptions (1.5) (for instance, for $\varkappa_F = 1.4\ \mathrm{fm}^{-1}$ it is found that $b_v = 0.15$ MeV(!) instead of $b_v = -15.5$ MeV as assumed by the authors); *b*) the calculation of the effective mass is incorrect and *c*) the interpolation formula, given in note (5), is wrong because the cubic power of the nucleon momentum is not allowed.

Let N be the maximum value of n in expansion (6.1), which we re-write in the form

$$v^{(N)}(p, \varkappa) = \sum_{n=0}^{N} B_n^{(N)}(\varkappa) p^{2n} \tag{6.6}$$

this means that all terms corresponding to $n > N$ are assumed to be zero, *i.e.*

$$B_{N+\nu}^{(N)}(\varkappa) = 0 \tag{6.7}$$

for $\nu = 1, 2, \ldots$. The functions $B_{(n)}^{n}(\varkappa)$ satisfy the recurrence relations

$$(2n + 2)(2n + 3) B_{n+1}^{(N)}(\varkappa) = \mathfrak{D} B_n^{(N)}(\varkappa) . \tag{6.8}$$

It follows that the determination of the functions $B_n^{(N)}(\varkappa)$ can be carried out by solving the system of differential equations

$$\left\{\begin{array}{rl} & 6 B_1^{(N)}(\varkappa) = \mathfrak{D} B_0^{(N)}(\varkappa) , \\ & \cdots\cdots\cdots , \\ & \cdots\cdots\cdots , \\ 2N(2N + 1) B_N^{(N)}(\varkappa) & = \mathfrak{D} B_{N-1}^{(N)}(\varkappa) , \\ & 0 = \mathfrak{D} B_N^{(N)}(\varkappa) . \end{array}\right. \tag{6.9}$$

The solutions of the system are

$$\left\{\begin{array}{l} B_N^{(N)}(\varkappa) = C_{2N+1}^{(N)} \varkappa^3 + C_{2N}^{(N)} , \\ B_{N-1}^{(N)}(\varkappa) = N(2N + 1)[\tfrac{1}{5} C_{2N+1}^{(N)} \varkappa^5 - C_{2N}^{(N)} \varkappa^3] + \tfrac{1}{3} C_{2N-1}^{(N)} \varkappa^3 + C_{2N-2}^{(N)} , \\ \cdots\cdots\cdots , \\ \cdots\cdots\cdots , \end{array}\right. \tag{6.10}$$

where $C_{2N+1}^{(N)}$ are $2N + 1$ arbitrary integration constants. Taking into account conditions (5.2) and (5.3), we must put equal to zero all the constants which are not multiplied by the limiting momentum $\varkappa$, thus reducing their number to $N + 1$; in conclusion, functions (6.10) are physically significant provided

$$C_{2N}^{(N)} = C_{2N-2}^{(N)} = \ldots = 0 . \tag{6.11}$$

It is worthwhile to remark that, owing to conditions (6.11), *function* $v(p, \varkappa)$ *depends only on the odd powers of* $\varkappa$ ($\varkappa^n$ with $n \geqslant 3$).

6·2. – We shall now solve eq. (5.5) assuming that the single-particle potential energy $v(p, \varkappa)$ is a separable function of p and $\varkappa$, *i.e.*

$$v(p, \varkappa) = v_1(p) v_2(\varkappa); \tag{6.12}$$

the analytic form (6.12) has never been used before in nuclear-matter calculations ([31]). Substituting (6.12) into eq. (5.5) one has

$$\nabla_p^2 v_1(p)/v_1(p) = \mathfrak{D} v_2(\varkappa)/v_2(\varkappa) , \tag{6.13}$$

where $\nabla_p^2 = \mathrm{d}^2/\mathrm{d}p^2 + (2/p)(\mathrm{d}/\mathrm{d}p)$ is the «radial» part of the Laplacian operator in momentum space. Both terms of eq. (6.13) must be equal to a real constant c_0; one has

$$(\nabla_p^2 - c_0) v_1(p) = 0 , \quad (\mathfrak{D} - c_0) v_2(\varkappa) = 0 . \tag{6.14}$$

It is *a priori* unknown whether c_0 is a real, negative quantity ($c_0 = -\alpha^2$) or a positive one ($c_0 = \beta^2$); thus the differential equations to be examined are

$$(\nabla_p^2 + \alpha^2) v_1(p) = 0 , \quad (\mathfrak{D} + \alpha^2) v_2(\varkappa) = 0 , \tag{6.15a}$$

$$(\nabla_p^2 - \beta^2) v_1(p) = 0 , \quad (\mathfrak{D} - \beta^2) v_2(\varkappa) = 0 . \tag{6.15b}$$

The solutions of eqs. (6.15) must be sought for by requiring that the following conditions be satisfied:

$$v_1(0) = \text{const} , \quad v_1(\infty) = 0 , \quad v_2(0) = 0 . \tag{6.16}$$

It follows that *a*) the first of eqs. (6.15*a*) for $p \leqslant \varkappa$ possesses the general integral

$$v_1(p) = C_1\{\sin(\alpha p)/p\} , \tag{6.17}$$

where C_1 is an arbitrary constant, whereas the solution for $p > \varkappa$ has to be rejected because asymptotically oscillatory; *b*) the first of eqs. (6.15*b*) is a Klein-Gordon equation in momentum space: for $p \leqslant \varkappa$ its solution is given by a Hulthén function, which turns out to be incompatible with the saturation prescriptions (1.5), and for $p > \varkappa$ is expressed by the Yukawa function

$$v_1(p) = C_2\{\exp[-\beta p]/p\} , \tag{6.18}$$

([31]) Only two-body factorable potentials have been used so far in nuclear-matter calculations, following a suggestion of K. M. WATSON to K. A. BRUECKNER (see K. A. BRUECKNER and W. WADA: *Phys. Rev.*, **103**, 1008 (1956)).

where C_2 is a new arbitrary integration constant. Thus the p-dependence of the factorable single-particle potential energy (6.12) in the interval $0 \leqslant p \leqslant \infty$ is obtained by linking by continuity at $p = \varkappa = \varkappa_F$ functions (6.17) and (6.18) and their first derivatives: this is ensured provided

$$C_1 = 1 , \qquad C_2 \equiv C_2(\varkappa_F) = \exp[\beta\varkappa_F] \sin(\alpha\varkappa_F) \tag{6.19}$$

and the parameters $\alpha \equiv \alpha(\varkappa_F)$ and $\beta \equiv \beta(\varkappa_F)$ satisfy the equation

$$(1 + \beta\varkappa_F) j_0(\alpha\varkappa_F) = \alpha\varkappa_F j_1(\alpha\varkappa_F) , \tag{6.20}$$

where $j_0(x)$ and $j_1(x)$ are the spherical Bessel functions specified in eqs. (5.38). In conclusion, the factorable single-particle potential energy (6.12) reads

$$v(p, \varkappa) = v_2(\varkappa)\{\sin(\alpha p)/p\} , \qquad p \leqslant \varkappa , \tag{6.21a}$$

$$v(p, \varkappa) = v_2(\varkappa) C_2(\varkappa_F)\{\exp[-\beta p]/p\} , \qquad p \geqslant \varkappa . \tag{6.21b}$$

From the preceding considerations it follows that $v_2(\varkappa)$ is the solution of the equation

$$(\mathfrak{D} + \alpha^2) v_2(\varkappa) = 0 . \tag{6.22}$$

By means of the transformation

$$v_2(\varkappa) = \varkappa z(\varkappa) \tag{6.23}$$

we eliminate the first-order derivative appearing in $\mathfrak{D}$ and obtain the equation

$$\boldsymbol{D}^2 z(\varkappa) + \{\alpha^2 - (2/\varkappa^2) z(\varkappa)\} = 0 . \tag{6.24}$$

A solution of eq. (6.22), vanishing at the origin as required by conditions (5.2) and (5.3), is

$$v_2(\varkappa) = C_3\{\sin(\alpha\varkappa) - (\alpha\varkappa)\cos(\alpha\varkappa)\} , \tag{6.25}$$

where C_3 is an integration constant. In the forthcoming discussion we shall examine the single-particle potential energy (6.21) with $v_2(\varkappa)$ expressed by eq. (6.25) ([32]).

([32]) General solutions of the differential equation of nuclear matter have been searched for by T. A. MINELLI: *Atti e Memorie dell'Accademia Patavina di Scienze, Lettere ed Arti*, Vol. LXXXV, Part. II (1972-1973).

7. – Constraints imposed by the saturation prescriptions on the solutions of the differential equation of nuclear matter.

As is well known, the saturation prescriptions (1.5) and the separation energy theorem (1.15) are necessary, but not sufficient, conditions for proving the capability of a theory to outline a realistic description of nuclear matter: any theory constructed for this goal only deserves limited credit *a priori* (actually such a goal has not yet been achieved satisfactorily). For this reason we shall regard b_v and r_0 as input parameters. Our aim is to build up a nuclear-matter description which, by construction, fulfils eqs. (1.5) and (1.15), and then use such a description in order to determine parameters b_v, r_0 and K, consistently with the phenomenological evidence on the energy dependence of the real part of the optical potential.

We shall adopt the following notation:

$$(7.1a) \qquad W^{(N)}(\varkappa_F, \varkappa_F) = \varepsilon_F + V^{(N)}(\varkappa_F, \varkappa_F) ,$$

$$(7.1b) \qquad V^{(N)}(\varkappa) = (3A/8\pi\varkappa^2)\int v^{(N)}(p, \varkappa)\, \mathrm{d}\boldsymbol{p} ,$$

$$(7.1c) \qquad K^{(N)} = (6/5)\varepsilon_F + (1/A)\{\varkappa^2 \boldsymbol{D} V^{(N)}(\varkappa)\}_{\varkappa=\varkappa_F} ,$$

where N indicates the order of approximation on solutions (6.6), obtained by series integration of eq. (5.5). We shall indicate with the notation $W(\varkappa_F, \varkappa_F)$, $V(\varkappa)$ and K the same quantities calculated with the solutions of eq. (5.5), obtained by variable separation.

For the reader's convenience we give in table I the values of the Fermi momentum and those of the Fermi energy as functions of the nuclear length r_0.

TABLE I. – *The Fermi momentum* $\varkappa_F$ (*in units* 1/fm) *and the Fermi energy* ε_F (*in* MeV) *calculated as functions of the nuclear length* r_0 (*in* fm *units*).

r_0	$\varkappa_F$	ε_F	r_0	$\varkappa_F$	ε_F
0.9	1.6925	59.38	1.2	1.2693	33.40
1.0	1.5232	48.09	1.3	1.1718	28.46
1.1	1.3848	39.75	1.4	1.0880	24.54

7·1. – The $N = 0$ approximation is not significant. Function $B_0^{(0)}(\varkappa)$ obeys the differential equation

$$(7.2a) \qquad \mathfrak{D} B_0^{(0)}(\varkappa) = 0 ;$$

the momentum dependence of $v(p, \varkappa)$ is lost and one has

$$v^{(0)}(p, \varkappa) = B_0^{(0)}(\varkappa) = \tfrac{1}{3} C_1^{(0)} \varkappa^3 , \tag{7.2b}$$

where we have put $C_1^{(0)} = 0$. Equation (7.2*b*) represents a potential well in momentum space: this description of the single-particle potential energy is not compatible with nuclear stability.

The $N = 1$ approximation is obtained by solving the system of differential equations

$$\mathfrak{D} B_0^{(1)}(\varkappa) = 6 B_1^{(1)}(\varkappa) , \qquad \mathfrak{D} B_1^{(1)}(\varkappa) = 0 ; \tag{7.3}$$

it is found

$$v^{(1)}(p, \varkappa) = B_0^{(1)}(\varkappa) + B_1^{(1)}(\varkappa) p^2 , \tag{7.4a}$$

$$B_0^{(1)}(\varkappa) = \tfrac{3}{5} C_3^{(1)} \varkappa^5 + C_1^{(1)} \varkappa^3 , \qquad B_1^{(1)}(\varkappa) = C_3^{(1)} \varkappa^3 , \tag{7.4b}$$

where we have put $C_2^{(1)} = 0$; note that an unessential factor $\frac{1}{3}$ has been included in the integration constants $C_3^{(1)}$ and $C_1^{(1)}$. It is readily verified that the function $Q(\varkappa)$, calculated with (7.4), is identically zero. The $\varkappa$-dependence found for $V^{(1)}(\varkappa)$ shows that the interpolation formula reported by BETHE [33] is wrong. The total energy of a single nucleon having momentum p in the Fermi sea is

$$w^{(1)}(p, \varkappa_F) = (p^2/2M) + v^{(1)}(p, \varkappa_F) = (p^2/2M^*) + B_0^{(1)}(\varkappa_F) , \tag{7.5a}$$

$$M/M^* = 1 + 2M B_1^{(1)}(\varkappa_F) ; \tag{7.5b}$$

in the considered approximation both definitions (3.5) and (3.6) lead to the analytical expression (7.5*b*) for the nucleon effective mass. According to eqs. (7.5), the nucleon is described as a free particle in motion in the potential well $B_0^{(1)}(\varkappa_F)$ with momentum $p \leqslant \varkappa_F$ and constant effective mass M^*: as is well known, this is the *effective-mass approximation*.

The two unknown constants $C_3^{(1)}$ and $C_1^{(1)}$ are determined from the saturation prescriptions (1.5). The following system of linear equation is found:

$$6\varkappa_F^5 C_3^{(1)} + 5\varkappa_F^3 C_1^{(1)} = 10 b_v - 6\varepsilon_F , \qquad 10\varkappa_F^2 C_3^{(1)} + 15\varkappa_F^3 C_1^{(1)} = -4\varepsilon_F , \tag{7.6}$$

the solutions of which are

$$\varkappa_F^5 C_3^{(1)} = (\varepsilon_F - 5b_v)/2 , \qquad \varkappa_F^3 C_1^{(1)} = (25 b_v - 9\varepsilon_F)/5 ; \tag{7.7}$$

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[33] H. A. BETHE: *Annu. Rev. Nucl. Sci.*, **21**, 93 (1971); see also W. KUNDT and E. T. NEWMAN: *J. Math. Phys. (N. Y.)*, **9**, 2193 (1967).

therefore, the coefficients of the single-particle potential energy (7.4) read

$$B_0^{(1)}(\varkappa_F) = (7b_v - 3\varepsilon_F)/2\,, \qquad \varkappa_F^2 B_1^{(1)}(\varkappa_F) = (\varepsilon_F - 5b_v)/2\,. \tag{7.8}$$

Using eqs. (7.7) one has

$$M^*/M = \varepsilon_F/\{\varepsilon_F + \varkappa_F^5 C_3^{(1)}\} = 2\varepsilon_F/(3\varepsilon_F - 5b_v)\,, \tag{7.9}$$

$$K^{(1)} = (6/5)\varepsilon_F + 12\varkappa_F^5 C_3^{(1)} + 3\varkappa_F^3 C_1^{(1)} = (3/5)(3\varepsilon_F - 25b_v)\,. \tag{7.10}$$

The total energy of a single nucleon at the Fermi surface is

$$w^{(1)}(\varkappa_F, \varkappa_F) = \varepsilon_F + (8/5)\varkappa_F^5 C_3^{(1)} + \varkappa_F^3 C_1^{(1)}\,. \tag{7.11a}$$

Substituting parameters (7.7) in eq. (7.11*a*) it is found that

$$w^{(1)}(\varkappa_F, \varkappa_F) = b_v\,; \tag{7.11b}$$

thus *the saturation prescriptions* (1.5) *are consistent with the Hugenholtz and Van Hove theorem* (1.15), *because the single-particle potential energy* (7.4), *although expressed in* $N = 1$ *approximation, is nevertheless an exact solution of the differential equation of nuclear matter.*

The numerical results of the $N = 1$ approximation are given in table II as functions of the input parameters r_0 and b_v. The values of the ratio M^*/M and those of the compressibility modulus express the ultimate result of the effective-mass approximation: to the author's knowledge they have never been evaluated before by taking exactly into account the saturation prescription and the separation energy theorem ([34]). The most appealing peculiarity is that both the effective mass and the compressibility modulus increase as r_0 decreases: this fact has stimulated the considerations developed in sect. **4**.

7·2. – The $N = 2$ approximation is obtained by solving the system of differential equations

$$\mathfrak{D}B_0^{(2)}(\varkappa) = 6B_1^{(2)}(\varkappa)\,, \qquad \mathfrak{D}B_1^{(2)}(\varkappa) = 20B_2^{(2)}(\varkappa)\,, \qquad \mathfrak{D}B_2^{(2)}(\varkappa) = 0\,; \tag{7.12}$$

([34]) The effective-mass approximation without the knowledge of the k-dependence of $v(p, \varkappa)$ has been extensively used by K. A. BRUECKNER and W. WADA: *Phys. Rev.*, **103**, 1008 (1958): it can be readily proved that the quoted results do not fulfil the first of equations (1.5) and the separation energy theorem (1.15).

TABLE II*a*. – $N = 1$ *approximation*. Numerical values (in MeV) of the coefficients of the single-particle potential energy (7.4) as functions of r_0 (in fm units) and b_v.

r_0	$b_v = -14.0$ MeV		$b_v = -15.0$ MeV		$b_v = -16.0$ MeV	
	$B_0^{(1)}(\varkappa_F)$	$\varkappa_F^2 B_1^{(1)}(\varkappa_F)$	$B_0^{(1)}(\varkappa_F)$	$\varkappa_F^2 B_1^{(1)}(\varkappa_F)$	$B_0^{(1)}(\varkappa_F)$	$\varkappa_F^2 B_1^{(1)}(\varkappa_F)$
0.9	−134.53	63.51	−138.03	66.01	−141.53	68.51
1.0	−118.27	58.09	−121.77	60.59	−125.27	63.09
1.1	−106.25	54.08	−109.75	56.58	−113.25	59.08
1.2	− 97.10	51.03	−100.06	52.99	−104.10	56.03
1.3	− 89.99	48.66	− 93.49	51.16	− 96.99	53.66
1.4	− 84.34	46.78	− 87.84	49.28	− 91.34	51.78

TABLE II*b*. – $N = 1$ *approximation*. Numerical values of the ratio M^*/M and the compressibility modulus $K^{(1)}$ (in MeV) as functions of r_0 (in fm units) and b_v.

r_0	$b_v = -14.0$ MeV		$b_v = -15.0$ MeV		$b_v = -16.0$ MeV	
	M^*/M	$K^{(1)}$	M^*/M	$K^{(1)}$	M^*/M	$K^{(1)}$
0.9	0.473	312.63	0.463	327.63	0.454	342.64
1.0	0.443	293.12	0.432	308.12	0.423	323.12
1.1	0.414	278.71	0.401	293.71	0.392	308.70
1.2	0.386	267.73	0.375	282.73	0.364	297.73
1.3	0.360	259.19	0.348	274.19	0.337	289.19
1.4	0.335	252.41	0.323	267.41	0.313	282.40

it is found that

$$v^{(2)}(p, \varkappa) = B_0^{(2)}(\varkappa) + B_1^{(2)}(\varkappa)p^2 + B_2^{(2)}(\varkappa)p^4 , \tag{7.13a}$$

$$\begin{cases} B_0^{(2)}(\varkappa) = \frac{3}{7} C_5^{(2)} \varkappa^7 + \frac{3}{5} C_3^{(2)} \varkappa^5 + C_1^{(2)} \varkappa^3 , \\ B_1^{(2)}(\varkappa) = 2 C_5^{(2)} \varkappa^5 + C_3^{(2)} \varkappa^3 , \\ B_2^{(2)}(\varkappa) = C_5^{(2)} \varkappa^3 , \end{cases} \tag{7.13b}$$

where we have put $C_2^{(2)} = C_4^{(2)} = 0$ as required by conditions (6.11) (note that an unessential factor $\frac{1}{3}$ has been included in the integration constants $C_5^{(2)}$ and $C_3^{(2)}$). We shall now bring to light once more the subtle link existing between solution (7.13) of the differential equation (5.5) and the Hugenholtz and Van Hove theorem. To this end, we use the single-particle potential energies (7.13) in order to deduce from eqs. (1.5) two equations in the un-

known parameters $C_5^{(2)}$, $C_3^{(2)}$ and $C_1^{(2)}$; they are

$$(7.14) \qquad \begin{cases} 72\varkappa_F^7 C_5^{(2)} + 42\varkappa_F^5 C_3^{(2)} + 35\varkappa_F^3 C_1^{(2)} = 70b_v - 42\varepsilon_F , \\ 72\varkappa_F^7 C_5^{(2)} + 30\varkappa_F^5 C_3^{(2)} + 15\varkappa_F^3 C_1^{(2)} = -12\varepsilon_F ; \end{cases}$$

the solutions $C_3^{(2)}$ and $C_1^{(2)}$ of system (7.14), expressed as functions of $C_5^{(2)}$, are

$$(7.15) \qquad \begin{cases} \varkappa_F^5 C_3^{(2)} = (1/2)\,\varepsilon_F - (5/2)\,b_v - (24/7)\,\varkappa_F^7 C_5^{(2)} , \\ \varkappa_F^3 C_1^{(2)} = 5b_v - (63/35)\,\varepsilon_F + (72/35)\,\varkappa_F^7 C_5^{(2)} . \end{cases}$$

The single-particle total energy at the Fermi surface is

$$(7.16a) \qquad W^{(2)}(\varkappa_F, \varkappa_F) = \varepsilon_F + (24/7)\,\varkappa_F^7 C_5^{(2)} + (8/5)\,\varkappa_F^5 C_3^{(2)} + \varkappa_F^3 C_1^{(2)};$$

substituting eqs. (7.15) into eq. (7.16*a*) it is found that

$$(7.16b) \qquad W^{(2)}(\varkappa_F, \varkappa_F) = b_v ,$$

thus the Hugenholtz and Van Hove theorem is fulfilled for any value of $\varkappa_F^7 C^{(2)}$. Let us suppose we ignore the interlacement existing among the solutions of the differential equation of nuclear matter and the separation energy theorem. Then, one could be tempted to determine the three unknown parameters $C_5^{(2)}$, $C_3^{(2)}$ and $C_1^{(2)}$ as solutions of the system of three linear equations composed by eqs. (7.14) and eq. (7.16), which reads

$$(7.17) \qquad 120\varkappa_F^7 C_5^{(2)} + 56\varkappa_F^5 C_3^{(2)} + 35\varkappa_F^3 C_1^{(2)} = 35(b_v - \varepsilon_F) .$$

To reach this goal one needs to evaluate determinant $\Delta(\varkappa_F)$ of the system, which must be different from zero; it is found that

$$(7.18) \qquad \Delta(\varkappa_F) = \begin{vmatrix} 72\varkappa_F^7 & 42\varkappa_F^5 & 35\varkappa_F^3 \\ 72\varkappa_F^7 & 30\varkappa_F^5 & 15\varkappa_F^3 \\ 120\varkappa_F^7 & 56\varkappa_F^5 & 35\varkappa_F^3 \end{vmatrix} = 0 .$$

The result expressed in eq. (7.18) is self-explanatory: *saturation prescriptions* (1.5) *are consistent with the Hugenholtz and Van Hove theorem* (1.15), *because the single-particle potential energy* (7.13), *although expressed in* $N = 2$ *approximation, is nevertheless an exact solution of the differential equation of nuclear matter.*

As a third equation we choose the analytic form of the single-particle potential energy of zero momentum at the Fermi sphere. From eq. (7.13)

it is found that

$$v^{(2)}(0, \varkappa_F) = (3/7)\varkappa_F^7 C_5^{(2)} + (3/5)\varkappa_F^5 C_3^{(2)} + \varkappa_F^3 C_1^{(2)}, \tag{7.19}$$

where $v^{(2)}(0, \varkappa_F) \equiv v(0, \varkappa_F)$ is supposed to be a known parameter. The system of linear equations required for the determination of the parameters $C_5^{(2)}$, $C_3^{(2)}$ and $C_1^{(2)}$ is

$$\begin{cases} 72\varkappa_F^7 C_5^{(2)} + 42\varkappa_F^5 C_3^{(2)} + 35\varkappa_F^3 C_1^{(2)} = 70b_v - 42\varepsilon_F, \\ 72\varkappa_F^7 C_5^{(2)} + 30\varkappa_F^5 C_3^{(2)} + 15\varkappa_F^3 C_1^{(2)} = -12\varkappa_F, \\ 15\varkappa_F^7 C_5^{(2)} + 21\varkappa_F^5 C_3^{(2)} + 35\varkappa_F^3 C_1^{(2)} = 35v(0, \varkappa_F); \end{cases} \tag{7.20}$$

the solutions are

$$\begin{cases} \varkappa_F^7 C_5^{(2)} = (7/2)\varepsilon_F - (49/6)b_v + (7/3)v(0, \varkappa)_F, \\ \varkappa_F^5 C_3^{(2)} = (51/2)b_v - (23/2)\varepsilon_F - 8v(0, \varkappa_F), \\ \varkappa_F^3 C_1^{(2)} = (27/5)\varepsilon_F - (59/5)b_v + (24/5)v(0, \varkappa_F). \end{cases} \tag{7.21}$$

A check that solutions (7.21) are exact can be carried out by replacing them into eq. (7.16*a*): then eq. (7.16*b*) is found! The coefficients of the single-particle potential energy (7.13) are

$$\begin{cases} B_0^{(2)}(\varkappa_F) = v(0, \varkappa_F), \\ \varkappa_F^2 B_1^{(2)}(\varkappa_F) = (55/6)b_v - (9/2)\varepsilon_F - (10/3)v(0, \varkappa_F), \\ \varkappa_F^4 B_2^{(2)}(\varkappa_F) = (7/2)\varepsilon_F - (49/6)b_v + (7/3)v(0, \varkappa_F). \end{cases} \tag{7.22}$$

A check that coefficients (7.22) are exact can be carried out by putting $B_2^{(2)}(\varkappa_F) = 0$; it is found that

$$v(0, \varkappa_F) = (7/2)b_v - (3/2)\varepsilon_F \equiv B_0^{(1)}(\varkappa_F), \tag{7.23}$$

whereas, substituting eq. (7.23) into the second of eqs. (7.22), one finds the coefficient $B_1^{(1)}(\varkappa_F)$ which, together with $B_0^{(1)}(\varkappa_F)$, characterizes the effective-mass approximation. The compressibility modulus turns out to be

$$\begin{cases} K^{(2)} = (6/5)\varepsilon_F + (216/5)\varkappa_F^7 C_5^{(2)} + 12\varkappa_F^5 C_3^{(2)} + 3\varkappa_F^3 C_1^{(2)}, \\ K^{(2)} = (153/5)\varepsilon_F - (411/5)b_v + (96/5)v(0, \varkappa_F); \end{cases} \tag{7.24}$$

substituting eq. (7.23) ($C_5^{(2)} = 0$) into eqs. (7.24) one finds $K^{(1)}$ given by eqs. (7.10). The nucleon effective mass, calculated for $p = 0$ and $p = \varkappa_F$ ac-

cording to the quantum-mechanical definition (3.5), and the parameter M^{**} defined in (3.6) are, respectively,

$$M^*(0)/M = 6\varepsilon_F/\{55b - 21\varepsilon_F - 20v(0, \varkappa_F)\}, \tag{7.25a}$$

$$M^*(\varkappa_F)/M = 6\varepsilon_F/\{105\varepsilon_F - 239b_v + 64v(0, \varkappa_F)\}, \tag{7.25b}$$

$$M^{**}/M = 6\varepsilon_F/\{21\varepsilon_F - 43b_v + 8v(0, \varkappa_F)\}. \tag{7.26}$$

The preceding relations provide a useful tool for testing the validity of nuclear-matter calculations. As an example, let us extract the physical content of the empirical formula for the single-particle potential energy used by BRUECKNER ([35])

$$V(p) = -119 + 57.3(p/\varkappa_F)^2 - 7.18(p/\varkappa_F)^4. \tag{7.27}$$

The identification of $V(p)$ with $v^{(2)}(p, \varkappa_F)$, expressed by eq. (7.13*a*), leads to the system of equations

$$\begin{cases} B_0^{(2)}(\varkappa_F) = -119 \text{ MeV}, \\ \varkappa_F^2 B_1^{(2)}(\varkappa_F) = 57.3 \text{ MeV}, \\ \varkappa_F^4 B_2^{(2)}(\varkappa_F) = -7.18 \text{ MeV}, \end{cases} \tag{7.28}$$

where the coefficients $\varkappa_F^{2n} B_n^{(2)}(\varkappa_F)$ $(n = 0, 1, 2)$ are given by eqs. (7.22). It is found that $v^{(2)}(0, \varkappa_F) \equiv v(0, \varkappa_F) = -119$ MeV and

$$b_v = -6.28 \text{ MeV}, \quad \varepsilon_F = 62.61 \text{ MeV}, \quad r_0 = 0.86 \cdot 10^{-13} \text{ cm}, \tag{7.29a}$$

$$M^*(0)/M = 0.522, \quad M^*(\varkappa_F)/M = 0.817, \quad M^{**}/M = 0.593; \tag{7.29b}$$

the compressibility modulus is found to be $K^{(2)} = 147.47$ MeV. The values of b_v and r_0 do not correspond with those given by BRUECKNER: thus, the quoted paper is numerically erroneous and conceptually misleading.

Let us evaluate parameter $v(0, \varkappa_F)$ by taking the limit for $p \to 0$ of $v(p, \varkappa_F)$ given by eq. (2.1*a*), *i.e.*

$$v(0, \varkappa_F) = -(4/\pi M)\int_0^{\varkappa_F/2} F(q)\, q^2\, dq. \tag{7.30}$$

According to the classification (2.2) forward scattering amplitudes, involved in the construction of $F(q)$, must be separated into even and odd nucleon-

([35]) K. A. BRUECKNER: *Phys. Rev.*, **97**, 1353 (1955).

nucleon spatial states; the calculation was numerically performed taking into account S, P and D waves. We used the asymptotic phase shifts determined from the analyses of the nucleon-nucleon elastic-scattering data by McGREGOR, MORAVCSIK and NOYES ([36]), and by CLEMENTEL and myself. Extensive use has been made of singlet and triplet scattering lengths and effective ranges. The dominant contribution to the integral appearing in (7.30) arises from the triplet and singlet S-wave phase shifts. The major difficulties in evaluating $v(0, \varkappa_F)$ come *a*) from uncertainties concerning the slope of the singlet S-wave phase shift, which possesses a maximum at about 10 MeV (in the laboratory system); *b*) from the fourfold triplet-P ambiguity, and *c*) from the insufficient knowledge in the considered energy interval of the coupling parameters of S and D waves. As is well known, the phase shifts can be varied by few degrees in a correlated way without using undue violence to the data: the stability of $v(0, \varkappa_F)$ against such variations has not been investigated. This is a weak point, because small variations of $v(0, \varkappa_F)$ could change the sign of the coefficient $B_2^{(2)}(\varkappa_F)$. The numerical results are given in table III; I am unable to assess their degree of reliability.

TABLE III. – *Numerical values* (*in* MeV) *of the single-particle potential energy* $v(0, \varkappa_F)$ *as functions of* r_0 (*in* fm *units*).

r_0	$v(0, \varkappa_F)$	r_0	$v(0, \varkappa_F)$
0.9	−141.29	1.2	−107.31
1.0	−126.24	1.3	−100.59
1.1	−115.67	1.4	− 95.00

The numerical results characterizing the $N = 2$ approximation are given in table IV as functions of the input parameters r_0 and b_v. The values of $\varkappa_F^2 B_1^{(2)}(\varkappa_F)$ at 1.0 fm probably betray the lack of accuracy in the calculation of the integral appearing in eq. (7.30). Ratio $M^*(p)/M$ possesses a singularity at the Fermi sphere: it is seen that the singularity moves towards higher values of r_0 and disappears for $b_v = -16.0$ MeV. While in the $N = 1$ approximation the concept of momentum-independent nucleon effective mass has a clear-cut physical meaning, in the $N = 2$ approximation $M/M^*(p)$ is significant only because it is a linear function of the second-order derivative of the single-particle potential energy. The parameter M^{**}/M turns out to be a slowly varying function of r_0 and b_v; its values are not very different from those characterizing the $N = 1$ approximation: thus, not only is definition (3.6)

([36]) Use has been made of the information contained in the following reports issued by the Lawrence Radiation Laboratory of the University of California: UCRL 4947 (1957), UCRL 5348 (1958), UCRL 5566 (1959) and UCRL 6108-T (1960).

TABLE IV. – $N = 2$ *approximation*. Numerical values (in MeV) of the coefficients of the single-particle potential energy $v^{(2)}(p, \varkappa_F)$ and the compressibility modulus $K^{(2)}$ as functions of r_0 (in fm units) and b_v. The mass ratios $M^*(p)/M$ ($p = 0$, $p = \varkappa_F$) and M^{**}/M have been evaluated according to eqs. (7.25) and, respectively, (7.26).

$b_v = -14.0$ MeV

r_0	$B_0^{(2)}(\varkappa_F)$	$\varkappa_F^2 B_1^{(2)}(\varkappa_F)$	$\varkappa_F^4 B_2^{(2)}(\varkappa_F)$	$M^*(0)/M$	$M^*(\varkappa_F)/M$	M^{**}/M	$K^{(2)}$
0.9	−141.29	86.04	−15.77	0.398	1.177	0.519	182.84
1.0	−126.24	84.66	−18.60	0.353	2.398	0.493	140.10
1.1	−115.67	85.47	−21.97	0.309	−4.671	0.479	97.94
1.2	−107.31	85.05	−23.81	0.274	−1.249	0.461	71.79
1.3	−100.59	83.98	−24.72	0.245	−0.738	0.442	55.77
1.4	− 95.00	82.31	−24.87	0.222	−0.543	0.420	47.73

$b_v = -15.0$ MeV

r_0	$B_0^{(2)}(\varkappa_F)$	$\varkappa_F^2 B_1^{(2)}(\varkappa_F)$	$\varkappa_F^4 B_2^{(2)}(\varkappa_F)$	$M^*(0)/M$	$M^*(\varkappa_F)/M$	M^{**}/M	$K^{(2)}$
0.9	−141.29	76.87	− 7.60	0.426	0.645	0.480	265.04
1.0	−126.24	75.49	−10.43	0.379	0.780	0.458	222.30
1.1	−115.67	76.30	−13.80	0.333	1.202	0.439	180.14
1.2	−107.31	75.88	−15.64	0.297	2.251	0.418	153.99
1.3	−100.59	74.81	−16.55	0.267	9.429	0.396	137.97
1.4	− 95.00	73.14	−16.70	0.244	−6.828	0.372	129.93

$b_v = -16.0$ MeV

r_0	$B_0^{(2)}(\varkappa_F)$	$\varkappa_F^2 B_1^{(2)}(\varkappa_F)$	$\varkappa_F^4 B_2^{(2)}(\varkappa_F)$	$M^*(0)/M$	$M^*(\varkappa_F)/M$	M^{**}/M	$K^{(2)}$
0.9	−141.29	67.71	0.56	0.457	0.445	0.453	347.24
1.0	−126.24	66.32	− 2.26	0.410	0.467	0.428	304.50
1.1	−115.67	67.13	− 5.63	0.362	0.534	0.406	262.34
1.2	−107.31	66.72	− 7.48	0.325	0.594	0.382	236.19
1.3	−100.59	65.65	− 8.39	0.294	0.641	0.358	220.17
1.4	− 95.00	63.98	− 8.54	0.269	0.649	0.334	212.14

deprived of any theoretical background, but it is also heuristically useless. It is interesting to compare the predicted values of the compressibility modulus $K^{(2)}$ with those quoted in the literature: *a*) CHEID, LIGENSA and GREINER [37] have suggested value $K \simeq 100$ MeV in a research on heavy-ion

[37] W. CHEID, R. LIGENSA and W. GREINER: *Phys. Rev. Lett.*, **21**, 1479 (1968).

reactions; *b*) BOHR and MOTTELSON [38] and BETHE [39] have estimated $K \simeq 117$ MeV and, respectively, $K \simeq 135$ MeV; *c*) the values calculated by BRUECKNER and GAMMEL [40], using different two-body potentials, vary from $K \simeq 167$ MeV to $K \simeq 187$ MeV; *d*) value $K \simeq (210 \pm 30)$ MeV has been reported by BLAIZOT [41]; *e*) the investigation of Mach and head shock waves, occurring in the collisions of light high-energetic nuclei with heavy ones, has allowed the deduction of value $K \simeq 300$ MeV [42]; *f*) according to the Zamick formula [43] K varies from $\simeq 95$ MeV to $\simeq 379$ MeV; the compressibility modulus evaluated by using potentials of the Skyrme type [44] turns out to be $\simeq 370$ MeV. The conclusion to be drawn is obvious: the spread of the values of K is so large that none of them can be considered to be reliable. The criterion for the determination of K, outlined in sect. **9**, is an attempt to overcome such a discouraging situation.

The total energies of infinite nuclear matter in the $N = 1$ and $N = 2$ approximation are

$$(7.31a) \qquad W^{(1)}(\varkappa)/A = (3\varepsilon_F/5)\chi^2 + (1/2)\varkappa_F^3 C_1^{(1)}\chi^3 + (3/5)\varkappa_F^5 C_3^{(1)}\chi^5 ,$$

$$(7.31b) \qquad W^{(2)}(\varkappa)/A = (3\varepsilon_F/5)\chi^2 + (1/2)\varkappa_F^3 C_1^{(2)}\chi^3 + (3/5)\varkappa_F^5 C_3^{(2)}\chi^5 + (36/35)\varkappa_F^6 C_6^{(2)}\chi^7 ,$$

where $\chi = \varkappa/\varkappa_F$: clearly, functions (7.31) satisfy exactly eqs. (1.5), provided the parameters are evaluated according to eqs. (7.7) and, respectively, eqs. (7.21). The $\varkappa$-dependence of $W^{(1)}(\varkappa)$ and $W^{(2)}(\varkappa)$ is strongly affected by the nature (real or conjugate complex) and by the localization of the roots of the fifth- and, respectively, seventh-degree equations $W^{(1)}(\varkappa_v) = 0$ and $W^{(2)}(\varkappa_v) = 0$: in both cases there exists at the origin a root $K_0 = 0$ of multiplicity 2, while another root is localized within the interval $(0, 1)$. In fig. 2*a* we have plotted functions (7.31) assuming as input parameters $r_0 = 1.2$ fm and $b_v = -15.0$ MeV. For $\varkappa > \varkappa_F$ the behaviour of $W^{(2)}(\varkappa)$ is governed by two complex conjugate roots having real parts slightly larger than the considered Fermi

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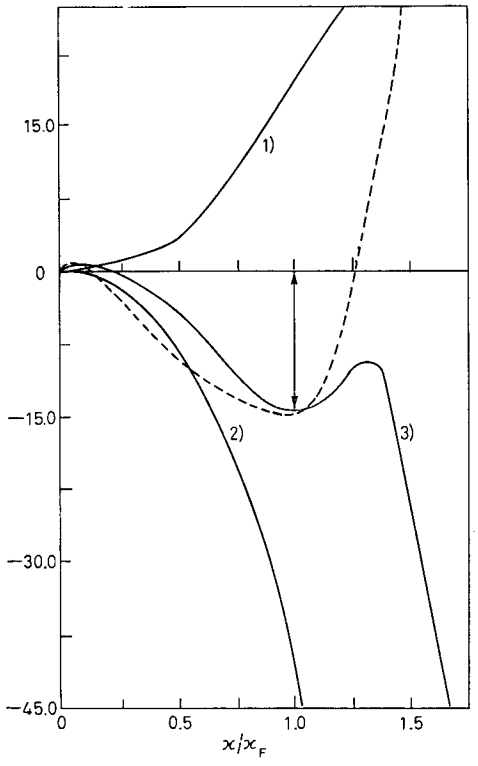

Fig. 2*a*.

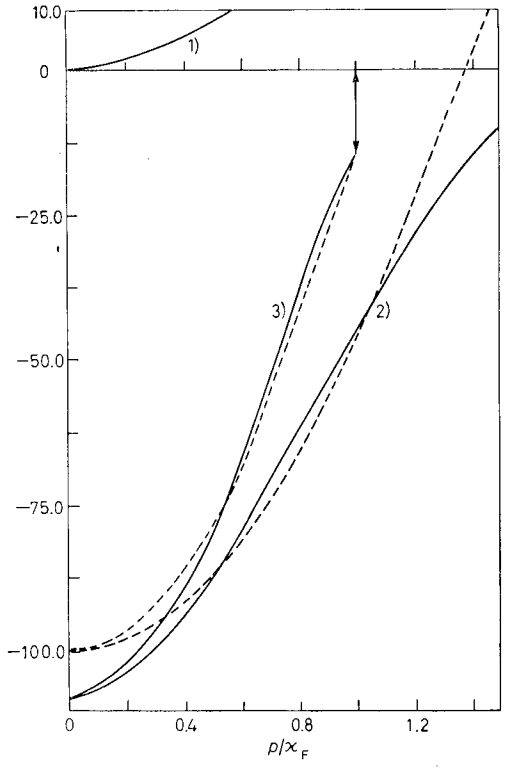

Fig. 2*b*.

Fig. 2*a*. – The saturation of infinitely extended nuclear matter described in $N = 1$ and $N = 2$ approximations, by assuming as input parameters $r_0 = 1.2$ fm and $b_v = -15.0$ MeV. The ordinate is expressed in MeV. Continuous lines: 1) $T(\varkappa)/A$, 2) $V^{(2)}(\varkappa)/A$ and 3) $W^{(2)}(\varkappa)/A$; dotted line: $W^{(1)}(\varkappa)/A$.

Fig. 2*b*. – Visualization of the Hugenholtz and Van Hove theorem in $N = 1$ and $N = 2$ approximations, assuming as input parameters $r_0 = 1.2$ fm and $b_v = -15.0$ MeV. The ordinate is expressed in MeV. Continuous lines: 1) $t(p)$, 2) $v^{(2)}(p, \varkappa_F)$ and 3) $w^{(2)}(p, \varkappa_F)$; dotted lines: 2) $v^{(1)}(p, \varkappa_F)$ and 3) $w^{(1)}(p, \varkappa_F)$.

momentum ($\varkappa_F = 1.5232/r_0$). In fig. 2*b* we visualize the Hugenholtz and Van Hove theorem; more specifically, we plot the single-particle total energies evaluated in the $N = 1$ and $N = 2$ approximation at the minimum of functions (7.31):

$$(7.32a) \qquad w^{(1)}(p, \varkappa_F) = B_0^{(1)}(\varkappa_F) + \{\varepsilon_F + \varkappa_F^2 B_1^{(1)}(\varkappa_F)\}(p/\varkappa_F)^2 ,$$

$$(7.32b) \qquad w^{(2)}(p, \varkappa_F) = B_0^{(2)}(\varkappa_F) + \{\varepsilon_F + \varkappa_F^2 B_1^{(2)}(\varkappa_F)\}(p/\varkappa_F)^2 + \varkappa_F^4 B_2^{(2)}(\varkappa_F)(p/\varkappa_F)^4 .$$

In fig. 2*b* we have also plotted the p-dependence of $v^{(1)}(p, \varkappa_F)$ and $v^{(2)}(p, \varkappa_F)$ for $p \geqslant \varkappa_F$. The importance of the momentum dependence of $v(p, \varkappa_F)$ outside the Fermi sphere will be discussed in sect. **9**.

7·3. – We shall now determine, as functions of the input parameters r_0 and b_v, the constraints imposed by the saturation prescriptions on the single-particle potential energy (6.21a). Recalling eq. (6.25), one has

$$v(p, \varkappa) = C_3\{\sin(\alpha\varkappa) - a\varkappa\cos(\alpha\varkappa)\}\{\sin(\alpha p)/p\}; \tag{7.33}$$

then, taking into account eqs. (1.4) and (2.3), one finds that the analytic expression of the total energy of infinite nuclear matter, brought about by the single-particle potential energy (7.33), reads

$$W(\varkappa)/A = (3/10M)\varkappa^2 + (3C_3/2\alpha^2\varkappa^3)\{\sin(\alpha\varkappa) - \alpha\varkappa\cos(\alpha\varkappa)\}^2 . \tag{7.34}$$

Function (7.34) must satisfy conditions (1.5) for a certain value $\varkappa_F$ of the limiting momentum $\varkappa$. From eq. (7.33) one finds

$$C_3 \equiv C_3(\varkappa_F) = (b_v - \varepsilon_F)/\alpha^3\varkappa_F^2 j_0(\alpha\varkappa_F) j_1(\alpha\varkappa_F) , \tag{7.35}$$

where the spherical Bessel functions $j_0(x)$ and $j_1(x)$ are defined in eqs. (5.38). From both saturation prescriptions (1.5) one deduces that, for given values of r_0 and b_v, the length α is determined by the equation

$$\mathrm{tg}(\alpha\varkappa_F) = \frac{15(b_v - \varepsilon_F)\alpha\varkappa_F}{15(b_v - \varepsilon_F) - 2(5b_v - 3\varepsilon_F)\alpha^2\varkappa_F^2} . \tag{7.36}$$

The nucleon effective masses calculated at $p = 0$ and at $p = \varkappa_F$ according to the quantum-mechanical definition (3.5) read

$$\frac{M^*(0)}{M} = \frac{6\varepsilon_F j_0(\alpha\varkappa_F)}{6\varepsilon_F j_0(\alpha\varkappa_F) - (b_v - \varepsilon_F)\alpha^2\varkappa_F^2} , \tag{7.37a}$$

$$\frac{M^*(\varkappa_F)}{M} = \frac{2\varepsilon_F j_0(\alpha\varkappa_F)}{2\varepsilon_F j_0(\alpha\varkappa_F) - \alpha\varkappa_F(b_v - \varepsilon_F)\{\alpha\varkappa_F j_0(\alpha\varkappa_F) - 2j_1(\alpha\varkappa_F)\}} ; \tag{7.37b}$$

the analytical expression of the ratio M^{**}/M evaluated according to definition (3.6) turns out to be

$$\frac{M^{**}}{M} = \frac{2\varepsilon_F\alpha\varkappa_F j_0(\alpha\varkappa_F)}{\varepsilon_F\{1 + 2\alpha\varkappa_F j_0(\alpha\varkappa_F)\} - b_v} . \tag{7.38}$$

The compressibility modulus implied by the total energy (7.34) can be readily calculated using eq. (5.23); in the considered case the first-order logarithmic derivative is

$$\psi(\varkappa_F) = -1 + 2\{j_1(2\alpha\varkappa_F)/j_0(\alpha\varkappa_F) j_1(\alpha\varkappa_F)\} . \tag{7.39}$$

It is trivially verified that *the saturation prescriptions* (1.5) *are consistent with the Hugenholtz and Van Hove theorem* (1.15), *because the factorable single-particle potential energy* (7.33) *is an exact solution of the differential equation of nuclear matter* (5.5).

The solutions of eq. (7.36) are to be searched for in the interval

$$\eta < \alpha\varkappa_F < 2\pi/3 , \tag{7.40a}$$

$$\eta^2 = 15(b_v - \varepsilon_F)/2(5b_v - 3\varepsilon_F) . \tag{7.40b}$$

The numerical values of $\alpha\varkappa_F$ as functions of r_0 and b_v are given in table V, together with the values of the parameters defined by eqs. (6.19), (6.20) and (7.35), which characterize the factorable single-particle potential energy (6.21) inside and outside the Fermi sphere: it is needless to say that all tabulated values fulfil exactly the saturation requirements of nuclear matter. The role of table V is auxiliary to the construction of table VI, which refers only to the single-particle potential energy (6.21*a*), brought about by the saturation prescriptions, conceived as inextricably connected with the separation energy theorem: we stress again that such a connection is made mathematically possible by virtue of eq. (5.5). The comparison of the results listed in table VI and in table IV discloses the origin of several misleading conclusions drawn from nuclear-matter calculations and uncritically reported in the literature. We shall limit ourselves to pointing out that *a*) the single-particle potential energy at zero momentum,

$$v(0, \varkappa_F) = (b_v - \varepsilon_F)/j_0(\alpha\varkappa_F) , \tag{7.41}$$

is a function of the Fermi momentum which also depends slightly on b_v (in contrast with the results obtained from the direct calculation of eq. (7.26), listed in table III; it has been ascertained that a self-consistent calculation of eq. (7.30) does not improve the $N = 2$ approximation discussed in subsect. 7·2); *b*) the pathological behaviour, exhibited in table IV, of the nucleon effective mass at the Fermi surface for $b_v = -14.0$ MeV and $b_v = -15.0$ MeV disappears; as already noted, parameter M^{**} is completely unrelated to the quantum-mechanical concept of effective mass: table VI gives an additional evidence of its heuristic uselessness in nuclear physics, since it turns out to be practically independent of both the input parameters r_0 and b_v; *c*) the most significant differences between the predictions contained in table IV and in table VI concern the dependence on r_0 and b_v of the compressibility modulus K.

In conclusion, the factorable single-particle potential energy (6.21) at the Fermi surface reads

$$v(p, \varkappa_F) = (b_v - \varepsilon_F)\{j_0(\alpha\varkappa_F\xi)/j_0(\alpha\varkappa_F)\} , \qquad p \leqslant \varkappa_F , \tag{7.42a}$$

$$v(p, \varkappa_F) = (b_v - \varepsilon_F)\{\exp[-\beta\varkappa_F(\xi - 1)]/\xi\} , \qquad p \geqslant \varkappa_F , \tag{7.42b}$$

TABLE V. – *Numerical values of the parameters of the factorable single-particle potential energy* (6.21) *as functions of* r_0 (*in* fm *units*) *and* b_v. Quantity $C_3(\varkappa_F)/\varkappa_F$ is expressed in MeV. For the reader's convenience we have also tabulated the spherical Bessel functions $j_0(\alpha\varkappa_F)$ and $j_1(\alpha\varkappa_F)$.

$b_v = -14.0$ MeV

r_0	$\alpha\varkappa_F$	$\beta\varkappa_F$	$C_2(\varkappa_F)$	$C_3(\varkappa_F)/\varkappa_F$	$j_0(\alpha\varkappa_F)$	$j_1(\alpha\varkappa_F)$
0.9	1.8830	0.6077	1.7475	−48.78	0.5054	0.4315
1.0	1.9293	0.7230	1.9295	−39.82	0.4854	0.4335
1.1	1.9698	0.8304	2.1139	−33.57	0.4678	0.4347
1.2	2.0000	0.9155	2.2714	−29.09	0.4546	0.4354
1.3	2.0489	1.0617	2.5671	−25.43	0.4333	0.4360
1.4	2.0765	1.1499	2.7625	−22.83	0.4213	0.4362

$b_v = -15.0$ MeV

r_0	$\alpha\varkappa_F$	$\beta\varkappa_F$	$C_2(\varkappa_F)$	$C_3(\varkappa_F)/\varkappa_F$	$j_0(\alpha\varkappa_F)$	$j_1(\alpha\varkappa_F)$
0.9	1.8950	0.6366	1.7915	−48.97	0.5002	0.4320
1.0	1.9418	0.7552	1.9831	−40.14	0.4799	0.4338
1.1	1.9846	0.8714	2.1885	−33.90	0.4613	0.4350
1.2	2.0252	0.9891	2.4158	−29.32	0.4436	0.4357
1.3	2.0621	1.1031	2.6570	−25.90	0.4275	0.4360
1.4	2.0949	1.2110	2.9060	−23.26	0.4133	0.4362

$b_v = -16.0$ MeV

r_0	$\alpha\varkappa_F$	$\beta\varkappa_F$	$C_2(\varkappa_F)$	$C_3(\varkappa_F)/\varkappa_F$	$j_0(\alpha\varkappa_F)$	$j_1(\alpha\varkappa_F)$
0.9	1.9080	0.6688	1.8420	−49.13	0.4946	0.4326
1.0	1.9573	0.7963	2.0537	−40.35	0.4732	0.4343
1.1	2.0000	0.9151	2.2705	−34.21	0.4546	0.4353
1.2	2.0403	1.0352	2.5110	−29.71	0.4370	0.4359
1.3	2.0766	1.1501	2.7631	−26.34	0.4212	0.4361
1.4	2.1102	1.2633	3.0348	−23.74	0.4066	0.4361

where $\xi = p/\varkappa_F$: it should be evident that from eq. (7.42*a*) one can construct a variety of descriptions of nuclear matter which fulfil *exactly* the saturation conditions (1.5) and the Hugenholtz and Van Hove theorem (1.15), provided the parameters $\alpha\varkappa_F$ and $\beta\varkappa_F$ depend on r_0 and b_v according to the numerical trend given in table V. The saturation of nuclear matter for $r_0 = 1.2$ fm and $b_v = -15.0$ MeV is shown in fig. 3*a*; the Hugenholtz and Van Hove theorem is visualized for the same input parameters in fig. 3*b*; for the reader's

TABLE VI. – *Quantities deduced, consistently with the saturation prescriptions* (1.5), *from the factorable single-particle potential energy* (7.33). The zero-momentum limit $v(0, \varkappa_F)$ and the compressibility modulus K, evaluated according to eq. (5.23), are expressed in MeV. The mass ratios have been evaluated according to eqs. (7.37) and (7.38). Length r_0 is given in fm units.

$b_v = -14.0$ MeV					
r_0	$v(0, \varkappa_F)$	$M^*(0)/M$	$M^*(\varkappa_F)/M$	M^{**}/M	K
0.9	−140.52	0.407	0.829	0.604	214.80
1.0	−123.98	0.375	0.847	0.590	198.16
1.1	−111.52	0.346	0.870	0.574	183.73
1.2	−101.34	0.322	0.892	0.559	168.49
1.3	− 95.38	0.290	0.946	0.540	167.07
1.4	− 89.15	0.269	0.990	0.523	157.53

$b_v = -15.0$ MeV					
r_0	$v(0, \varkappa_F)$	$M^*(0)/M$	$M^*(\varkappa_F)/M$	M^{**}/M	K
0.9	−143.98	0.398	0.833	0.600	221.62
1.0	−127.48	0.366	0.853	0.584	204.94
1.1	−115.26	0.335	0.880	0.568	191.71
1.2	−106.11	0.306	0.917	0.550	182.25
1.3	− 99.02	0.280	0.965	0.532	174.60
1.4	− 93.15	0.257	1.028	0.514	167.59

$b_v = -16.0$ MeV					
r_0	$v(0, \varkappa_F)$	$M^*(0)/M$	$M^*(\varkappa_F)/M$	M^{**}/M	K
0.9	−147.63	0.389	0.837	0.596	229.87
1.0	−131.40	0.353	0.862	0.579	214.55
1.1	−119.16	0.324	0.892	0.562	200.84
1.2	−110.00	0.296	0.935	0.543	190.82
1.3	−102.87	0.270	0.990	0.524	182.58
1.4	− 97.29	0.246	1.066	0.505	176.51

convenience we specify the analytic form of the function plotted in fig. 3*a*, *b*

$$W(\varkappa)/A = (3\varepsilon_F/5)\,\chi^2 + (3/2)(\chi/\alpha\varkappa_F)(b_v - \varepsilon_F)\,\{j_1^2(\alpha\varkappa_F\,\chi)/j_0(\alpha\varkappa_F)\,j_1(\alpha\varkappa_F)\}\,, \tag{7.43}$$

$$w(p, \varkappa_F) = \varepsilon_F\xi^2 + (b_v - \varepsilon_F)\,\{j_0(\alpha\varkappa_F\xi)/j_0(\alpha\varkappa_F)\}\,, \tag{7.44}$$

where $\chi = \varkappa/\varkappa_F$ and $\xi = p/\varkappa_F$. In fig. 3*b* we have also plotted the momentum

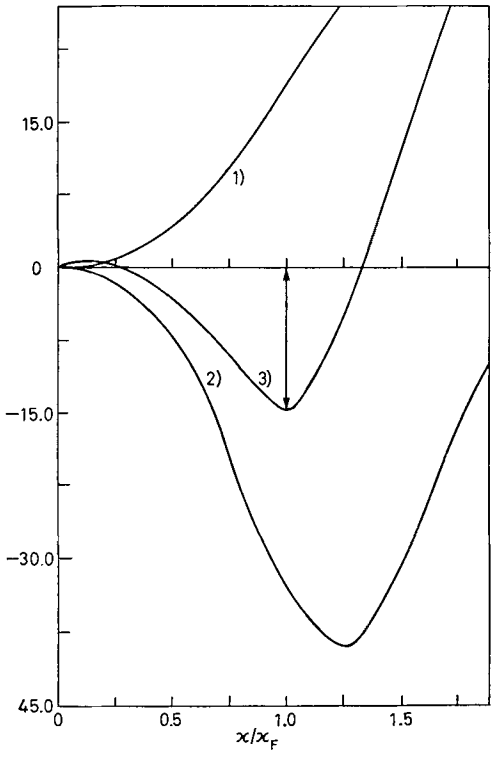

Fig. 3*a*.

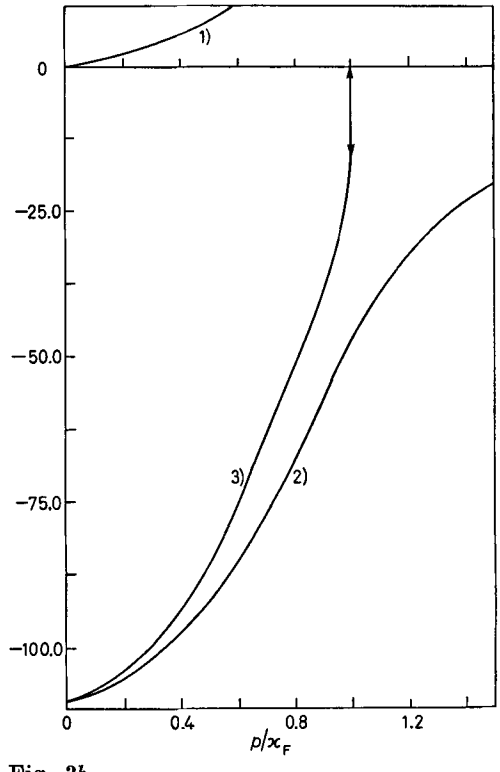

Fig. 3*b*.

Fig. 3*a*. – Description of the saturation of infinitely extended nuclear matter derived from the factorable single-particle potential energy (7.42*a*), by assuming as input parameters $r_0 = 1.2$ fm and $b_v = -15.0$ MeV. The ordinate is expressed in MeV; 1) $T(\varkappa)/A$, 2) $V(\varkappa)$ and 3) $W(\varkappa)/A$, given by eq. (7.43).

Fig. 3*b*. – Visualization of the Hugenholtz and Van Hove theorem for the factorable single-particle energy (7.42), by assuming as input parameters $r_0 = 1.2$ fm and $b_v = -15.0$ MeV. The ordinate is expressed in MeV; 1) $t(p)$, 2) eqs. (7.42) and 3) eq. (7.43).

dependence of $v(p, \varkappa_F)$ predicted by eqs. (7.42) for $r_0 = 1.2$ and $b_v = -15.0$ MeV, inside and outside the Fermi sphere.

7·4. – Our choice of describing the saturation properties of infinite nuclear matter in momentum space deserves further comments. According to current opinions, the strong short-range repulsion in the nucleon-nucleon potential makes the use of co-ordinate space preferable; in fact, short-range repulsion is usually approximated by an infinitely repulsive core and one of the most important properties of the correlated wave functions is that they vanish inside the hard core: this condition takes a simple form in co-ordinate space, whereas it turns out to be very complicated in momentum space. We have

already pointed out at the end of subsect. 5·2 that the single-particle potential energy, calculated in first-order perturbation theory, fulfils the hyperbolic differential equation (5.5), provided the nucleon-nucleon potential is a continuous function of the internuclear distance: the use of the Heaviside function in the description of the repulsive hard core would be catastrophic. We have also mentioned that the treatments used to overcome hard-core difficulties together with the procedures based on the concept of healing distance violate the Hungenholtz and Van Hove theorem and, consequently, the differential equation of nuclear matter. The mathematical troubles brought about by the hard core have always systematically biassed the theoretical outline of the nuclear-matter problem: this is indeed disconcerting, because the hard core is nothing but a phenomenological invention quantum-mechanically inconsistent, which compensates our ignorance of the spatial extension of the meson source: in this connection it is worthwhile to recall that a proper choice of the meson density distribution leads to nucleon-nucleon potentials which, although continuous functions of the internucleon distance, simulate the repulsive short-range properties otherwise grossly attributed to the hard core ([45]). The fundamental implication of our mathematical scheme, based on eq. (5.5), is that two-nucleon potentials inside the nuclear matter do not possess a hard core. The suspicion that hard-core nucleon-nucleon potentials, although fitting the elastic-scattering data, cannot give the correct volume energy and equilibrium density of nuclear matter was originally expressed by BETHE ([46]).

8. – The energy dependence of the real part of the optical potential.

The genesis of the real part of the optical potential has been outlined in sect. **1**. Potential $\mathcal{U}_R(E, \varkappa_F)$ deduced in such a way contains information on nuclear matter and inherits its characteristic features: this will be the *leitmotiv* of our further considerations.

([45]) See ref. ([25]).

([46]) Private communication from H. A. BETHE to B. D. DAY (see B. D. DAY: *Rev. Mod. Phys.*, **4**, 719 (1967), p. 743). The opinion expressed by BETHE can be readily understood in the framework of the effective-mass approximation. Point nucleons obeying the Pauli exclusion principle have the kinetic energy $T(\varkappa)$ due to the filling of free particle energy levels. The requirement that the wave function should vanish whenever any two nucleons approach within a separation distance equal to the hard core radius r_c increases the wave function curvature and, hence, the kinetic energy above the Fermi value at $\varkappa = \varkappa_F$. The Lenz correction (W. LENZ: *Z. Phys.*, **56**, 778 (1929), see also S. D. DRELL and K. HUANG: *Phys. Rev.*, **91**, 1527 (1953)) accounts for this effect: the kinetic energy turns out to be $T'(\varkappa) = T(\varkappa)(1 + \alpha\varkappa)$, where α is a constant proportional to r_c. It follows that the corrected kinetic energy depends also on a term proportional to $\varkappa^3$. This positive term modifies quantitatively the χ^3-dependence of the total energy expressed by eq. (7.31*a*), thus preventing one from obtaining simultaneously the correct volume energy and equilibrium density.

According to eq. (1.19) the real optical potential in the $N = 1$ approximation is

$$\mathfrak{V}_{\mathrm{R}}^{(1)}(E, \varkappa_{\mathrm{F}}) = v^{(1)}\{p(E), \varkappa_{\mathrm{F}}\}\,, \tag{8.1}$$

where $v^{(1)}(p, \varkappa_{\mathrm{F}})$ is given by eqs. (7.4); it is found that

$$\mathfrak{V}_{\mathrm{R}}^{(1)}(E, \varkappa_{\mathrm{F}}) = \mathfrak{V}_{0}^{(1)}(\varkappa_{\mathrm{F}}) + \mathcal{A}_{1}^{(1)}(\varkappa_{\mathrm{F}})E\,, \tag{8.2a}$$

$$\begin{cases} \mathfrak{V}_{0}^{(1)}(\varkappa_{\mathrm{F}}) = \dfrac{\mathrm{B}_{0}^{(1)}(\varkappa_{\mathrm{F}})}{1 + 2M\mathrm{B}_{1}^{(1)}(\varkappa_{\mathrm{F}})} = \dfrac{M^*}{M}\,\mathrm{B}_{0}^{(1)}(\varkappa_{\mathrm{F}})\,, \\ \mathcal{A}_{1}^{(1)}(\varkappa_{\mathrm{F}}) = \dfrac{\varkappa_{\mathrm{F}}^2\,\mathrm{B}_{1}^{(1)}(\varkappa_{\mathrm{F}})}{\varepsilon_{\mathrm{F}}\{1 + 2M\mathrm{B}_{1}^{(1)}(\varkappa_{\mathrm{F}})\}} = \dfrac{M^*}{M}\left(\dfrac{M}{M^*} - 1\right). \end{cases} \tag{8.2b}$$

Taking into account eqs. (7.8), the explicit expressions of (8.2*b*), as functions of the input parameters r_0 and b_{v}, turn out to be

$$\mathfrak{V}_{0}^{(1)}(\varkappa_{\mathrm{F}}) = \frac{\varepsilon_{\mathrm{F}}(7b_{\mathrm{v}} - 3\varepsilon_{\mathrm{F}})}{3\varepsilon_{\mathrm{F}} - 5b_{\mathrm{v}}}\,, \qquad \mathcal{A}_{1}^{(1)}(\varkappa_{\mathrm{F}}) = \frac{\varepsilon_{\mathrm{F}} - 5b_{\mathrm{v}}}{3\varepsilon_{\mathrm{F}} - 5b_{\mathrm{v}}}\,. \tag{8.3}$$

A real optical potential formally similar to (8.2*a*) has been semi-empirically deduced by BRUECKNER, EDEN and FRANCIS ([47]) in the framework of their theory on neutron reactions with nuclei; the potential is

$$V(E) = -\,41\ \mathrm{MeV} + 0.4E\,. \tag{8.4}$$

The mathematical scheme developed in the preceding subsections enables us to disclose the theoretical contents concealed behind potential (8.4). To this end we identify V with $\mathfrak{V}_{\mathrm{R}}^{(1)}(E)$ and solve the two equations (8.3) in the unknown quantities r_0 and b_{v}, *i.e.*

$$\mathfrak{V}_{\mathrm{R}}^{(1)}(\varkappa_{\mathrm{F}}) = -\,41.0\ \mathrm{MeV}\,, \qquad \mathcal{A}_{1}^{(1)}(\varkappa_{\mathrm{F}}) = 0.4\,. \tag{8.5}$$

It is found that

$$b_{\mathrm{v}} = -\,2.63\ \mathrm{MeV}\,, \qquad \varepsilon_{\mathrm{F}} = -\,39.42\ \mathrm{MeV}\,, \qquad r_0 = 1.082\cdot 10^{-13}\ \mathrm{cm}\,; \tag{8.6}$$

it is seen that the volume energy turns out to be ludicrously small. The very theoretical scenery disguised by the apparently innocent potential (8.4) is even worse than may appear from the predicted value of b_{v}. In fact, it can be readily proved that potential (8.4) is the optical transformation of a single-particle potential energy which is not compatible either with the satu-

([47]) K. A. BRUECKNER, R. J. EDEN and N. C. FRANCIS: *Phys. Rev.*, **100**, 891 (1955).

ration prescriptions of nuclear matter or with the Hugenholtz and Van Hove theorem: thus its value $b_v = -2.63$ MeV is not only too small, but also physically meaningless because it does not correspond to the minimum of the total energy at the value r_0, given in eq. (8.6)!

The $N = 2$ approximation of $\mathfrak{V}_R(E)$ is

$$\mathfrak{V}_R^{(2)}(E, \varkappa_F) = v^{(2)}\{p(E), \varkappa_F\}, \tag{8.7a}$$

where $v^{(2)}(p, \varkappa_F)$ is given by eqs. (7.13); function (8.7a) is the solution of the quadratic equation

$$\mathfrak{B}_1(\varkappa_F)[\mathfrak{V}_R^{(2)}(E, \varkappa_F)]^2 - \mathfrak{B}_2(E, \varkappa_F)\,\mathfrak{V}_R^{(2)}(E, \varkappa_F) + \mathfrak{B}_3(E, \varkappa_F) = 0, \tag{8.7b}$$

$$\begin{cases} \mathfrak{B}_1(\varkappa_F) = (1/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F), \\ \mathfrak{B}_2(E, \varkappa_F) = 1 + (1/\varepsilon_F)\varkappa_F^2 B_1^{(2)}(\varkappa_F) + (2/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F) E, \\ \mathfrak{B}_3(E, \varkappa_F) = B_0^{(2)}(\varkappa_F) + (1/\varepsilon_F)\varkappa_F^2 \mathfrak{B}_1^{(2)}(\varkappa_F) E + (1/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F) E^2. \end{cases} \tag{8.7c}$$

The physical solutions of eq. (8.7) have to be selected according to the obvious criterion expressed by the inequalities

$$\mathfrak{V}_R^{(2)}(E, \varkappa_F) < 0, \quad |\mathfrak{V}_R^{(2)}(E, \varkappa_F)| < |\mathfrak{V}_R^{(2)}(0, \varkappa_F)|. \tag{8.8}$$

The zero limit of $\mathfrak{V}_R^{(2)}(E)$ is obtained from the equation

$$(1/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F)[\mathfrak{V}_0^{(2)}(\varkappa_F)]^2 - \{1 + (1/\varepsilon_F)\varkappa_F^2 B_1^{(2)}(\varkappa_F)\}\,\mathfrak{V}_0^{(2)}(\varkappa_F) + B_0^{(2)}(\varkappa_F) = 0; \tag{8.9}$$

clearly, $\mathfrak{V}_0^{(2)}(\varkappa_F) \neq \mathfrak{V}_0^{(1)}(\varkappa_F)$ if $B_2^{(2)}(\varkappa_F) \neq 0$. Potential (8.7) has never been considered in the literature.

The optical transform of the single-particle potential energy (6.21b), obtained from eq. (5.5) by variable separation, is

$$\mathfrak{V}_R(E, \varkappa_F) = v\{p(E), \varkappa_F\}, \tag{8.10a}$$

$$\mathfrak{V}_R(E, \varkappa_F) = \mathfrak{V}_0(\varkappa_F)\left\{\frac{\varepsilon_F}{E - \mathfrak{V}_R(E, \varkappa_F)}\right\}^{\frac{1}{2}} \exp\left[-\beta\varkappa_F\left\{\frac{E - \mathfrak{V}_R(E, \varkappa_F)}{\varepsilon_F}\right\}^{\frac{1}{2}}\right], \tag{8.10b}$$

$$\mathfrak{V}_0(\varkappa_F) = (b_v - \varepsilon_F)\exp[\beta\varkappa_F]. \tag{8.10c}$$

The zero energy limit of potential (8.10) is given by the equation

$$a^2 \exp[-\beta\varkappa_F y] = y^3, \tag{8.11}$$

where

$$a^2 = -\mathfrak{V}_0(\varkappa_F)/\varepsilon_F, \quad y^2 = -\mathfrak{V}_R(0, \varkappa_F)/\varepsilon_F. \tag{8.12}$$

The numerical values of quantities $\mathfrak{V}_0(\varkappa_F)$ and $\mathfrak{V}_R(0, \varkappa_F)$, consistent with nuclear saturation, are given in table VII as functions of r_0 and b_v; use has been made of the values of parameter $\beta\varkappa_F$ listed in table V.

TABLE VII. – *Numerical values of quantities* $\mathfrak{V}_0(\varkappa_F)$ *and* $\mathfrak{V}_R(0, \varkappa_F)$ *(in* MeV*) as functions of* r_0 *(in* fm *units) and* b_v.

r_0	$b_v = -14.0$ MeV		$b_v = -15.0$ MeV		$b_v = -16.0$ MeV	
	$\mathfrak{V}_0(\varkappa_F)$	$\mathfrak{V}_R(0, \varkappa_F)$	$\mathfrak{V}_0(\varkappa_F)$	$\mathfrak{V}_R(0, \varkappa_F)$	$\mathfrak{V}_0(\varkappa_F)$	$\mathfrak{V}_R(0, \varkappa_F)$
0.9	−130.41	−64.43	−136.12	−64.67	−142.53	−65.16
1.0	−124.01	−53.17	−130.19	−53.43	−137.87	−53.91
1.1	−119.69	−44.77	−127.10	−45.23	−135.26	−45.64
1.2	−115.08	−38.45	−126.56	−38.68	−135.34	−38.98
1.3	−116.50	−32.98	−127.56	−33.76	−136.86	−33.98
1.4	−118.61	−29.34	−129.43	−29.55	−139.93	−29.73

We shall show in sect. **9** that the determination of the parameters b_v, K and ε_F (*i.e.* the length r_0) can be carried out, consistently with the saturation prescriptions and the Hugenholtz and Van Hove theorem, by taking advantage of the information extracted from the nucleon-nucleon elastic-scattering data.

9. – A criterion for the determination of the compressibility modulus of infinite nuclear matter.

9'1. – We re-formulate the $N = 2$ approximation of the single-particle potential energy by replacing the third of eqs. (7.20) with the first of eqs. (7.24). The new system of equations is

$$\text{(9.1)} \qquad \begin{cases} 72\varkappa_F^7 C_5^{(2)} + 42\varkappa_F^5 C_3^{(2)} + 35\varkappa_F^3 C_1^{(2)} = 70b_v - 42\varepsilon_F\,, \\ 72\varkappa_F^7 C_5^{(2)} + 30\varkappa_F^5 C_3^{(2)} + 15\varkappa_F^3 C_1^{(2)} = -12\varepsilon_F\,, \\ 216\varkappa_F^7 C_5^{(2)} + 60\varkappa_F^5 C_3^{(2)} + 15\varkappa_F^3 C_1^{(2)} = 6K - 6\varepsilon_F\,, \end{cases}$$

where $K^{(2)} \equiv K$; the solutions are

$$\text{(9.2)} \qquad \begin{cases} \varkappa_F^7 C_5^{(2)} = (175/96)\, b_v - (7/32)\,\varepsilon_F + (35/288)\,K\,, \\ \varkappa_F^5 C_5^{(3)} = (5/4)\,\varepsilon_F - (35/4)\,b_v - (5/12)\,K\,, \\ \varkappa_F^3 C_5^{(2)} = (35/4)\,b_v - (9/4)\,\varepsilon_F + (1/4)\,K\,. \end{cases}$$

Coefficients (7.18) of $v^{(2)}(p, \varkappa_F)$ become

$$(9.3)\qquad \begin{cases} B_0^{(2)}(\varkappa_F) &= (137/32)\,b_v - (51/32)\,\varepsilon_F + (5/96)\,K\,, \\ \varkappa_F^2 B_1^{(2)}(\varkappa_F) &= (117/144)\,\varepsilon_F - (245/48)\,b_v - (25/144)\,K\,, \\ \varkappa_F^4 B_2^{(2)}(\varkappa_F) &= (175/96)\,b_v - (7/32)\,\varepsilon_F + (35/288)\,K\,. \end{cases}$$

It is readily verified that potential (7.13), characterized by the coefficients (9.3), fulfils the Hugenholtz and Van Hove theorem (1.15).

Let us express the real optical potential (8.7) in power series of the incident energy E; the first three terms of the series are

$$(9.4)\qquad \mathfrak{V}_R^{(2)}(E, \varkappa_F) = \mathfrak{V}_0^{(2)}(\varkappa_F) + \mathcal{A}_1^{(2)}(\varkappa_F)E + \mathcal{A}_2^{(2)}(\varkappa_F)E^2\,,$$

where $\mathfrak{V}_0^{(2)}(\varkappa_F) \equiv \mathfrak{V}_R^{(2)}(0, \varkappa_F)$ is a solution of eq. (8.9). The coefficients of the McLaurin expansion are directly obtained from eq. (9.4); one has

$$(9.5)\qquad \mathcal{A}_1^{(2)}(\varkappa_F) = \{\mathrm{d}\mathfrak{V}_R^{(2)}(E, \varkappa_F)/\mathrm{d}E\}_{E=0}\,, \qquad \mathcal{A}_2^{(2)}(\varkappa_F) = \tfrac{1}{2}\{\mathrm{d}^2\mathfrak{V}_R^{(2)}(E, \varkappa_F)/\mathrm{d}E^2\}_{E=0}\,.$$

Using eq. (8.7*b*), we define the two-variable function

$$(9.6)\qquad F(E, \mathfrak{V}_R^{(2)}) = \mathcal{B}_1(\varkappa_F)[\mathfrak{V}_R^{(2)}]^2 - \mathcal{B}_2(E, \varkappa_F)\,\mathfrak{V}_R^{(2)} + \mathcal{B}_3(E, \varkappa_F) = 0\,;$$

from the theory of implicit functions one has

$$(9.7a)\qquad \{\mathrm{d}\mathfrak{V}_R^{(2)}(E, \varkappa_F)/\mathrm{d}E\}_{E=0} = -[\{\partial F/\partial E\}/\{\partial F/\partial \mathfrak{V}_R^{(2)}\}]_{E=0}\,,$$

$$(9.7b)\qquad \left\{\frac{\mathrm{d}^2\mathfrak{V}_R^{(2)}(E, \varkappa_F)}{\mathrm{d}E^2}\right\}_{E=0} =$$

$$= -\left[\frac{\dfrac{\partial^2 F}{\partial E^2}\left(\dfrac{\partial F}{\partial \mathfrak{V}_R^{(2)}}\right)^2 - 2\,\dfrac{\partial^2 F}{\partial E\,\partial \mathfrak{V}_R^{(2)}}\,\dfrac{\partial F}{\partial E}\,\dfrac{\partial F}{\partial \mathfrak{V}_R^{(2)}} + \dfrac{\partial^2 F}{\partial \mathfrak{V}_R^{(2)2}}\left(\dfrac{\partial F}{\partial E}\right)^2}{\left\{\dfrac{\partial F}{\partial \mathfrak{V}_R^{(2)}}\right\}^3}\right]_{E=0}.$$

Taking into account that

$$(9.8a)\qquad \{\partial F/\partial E\}_{E=0} = -1 - \{\partial F/\partial \mathfrak{V}_R^{(2)}\}_{E=0} = (1/\varepsilon_F)\varkappa_F^2 B_1^{(2)}(\varkappa_F) - (2/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F)\,,$$

$$(9.8b)\qquad \{\partial^2 F/\partial E^2\}_{E=0} = \{\partial^2 F/\partial \mathfrak{V}_2^{(2)2}\}_{E=0} = -\{\partial^2 F/\partial E\,\partial \mathfrak{V}_R^{(2)}\}_{E=0} = (2/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F)\,,$$

it is found that

$$(9.9a)\qquad \mathcal{A}_1^{(2)}(\varkappa_F) = \frac{(1/\varepsilon_F)\varkappa_F^2 B_1^{(2)}(\varkappa_F) - (2/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F)\,\mathfrak{V}_0^{(2)}(\varkappa_F)}{1 + (1/\varepsilon_F)\varkappa_F^2 B_1^{(2)}(\varkappa_F) - (2/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F)\,\mathfrak{V}_0^{(2)}(\varkappa_F)}\,,$$

$$(9.9b)\qquad \mathcal{A}_2^{(2)}(\varkappa_F) = \frac{(1/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F)}{\{1 + (1/\varepsilon_F)\varkappa_F^2 B_1^{(2)}(\varkappa_F) - (2/\varepsilon_F^2)\varkappa_F^4 B_2^{(2)}(\varkappa_F)\,\mathfrak{V}_0^{(2)}(\varkappa_F)\}^3}\,,$$

TABLE VIII. – *Numerical values of the parameters of the real optical potential* (9.4) *as functions of* r_0 (*in* fm *units*) *and* b_v. Quantities ε_F, K and $\mathfrak{V}_0^{(2)}(\varkappa_F)$ are expressed in MeV and parameter $\mathcal{A}_2^{(2)}(\varkappa_F)$ in $(\text{MeV})^{-1}$ (see table IV).

$b_v = -14.0$ MeV					
r_0	ε_F	$K^{(2)}$	$\mathfrak{V}_0^{(2)}(\varkappa_F)$	$\mathcal{A}_1^{(2)}(\varkappa_F)$	$\mathcal{A}_2^{(2)}(\varkappa_F)$
0.9	57.02	182.84	−58.36	0.4852	−0.000 661
1.0	46.18	140.10	−53.27	0.4748	−0.001 263
1.1	38.17	97.94	−45.26	0.4664	−0.002 290
1.2	32.07	71.79	−39.01	0.4582	−0.003 681
1.3	27.33	55.77	−34.21	0.4470	−0.005 596
1.4	23.56	47.73	−30.28	0.4383	−0.007 942

$b_v = -15.0$ MeV					
r_0	ε_F	$K^{(2)}$	$\mathfrak{V}_0^{(2)}(\varkappa_F)$	$\mathcal{A}_1^{(2)}(\varkappa_F)$	$\mathcal{A}_2^{(2)}(\varkappa_F)$
0.9	57.02	265.04	−64.28	0.5116	−0.000 272
1.0	46.18	222.30	−53.17	0.5271	−0.000 517
1.1	38.17	180.14	−44.98	0.5342	−0.000 957
1.2	32.07	153.99	−38.61	0.5438	−0.001 444
1.3	27.33	137.97	−33.58	0.5554	−0.001 947
1.4	23.56	129.93	−29.55	0.5701	−0.002 390

$b_v = -16.0$ MeV					
r_0	ε_F	$K^{(2)}$	$\mathfrak{V}_0^{(2)}(\varkappa_F)$	$\mathcal{A}_1^{(2)}(\varkappa_F)$	$\mathcal{A}_2^{(2)}(\varkappa_F)$
0.9	57.02	347.24	−64.26	0.5474	0.000 096
1.0	46.18	304.50	−53.04	0.5696	−0.000 084
1.1	38.17	262.34	−44.73	0.5856	−0.000 275
1.2	32.07	236.19	−38.30	0.6037	−0.000 452
1.3	27.33	220.17	−33.19	0.6236	−0.000 599
1.4	23.56	212.14	−29.07	0.6455	−0.000 685

where, as already pointed out, $\mathfrak{V}_0^{(2)}(\varkappa_F)$ is given by eq. (8.9). Quantity $\mathfrak{V}_0^{(2)}(\varkappa_F)$ and parameters (9.9) are calculated in table VIII as functions of r_0 by using the coefficients of the single-particle potential energy (7.22), listed in table IV.

Let us suppose that the elastic scattering of neutrons from nuclei are analysed on the basis of the real optical potential having the empirical form

$$V_0(E) = a_0 + b_0 E + c_0 E^2 , \tag{9.10}$$

where parameters (a_0, b_0, c_0) are determined by fitting the data. Should such a circumstance occur, then one could compare potential (9.4) with (9.10) by putting

$$\mathfrak{V}_0^{(2)}(\varkappa_F) = a_0\,, \qquad \mathcal{A}_1^{(2)}(\varkappa_F) = b_0\,, \qquad \mathcal{A}_2^{(2)}(\varkappa_F) = c_0; \tag{9.11}$$

clearly, *the three quantities* (b_v, K, ε_F), *deduced from the system of equations* (9.11), *are consistent with the saturation prescriptions and with the Hugenholtz and Van Hove theorem.* From the first and second of eqs. (9.11), taking into account eqs. (8.9) and (9.9*a*), one obtains

$$\begin{cases} B_0^{(2)}(\varkappa_F) - (a_0/\varepsilon_F)\varkappa_F^2 B_1^{(2)}(\varkappa_F) + (a_0/\varepsilon_F)^2\varkappa_F^4 B_2^{(2)}(\varkappa_F) = a_0\,, \\ \varkappa_F^2 B_1^{(2)}(\varkappa_F) + (2a_0/\varepsilon_F)\varkappa_F^4 B_2^{(2)}(\varkappa_F) = \gamma\varepsilon_F\,, \end{cases} \tag{9.12}$$

where $\gamma = b_0/(1-b_0)$. Substituting into system (9.12) the expressions given by eqs. (9.3) for the coefficient, the system of linear equations in the unknown parameters b_v and K is found to be

$$a_{11}b_v + a_{12}K = \mathfrak{A}_1\,, \qquad a_{21}b_v + a_{22}K = \mathfrak{A}_2\,, \tag{9.13a}$$

$$\begin{cases} a_{11} = 3(411 + 490x + 175x^2)\,, & a_{21} = 105(7 + 5x)\,, \\ a_{12} = 5(3 + 10x + 7x^2)\,, & a_{22} = 5(5 + 7x), \end{cases} \tag{9.13b}$$

$$\begin{cases} \mathfrak{A}_1 = 3(153 + 174x + 21x^2)\varepsilon_F\,, \\ \mathfrak{A}_2 = (117 + 63x - 144\gamma)\varepsilon_F\,, \end{cases} \tag{9.13c}$$

where $x = a_0/\varepsilon_F$. For given values of parameters (a_0, b_0) the solutions of system (9.13) are expressed as functions of the Fermi energy only, *i.e.*

$$\begin{cases} b_v = \varepsilon_F f_1(a_0/\varepsilon_F)\,, \\ f_1(x) = \dfrac{3(9 + 2\gamma) + 2(31 + 10\gamma)x + 7(5 + 2\gamma)x^2}{55 + 98x + 35x^2}\,, \end{cases} \tag{9.14a}$$

$$\begin{cases} K = -(6\varepsilon_F/5) f_2(a_0/\varepsilon_F)\,, \\ f_2(x) = \dfrac{3(149 + 137\gamma) + 14(62 + 35\gamma)x + 35(11 + 5\gamma)x^2}{55 + 98x + 35x^2}; \end{cases} \tag{9.14b}$$

the Fermi energy is determined from the third of eqs. (9.11):

$$\begin{cases} c_0\varepsilon_F = f_3(a_0/\varepsilon_F)\,, \\ f_3(x) = \dfrac{72\,576\{9 - 75(b_v/\varepsilon_F) - 5(K/\varepsilon_F)\}}{\{3(87 + 21x) - 105(7 + 5x)(b_v/\varepsilon_F) - 5(5 + 7x)(K/\varepsilon_F)\}^3}\,, \end{cases} \tag{9.14c}$$

where ratios b_v/ε_F and K/ε_F are given by eqs. (9.14*a*), (9.14*b*).

The calculations performed by WILMORE and HODGSON ([48]) for nonlocal potentials, appropriate to $E = 1, 5, 10$ and 15 MeV neutrons elastically scattered by nuclei of masses $A = 50, 100, 150$ and 200, have been fitted assuming an energy-dependent potential well of the form (9.10) with

$$(9.15) \qquad a_0 = -47.01 \text{ MeV}, \quad b_0 = 0.267, \quad c_0 = 0.00118 \text{ (MeV)}^{-1}.$$

Using the set of parameters (9.15) in eqs. (9.14), it is found that the upper limit of the variability interval of r_0 is fixed by the quadratic equations

$$(9.16) \qquad f_1(x_1) = 0, \quad f_2(x_2) = 0;$$

in the considered case ($\gamma = 0.2642$), the roots of eqs. (9.16) are

$$(9.17) \qquad x_1' = 1, \quad x_2' = -0.7278, \quad x_2'' = -0.9932, \quad x_2''' = -1.3387.$$

Only roots x_1' and x_2' are significant for our purposes: the volume energy b_v turns out to be a positive quantity for $x > x_1'$, and the compressibility modulus is negative for $x > x_2'$. The numerical trend of the results thus obtained is shown in table IX. It is found that eq. (9.14*c*) is satisfied for $\chi = -0.9537$; then, one deduces that

$$(9.18) \qquad \begin{cases} b_v = -3.10 \text{ MeV}, & K = 61.13 \text{ MeV}, \\ \varepsilon_F = 49.29 \text{ MeV}, & r_0 = 0.968 \cdot 10^{-13} \text{ cm}. \end{cases}$$

TABLE IX. – *Numerical results deduced from the Wilmore and Hodgson optical potential.* Quantities ε_F, b_v and K are expressed in MeV; length r_0 is expressed in fm units.

r_0	ε_F	x	b_v	K	$c_0\varepsilon_F$	$f_3(x)$
0.900	57.02	−0.8244	−19.29	1333.1	0.0673	−0.0926
0.925	53.98	−0.8709	−12.75	436.81	0.0637	−0.0316
0.950	51.18	−0.9185	− 5.79	958.04	0.0604	0.0092
0.975	48.59	−0.9675	− 2.16	35.16	0.0573	0.0748
1.000	46.18	−1.0180	1.13	−23.38	0.0544	0.1083

These horrible results are physically meaningless, but nonetheless mathematically consistent with eqs. (1.5) (the minimum of the total energy of infinite nuclear matter, evaluated equal to − 3.10 MeV, occurs at $r_0 = -0.968$ fm) and with eq. (1.15) (the evaluated average volume energy − 3.10 MeV is equal to the total energy of a single nucleon at the Fermi surface corresponding to

([48]) D. WILMORE and P. E. HODGSON: *Nucl. Phys.*, **55**, 673 (1964).

$r_0 = 0.968$ fm); thus, the theoretical implications of the saturation problem are safe and only the predicted values of b_v and K are nonsensical. This is probably due to the fact that c_0 should be a negative quantity: indeed, the WH potential implies $B_2^{(2)}(\varkappa_F) > 0$, which conflicts with the $N = 2$ approximation (see table VIII). Let us tentatively modify the WH potential by assuming

$$(9.19) \qquad a_0 = -47.01 \text{ MeV}, \quad b_0 = 0.267, \quad c_0' = -0.00118 \text{ (MeV)}^{-1};$$

from eqs. (9.14) it is found that

$$(9.20) \qquad \begin{cases} b_v = -14.43 \text{ MeV}, & K = 782.77 \text{ MeV}, \\ \varepsilon_F = 55.65 \text{ MeV}, & r_0 = 0.911 \cdot 10^{-13} \text{ cm}, \end{cases}$$

a very unsatisfactory result because the value of the compressibility modulus turns out to be too large. Finally, we consider the set of parameters

$$(9.21) \qquad a_0 = -47.01 \text{ MeV}, \quad b_0 = 0.267, \quad c_0'' = 0,$$

which lead to the following result:

$$(9.22) \qquad \begin{cases} b_v = -6.82 \text{ MeV}, & K = 203.59 \text{ MeV}, \\ \varepsilon_F = 51.83 \text{ MeV}, & r_0 = 0.944 \cdot 10^{-13} \text{ cm}. \end{cases}$$

The reliability of the nuclear-matter quantities (b_v, K, ε_F), determined from the phenomenological parameters of the real optical potential expressed by eq. (9.10), critically depends on the reliability of the zero-energy limit of the potential itself. This crucial point should be kept in mind in handling the criterion previously outlined, which is strictly based on the intimate connection existing between the optical model and the model of infinitely extended nuclear matter. The Wilmore and Hodgson analyses are heavily based on the use of electronic computer techniques: this circumstance prevents one from reaching a critical understanding of the results thus obtained. It would be desirable to know if set (9.15) is univocally determined and what is the degree of confidence one can attribute to value $a_0 = -47.01$ MeV. Unfortunately, such information is not available.

9'2. – Let us express the optical potential (8.10) in power series of the incident energy E; the first three terms of the series are

$$(9.23) \qquad \mathcal{U}_R(E, \varkappa_F) = \mathcal{U}_0(\varkappa_F) + \mathcal{A}_1(\varkappa_F) E + \mathcal{A}_2(\varkappa_F) E^2.$$

The values of $\mathcal{A}_0(\varkappa_F) = \mathcal{U}_R(0, \varkappa_F)$ are given in table VII; the coefficients $\mathcal{A}_1(\varkappa_F)$

and $\mathcal{A}_2(\varkappa_F)$ of expansion (9.23) are given by

$$(9.24) \qquad \mathcal{A}_1(\varkappa_F) = \{\mathrm{d}\mathfrak{V}_R(E, \varkappa_F)/\mathrm{d}E\}_{E=0}, \qquad \mathcal{A}_2(\varkappa_F) = \tfrac{1}{2}\{\mathrm{d}^2\mathfrak{V}_R(E, \varkappa_F)/\mathrm{d}E^2\}_{E=0}.$$

Using eq. (8.10*b*), we define the two-variable function

$$(9.25) \qquad G(E, \mathfrak{V}_R) = \mathfrak{V}_R - \mathfrak{V}_0(\varkappa_F)\left\{\frac{\varepsilon_F}{E - \mathfrak{V}_R}\right\}^{\frac{1}{2}} \exp\left[-\beta\varkappa_F\left\{\frac{E - \mathfrak{V}_R}{\varepsilon_F}\right\}^{\frac{1}{2}}\right] = 0.$$

From the theory of implicit functions one has

$$(9.26a) \qquad \{\mathrm{d}\mathfrak{V}_R(E, \varkappa_F)/\mathrm{d}E\}_{E=0} = -\left[\{\partial G/\partial E\}/\{\partial G/\partial \mathfrak{V}_R\}\right]_{E=0},$$

$$(9.26b) \qquad \left\{\frac{\mathrm{d}^2\mathfrak{V}_R(E, \varkappa_F)}{\mathrm{d}E^2}\right\}_{E=0} =$$

$$= -\left[\frac{\dfrac{\partial^2 G}{\partial E^2}\left(\dfrac{\partial G}{\partial \mathfrak{V}_R}\right)^2 - 2\dfrac{\partial^2 G}{\partial E\,\partial \mathfrak{V}_R}\dfrac{\partial G}{\partial E}\dfrac{\partial G}{\partial \mathfrak{V}_R} + \dfrac{\partial^2 G}{\partial \mathfrak{V}_R^2}\left(\dfrac{\partial G}{\partial E}\right)^2}{\left\{\dfrac{\partial G}{\partial \mathfrak{V}_R}\right\}^3}\right]_{E=0}.$$

Taking into account that

$$(9.27a) \qquad \frac{\mathrm{d}\mathfrak{V}_R(E, \varkappa_F)}{\mathrm{d}E} = -\frac{\partial G/\partial E}{1 - \partial G/\partial E},$$

$$(9.27b) \qquad \frac{\mathrm{d}^2\mathfrak{V}_R(E, \varkappa_F)}{\mathrm{d}E^2} = -\frac{\partial^2 G/\partial E^2}{\{1 - \partial G/\partial E\}^3},$$

TABLE X. – *Parameters characterizing the real optical potential* (9.23) *obtained as optical transform of the factorable single-particle potential energy* (6.21*b*): *the parameters are consistent with the saturation prescriptions and with the Hugenholtz and Van Hove theorem.* Length r_0 is expressed in fm units (see tables V and VII).

r_0	$b_v = -14.0$ MeV		$b_v = -15.0$ MeV		$b_v = -16.0$ MeV	
	γ	$\beta\varkappa_F\gamma$	γ	$\beta\varkappa_F\gamma$	γ	$\beta\gamma_F\gamma$
0.9	1.0630	0.6460	1.0649	0.6779	1.0689	0.7149
1.0	1.0730	0.7758	1.0756	0.8123	1.0804	0.8603
1.1	1.0830	0.8993	1.0885	0.9485	1.0935	1.0007
1.2	1.0949	1.0024	1.0982	1.0862	1.1025	1.1413
1.3	1.0985	1.1663	1.1114	1.2260	1.1150	1.2824
1.4	1.1159	1.2832	1.1199	1.3562	1.1233	1.4191

it is found that

$$\mathcal{A}_1(\varkappa_F) = \frac{1 + \beta\varkappa_F y}{3 + \beta\varkappa_F y}, \tag{9.28a}$$

$$\mathcal{A}_2(\varkappa_F) = -\frac{3 + 3\beta\varkappa_F y^2 + \beta^2\varkappa_F^2 y}{\varepsilon_F y(3 + \beta\varkappa_F y)^3}, \tag{9.28b}$$

where the parameter y is defined in eqs. (8.12). The values of $\beta\varkappa_F y$ as functions of r_0 and b_v are given in table X; in table XI we evaluate, consistently

TABLE XI. – *The nuclear-matter quantities* b_v, K *and* ε_F (*in* MeV) *and the coefficients* $\mathcal{A}_i(\varkappa_F)$ $(i = 0, 1, 2)$ *of the real optical potential* (9.23) *are evaluated as functions of* r_0 (*in* fm *units*), *consistently with the saturation prescriptions and the Hugenholtz and Van Hove theorem.* Quantities $\mathcal{A}(\varkappa_F)$ and $\mathcal{A}_2(\varkappa_F)$ are given in MeV and, respectively, in $(\text{MeV})^{-1}$. (See tables V, VI and VII.)

$b_v = -14.0$ MeV					
r_0	ε_F	K	$\mathcal{A}_0(\varkappa_F)$	$\mathcal{A}_1(\varkappa_F)$	$\mathcal{A}_2(\varkappa_F)$
0.9	57.02	374.28	−64.43	0.4514	−0.001 856
1.0	46.18	320.87	−53.17	0.4703	−0.002 271
1.1	38.17	283.09	−44.77	0.4871	−0.002 721
1.2	32.07	249.57	−38.45	0.5003	−0.003 203
1.3	27.33	228.99	−32.98	0.5195	−0.003 722
1.4	23.56	207.60	−29.34	0.5330	−0.004 246

$b_v = -15.0$ MeV					
r_0	ε_F	K	$\mathcal{A}_0(\varkappa_F)$	$\mathcal{A}_1(\varkappa_F)$	$\mathcal{A}_2(\varkappa_F)$
0.9	57.02	380.60	−64.67	0.4582	−0.001 853
1.0	46.18	328.97	−53.43	0.4754	−0.002 265
1.1	38.17	286.62	−45.23	0.4975	−0.002 707
1.2	32.07	258.84	−38.68	0.5093	−0.003 156
1.3	27.33	232.26	−33.76	0.5267	−0.003 682
1.4	23.56	211.48	−29.55	0.5409	−0.004 217

$b_v = -16.0$ MeV					
r_0	ε_F	K	$\mathcal{A}_0(\varkappa_F)$	$\mathcal{A}_1(\varkappa_F)$	$\mathcal{A}_2(\varkappa_F)$
0.9	57.02	387.53	−65.16	0.4616	−0.001 847
1.0	46.18	336.29	−53.91	0.4819	−0.002 255
1.1	38.17	296.19	−45.64	0.5001	−0.002 693
1.2	32.07	265.22	−38.98	0.5170	−0.003 138
1.3	27.33	240.42	−33.98	0.5330	−0.003 662
1.4	23.56	220.43	−29.73	0.5474	−0.004 192

with the saturation prescriptions and the Hugenholtz and Van Hove theorem, the coefficients of the real optical potential, expressed by eq. (9.23).

Let us suppose that the elastic-scattering data of nucleons by nuclei has been fitted using a real optical potential having the empirical form

$$V_1(E) = a_1 + b_1 E + c_1 E^2 . \tag{9.29}$$

We arbitrarily assume that such an analytical expression is valid also in the energy intervals where the $N = 2$ approximation certainly fails: for this reason we have modified the notation adopted for the empirical potential (9.10). The comparison between potentials (9.23) and (9.29) leads to the system of equations

$$\mathcal{A}_0(\varkappa_F) = a_1 , \quad \mathcal{A}_1(\varkappa_F) = b_1 , \quad \mathcal{A}_2(\varkappa_F) = c_1 . \tag{9.30}$$

The basic idea of the criterion devised for penetrating theoretically into the Fermi sphere, using the experimental data concerning the elastic scattering of neutrons by nuclei, consists in determining the parameters γ, $\beta\varkappa_F$ and ε_F from the phenomenologically known parameters (a_1, b_1, c_1) and, then, using eq. (6.20), determining parameter $\alpha\varkappa_F$ (see table V). From the second of eqs. (9.30) one has

$$\beta\varkappa_F \gamma = (3b_1 - 1)/(1 - b_1); \tag{9.31}$$

since $\gamma > 0$, eq. (9.31) implies that it must be

$$\tfrac{1}{3} \leqslant b_1 < 1 . \tag{9.32}$$

Inequality (9.32) shows that the Wilmore and Hodgson potential, discussed in subsect. 9·1, conflicts with the optical scheme constructed on the basis of the factorable single-particle potential energy (6.21*b*), not only because $c_0 > 0$ (note that $\mathcal{A}_2(\varkappa_F)$ is a negative quantity, as is shown in table XI), but also because $b_0 < \frac{1}{3}$. In conclusion, it is found that γ depends on the parameters (a_1, b_1, c_1) through the quadratic equation

$$\gamma^2 + \xi_1 \gamma + \xi_2 = 0 , \tag{9.33a}$$

$$\xi_1 = \frac{1 - b_1}{3b_1 - 1} , \qquad \xi_2 = \frac{(3b_1 - 1)^2(1 - b_1) - 8a_1 c_1}{3(3b_1 - 1)(1 - b_1)^2} ; \tag{9.33b}$$

then, parameter $\beta\varkappa_F$ is given by eq. (9.31) and the Fermi energy is obtained from the relation

$$\varepsilon_F = - a_1/\gamma^2 . \tag{9.34}$$

GIANNINI, RICCO and ZUCCHIATTI ([49]) have successfully fitted the energy dependence of the real optical potential up to $E \simeq 140$ MeV by using the analytical formula (9.29) with

$$(9.35) \qquad a_1 = -48.5 \text{ MeV}, \quad b_1 = 0.36, \quad c_1 = -0.000\,855 \text{ (MeV)}^{-1}.$$

From eq. (9.31) it follows that $\beta\varkappa_F Y = 0.1250$: from table X it is seen that this value is not compatible with nuclear stability. The description of nuclear matter inferred *sic et simpliciter* from the set of parameters (9.35) looks cumbersome; in fact, from eqs. (9.31), (9.32) and (9.34) it is found that

$$(9.36) \qquad y = 0.3968, \quad \beta\varkappa_F = 0.3150, \quad \varepsilon_F = 308.03(!) \text{ MeV}.$$

The paradoxical results in (9.36) show that the GRZ formula, although useful for fitting the data in a relatively wide energy interval, is deprived of any physical meaning in the energy region where the McLaurin expansion of any function describing the energy dependence of the real part of the optical potential is presumably valid: of course, this is also true for the optical transform (8.10) of the factorable potential (6.21*b*). We give credit to the analyses performed by GIANNINI, RICCO and ZUCCHIATTI and restrict the validity of their formula up to $E = 50.0$ MeV. As is shown in table XI, the nuclear stability implies that

$$(9.37) \qquad 0.400 < \mathcal{A}_2(\varkappa_F) < 0.550;$$

then, taking into account prescription (9.37), we specify the GRZ potential with the following sets of parameters:

$$(9.38) \quad \begin{cases} \text{set I:} & a_1 = -49.16 \text{ MeV}, & b_1 = 0.400, & c_1 = -0.001\,387 \text{ (MeV)}^{-1}; \\ \text{set II:} & a_1 = -49.57 \text{ MeV}, & b_1 = 0.425, & c_1 = -0.001\,723 \text{ (MeV)}^{-1}; \\ \text{set III:} & a_1 = -50.00 \text{ MeV}, & b_1 = 0.450, & c_1 = -0.002\,053 \text{ (MeV)}^{-1}; \\ \text{set IV:} & a_1 = -50.41 \text{ MeV}, & b_1 = 0.475, & c_1 = -0.002\,389 \text{ (MeV)}^{-1}; \\ \text{set V:} & a_1 = -50.83 \text{ MeV}, & b_1 = 0.500, & c_1 = -0.002\,720 \text{ (MeV)}^{-1}; \\ \text{set VI:} & a_1 = -51.66 \text{ MeV}, & b_1 = 0.550, & c_1 = -0.003\,387 \text{ (MeV)}^{-1}. \end{cases}$$

Except at the zero-energy limit, sets (9.38) lead to the same results otherwise obtainable by using the original GRZ formula. It is worthwhile to point out that the GRZ formula, modified by sets (9.38), looks like a perturbed optical

([49]) M. M. GIANNINI, G. RICCO and A. ZUCCHIATTI: *Microscopic Optical Potentials, Proceedings of the Hamburg Conference*, edited by H. V. VON GERAMB (Berlin, 1978), p. 126.

transform of the single-particle potential energy described in the effective-mass approximation: this is a very naïve clue for understanding many conceptually twisted treatments of the optical model (which, incidentally, have never produced significant results). The parameters expressed by eqs. (9.31), (9.32) and (9.34), calculated using sets (9.38), are given in table XII. The comparison of table XII with tables X and XI shows that the values of the parameters (a_1, b_1, c_1) consistent with nuclear stability and with the Hugenholtz and Van Hove theorem are included between those characterizing set IV and those characterizing set V. Taking into account table VI, one finds that it must be

$$\text{(9.39)} \qquad \begin{cases} r_0 = (1.1 \pm 0.1)\cdot 10^{-13}\ \text{cm}\,, \\ b_v = -(15.0 \pm 1.0)\ \text{MeV}\,, \\ K \simeq 220.0 \pm 50.1\,. \end{cases}$$

TABLE XII. – *Parameters* (9.31), (9.33) *and* (9.34), *characterizing the McLaurin expansion of the real optical potential* (8.10), *are evaluated as functions of sets* (9.38).

Set	y	$\beta\varkappa_F y$	$\beta\varkappa_F$	ε_F
I	0.6597	0.3333	0.5052	112.96
II	0.8089	0.4783	0.5912	75.76
III	0.9437	0.6364	0.6743	56.14
IV	1.0715	0.8095	0.7555	43.91
V	1.1930	1.0000	0.8382	35.71
VI	1.4382	1.4444	1.0043	24.97

This result is very gratifying; we shall not bother to determine the exact value of the nuclear-matter parameters (b_v, K, ε_F).

We shall now examine the real optical potential (8.10):

$$\text{(9.40}a\text{)} \qquad \mathfrak{V}_R(E, \varkappa_F) = \omega_1(\varkappa_F)\{E - \mathfrak{V}_R(E, \varkappa_F)\}^{-\frac{1}{2}} \exp\left[-\omega_2(\varkappa_F)\{E - \mathfrak{V}_R(E, \varkappa_F)\}^{\frac{1}{2}}\right],$$

$$\text{(9.40}b\text{)} \qquad \omega_1(\varkappa_F) = \mathfrak{V}_0(\varkappa_F)\sqrt{\varepsilon_F}\,, \qquad \omega_2(\varkappa_F) = \beta\varkappa_F/\sqrt{\varepsilon_F}\,.$$

The unknown parameters $\omega_1(\varkappa_F)$ and $\omega_2(\varkappa_F)$ have been determined by minimizing the mean square deviation

$$\text{(9.41)} \qquad \mathcal{M}(\omega_1, \omega_2) = \int_0^{E_0} \{V_1(E) - \mathfrak{V}_R(E, \varkappa_F)\}^2\, dE\,,$$

where $V_1(E)$ is the GRZ potential and $E_0 = 140$ MeV. Several mathematical difficulties have been overcome in order to deduce in explicit form the system

of equations

$$\partial \mathcal{M}/\partial \omega_i = 0 \,, \tag{9.42}$$

where $i = 1, 2$. A lengthy calculation has led to the solutions

$$\omega_1(\varkappa_F) = -\,899.82\ (\mathrm{MeV})^{\frac{3}{2}} \,, \qquad \omega_2(\varkappa_F) = 0.1310\ (\mathrm{MeV})^{-\frac{1}{2}} \,. \tag{9.43}$$

The minimum of eq. (9.41) is found to be $\mathcal{M}_0 = 55.98\ (\mathrm{MeV})^3$; it is worthwhile to point out that the minimum becomes $\mathcal{M}_0 = 1.50\ (\mathrm{MeV})^3$ if the integral is calculated by assuming $E \geqslant 50.0$ MeV. The numerical values of potential (9.40) are given in table XIII; the behaviour of the potential, governed by parameters (9.43), is plotted in fig. 4*a*), *b*) *vs.* the incident energy E ([50]).

TABLE XIII. – *Numerical values* (*in* MeV) *of the real optical potential* (9.40) *governed by parameters* (9.43).

E	$\mathcal{U}_R(E, \varkappa_F)$	E	$\mathcal{U}_R(E, \varkappa_F)$
0	−50.20	25.0	−40.96
2.5	−49.22	30.0	−39.27
5.0	−48.25	50.0	−33.00
10.0	−46.35	100.0	−20.89
15.0	−44.50	150.0	−13.86
20.0	−42.71	200.0	−11.26

We have now sufficient information for the determination of the nuclear-matter parameters (b_v, K, ε_F). From (9.43), taking into account definitions (9.40*b*), one has

$$\mathcal{U}_0(\varkappa_F)\beta\varkappa_F = -\,117.88\ \mathrm{MeV} \,. \tag{9.44}$$

The same quantity is evaluated in table XIV, by using $\mathcal{U}_0(\varkappa_F)$ and $\beta\varkappa_F$ given in tables VII and, respectively, V. It is seen that for $b_v = -14.0$ MeV value (9.44) corresponds to r_0 included between 1.1 and 1.2 fm, for $b_v = -15.0$ MeV it corresponds to r_0 varying in the interval between 1.1 and 1.0 fm and for $b_v = -16.0$ MeV r_0 appears to be included between 1.0 and 1.1 fm. These results are substantially consistent with the values of the nuclear-matter quantities given by eq. (9.39). We shall not seek for exact numerical solutions and assume $b_v = -15.0$ MeV. The final result, obtained by means of tables V

([50]) The real optical potentials, so far considered, do not reproduce with the same set of parameters the low- and high-energy data; this circumstance was first noticed by FRAHN (W. E. FRAHN: *Nuovo Cimento*, **5**, 393 (1957); see also W. E. FRAHN: *Nuovo Cimento*, **4**, 314 (1956)).

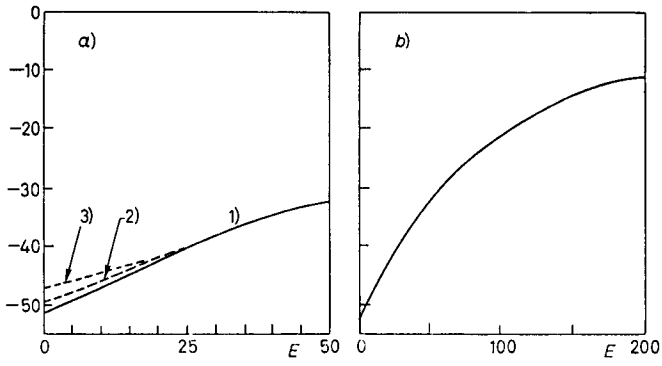

Fig. 4. – Energy dependence of the real part of the optical potential (9.40) characterized by parameters (9.43); *a*) energy interval $0 \leqslant E \leqslant 50$ MeV: potential (9.40) is expressed by curve 1); the empirical potential of Giannini, Ricco and Zucchiatti and the empirical potential of Wilmore and Hodgson are represented by curves 2) and 3), respectively; *b*) potential (9.40) in the energy interval $0 \leqslant E \leqslant 200$ MeV. The ordinate scale is given in MeV.

TABLE XIV. – *Numerical values (in* MeV*) of quantity* $\mathcal{U}_0(\varkappa_F)\beta\varkappa_F$ *calculated consistently with nuclear saturation and with the Hugenholtz and Van Hove theorem, by using the values of* $\mathcal{U}_0(\varkappa_F)$ *and* $\beta\varkappa_F$ *given in table VII and, respectively, in table V.* Length r_0 is given in fm units.

r_0	$\mathcal{U}_0(\varkappa_F)\beta\varkappa_F$		
	$b_v = -14.0$ MeV	$b_v = -15.0$ MeV	$b_v = -16.0$ MeV
0.9	— 79.25	— 86.65	— 95.32
1.0	— 96.59	— 98.32	—108.78
1.1	— 99.39	—110.75	—123.78
1.2	—105.35	—125.18	—140.10
1.3	—123.69	—140.71	—157.40
1.4	—136.39	—156.74	—176.77

and VII and using eqs. (6.20), (5.23) and (7.41), is

$$(9.45)\qquad \begin{cases} b_v = -15.0\,\text{MeV}\,, & r_0 = 1.171\cdot 10^{-13}\,\text{cm}\,, & \varkappa_F = 1.3008\cdot 10^{13}\,\text{cm}^{-1}\,, \\ \varepsilon_F = 35.08\,\text{MeV}\,, & \alpha\varkappa_F = 2.0184\,, & \beta\varkappa_F = 0.9276\,, \\ v(0, \varkappa_F) = -109.00\,\text{MeV}\,, & K = 187.48\,\text{MeV}\,. & \end{cases}$$

In conclusion, apart from conceptually irrelevant numerical approximations, quantities (9.45) are inferred from the phenomenological information on the elastic scattering of nucleons by nuclei according to an exact mathematical procedure consistent with the saturation prescriptions (1.5) and the Hugenholtz and Van Hove theorem (1.15). The obtained result is very satisfactory;

it should be clear that this goal has been achieved only by virtue of the differential equation of nuclear matter (5.5).

The total energy (7.34) can be written as

$$W(\varkappa)/A = \tfrac{3}{5}\varepsilon_F \chi^2 + \tfrac{3}{2}\{(b_v - \varepsilon_F)/\alpha\varkappa_F\}\{j_1^2(\alpha\varkappa_F\chi)/j_0(\alpha\varkappa_F)j_1(\alpha\varkappa_F)\}\chi\,, \tag{9.46}$$

where $\chi = \varkappa/\varkappa_F$. Function (9.46), calculated with parameters (9.45), is plotted *vs.* χ in fig. 5*a*: it is seen that $W(\varkappa)/A$ possesses a minimum equal to $b_v = -15.0$ MeV at $r_0 = 1.171$ fm. The momentum dependence of the factorable single-particle potential energy (7.42), calculated at the minimum of the total energy, is shown in fig. 5*b*.

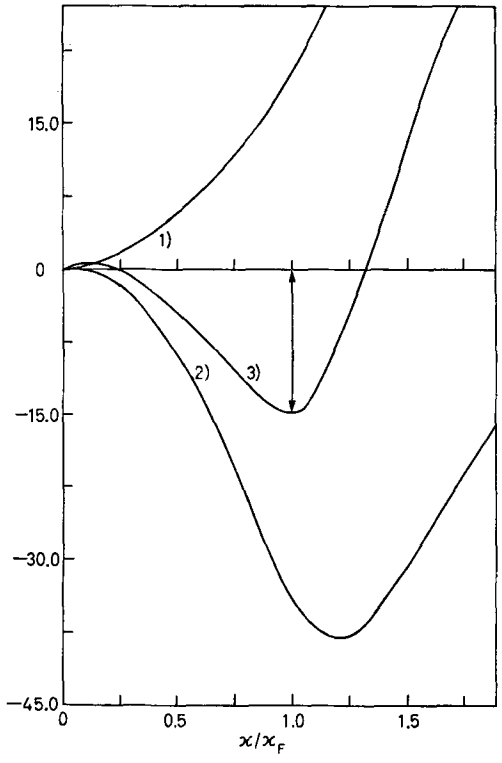

Fig. 5*a*.

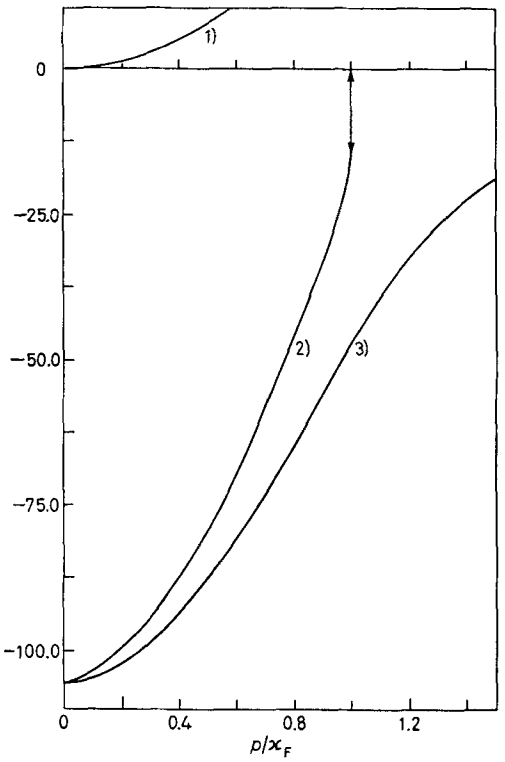

Fig. 5*b*.

Fig. 5*a*. – The saturation of infinitely extended nuclear matter described by the total energy (9.46), calculated using parameters (9.45) deduced from the analysis of the real optical potential. The ordinate is expressed in MeV. Curves 1), 2) and 3) correspond to $T(\varkappa)/A$, $V(\varkappa)/A$ and $W(\varkappa)/A$, respectively.

Fig. 5*b*. – The momentum dependence of the single-particle potential energy (7.42) at the minimum of the total energy expressed by eq. (9.46) (see fig. 2*b*). The Hugenholtz and Van Hove theorem is also visualized. The ordinate is expressed in MeV. Curves 1), 2) and 3) refer to $t(p) = \varepsilon_F\xi^2$, $v(p, \varkappa_F)$ and $w(p, \varkappa_F)$, respectively.

10. – The energy dependence of the imaginary part of the optical potential.

The pessimistic perspectives of sophisticated theories of the imaginary part of the optical potential ([51]), and the scanty results obtained from them, will probably stimulate the revival of the theoretical outlook on which the so-called frivolous models ([52]) are based. Our purpose is to re-formulate the simple model constructed by CLEMENTEL and VILLI ([53]) and calculate $\mathcal{U}_I(E, \varkappa_F)$ consistently with the saturation requirements of infinitely extended nuclear matter and with the Hugenholtz and Van Hove theorem. The achievement of this ambitious goal has never been attempted before: of course, it is significant only as a test of the internal logic of the whole nuclear-matter problem.

10·1. – As is well known, according to the clouded crystal ball model ([54]) the nucleon-nucleus scattering problem is reduced to a one-body problem by describing the nucleus by means of a complex attractive potential

$$\mathcal{U}(E, \varkappa_F) = \mathcal{U}_R(E, \varkappa_F) - i\mathcal{U}_I(E, \varkappa_F) , \tag{10.1}$$

acting upon the incoming particle. It is easy to see from the continuity equation that the introduction of a negative imaginary potential energy in the Schrödinger equation is a theoretical device for describing absorption of particles: in fact, the optical potential (10.1) corresponds to an absorption probability $2\mathcal{U}_I/\hbar$ per unit time as long as the particle is within the nucleus. It follows that the target nucleus can act upon the incoming nucleon as a potential well, because the formation of a compound state, described by the potential $\mathcal{U}_I(E, \varkappa_F)$ as an absorption, occurs inside the nucleus with a probability smaller than unity. In terms of the mean free path λ, the absorption probability per unit time is given by $v(E)/\lambda$, where $v(E)$ is the velocity in nuclear matter of a nucleon incident on the target nucleus with energy E. Trivial statistical considerations and elementary quantum-mechanical arguments lead to the general conclusion that the imaginary part of the optical potential depends on the density ϱ of nuclear matter, on the nucleon velocity $v(E)$ and on the neutron-proton cross-section $\langle\sigma_{np}\rangle$, averaged over the A nucleons of the nucleus and

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([51]) B. SINHA: *Phys. Rep. C*, **20**, 1 (1975).

([52]) E. CLEMENTEL and C. VILLI: *Nuovo Cimento*, **1**, 176 (1955); A. M. LANE and C. F. WANDEL: *Phys. Rev.*, **98**, 1524 (1955); I thank Prof. F. ZARDI for a useful discussion on the recent developments of this topic.

([53]) J. P. JENKENUE, A. LEJEUNE and C. MAHAUX: *Nuclear Optical Model Potential*, edited by S. BOFFI and G. PASSATORE (Berlin, 1976), p. 72; B. SINHA and F. DUGGAN: *Phys. Lett. B*, **47**, 389 (1973); B. SINHA and F. DUGGAN: *Nucl. Phys. A*, **226**, 31 (1974).

([54]) H. FESHBACH, C. E. PORTER and V. F. WEISSKOPF: *Phys. Rev.*, **96**, 448 (1954).

over their momenta within the Fermi sphere, according to the relation

$$\mathfrak{V}_{\mathrm{I}}(E, \varkappa_{\mathrm{F}}) = (5/16)\hbar\varrho v(E)\langle\sigma_{\mathrm{np}}\rangle , \tag{10.2}$$

where $\langle\sigma_{\mathrm{np}}\rangle$ is evaluated in the centre-of-mass system; potential (10.2) has been deduced taking into account the implications of charge symmetry and charge independence and assuming

$$\sigma_{\mathrm{pp}} = \sigma_{\mathrm{nn}} = (1/4)\sigma_{\mathrm{np}} , \tag{10.3}$$

where σ_{nn} and σ_{pp} are the elastic proton-proton and, respectively, neutron-neutron cross-sections. The validity of conditions (10.3) is a decreasing function of the scattering energy and is based upon the fact that the integral cross-section related to the pure isobaric spin state $T = 0$ is negligible as compared with that corresponding to the pure isobaric spin state $T = 1$.

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be the momenta of the nucleon incident on the nucleus with energy $E = k^2/2M$ in the laboratory system ($p > \varkappa_{\mathrm{F}}$) and, respectively, the momentum of a nucleon embedded in nuclear matter ($q \leqslant \varkappa_{\mathrm{F}}$); the momentum p will be defined according to the dispersive relation

$$p = p(E) = k\{1 - \mathfrak{V}_{\mathrm{R}}(E, \varkappa_{\mathrm{F}})/E\}^{\frac{1}{2}} , \tag{10.4}$$

already introduced in sect. **8**. The relative momenta of the two-nucleon systems before and after the elastic collision are

$$\boldsymbol{P} = \tfrac{1}{2}(\boldsymbol{p} - \boldsymbol{q}) , \qquad \boldsymbol{P}' = \tfrac{1}{2}(\boldsymbol{p}' - \boldsymbol{q}') ; \tag{10.5}$$

the energy conservation requires that

$$P' = P = \tfrac{1}{2}|\boldsymbol{p} - \boldsymbol{q}| . \tag{10.6}$$

The cross-section σ_{np} depends on P and on the angle between $\boldsymbol{P}$ and $\boldsymbol{P}'$, *i.e.*

$$\sigma_{\mathrm{np}} = \sigma_{\mathrm{np}}(P, \boldsymbol{P}\cdot\boldsymbol{P}') . \tag{10.7}$$

The procedure for calculating $\langle\sigma_{\mathrm{np}}\rangle$ is the following: *a*) quantity (10.7) is multiplied by the flux of incident particles of velocity $|\boldsymbol{p} - \boldsymbol{q}|/M$; *b*) the quantity thus obtained is integrated over the solid angle $\mathrm{d}\Omega = \sin\alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta$ defined by the relative momenta $\boldsymbol{P}$ and $\boldsymbol{P}'$, keeping momenta $\boldsymbol{p}$ and $\boldsymbol{q}$ fixed; *c*) then, by averaging over the Fermi sphere one obtains the number of particles scattered per unit time in the whole solid angle for any value of $\boldsymbol{q}$; *d*) finally, such a result has to be divided by the flux of incident particles

corresponding to the velocity k/M. In conclusion, it is found that

$$\langle\sigma_{np}\rangle = (1/p)(3/4\pi\varkappa_F^3)\int|\boldsymbol{p}-\boldsymbol{q}|\sigma_{np}(P, \boldsymbol{P}\cdot\boldsymbol{P}')\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega\,. \tag{10.8}$$

The calculation of eq. (10.8) cannot be performed without resorting to an electronic computer: a prospect which, at least for me, holds little excitement. At this stage, to avoid nonessential refinements, we shall simplify eq. (10.8) by neglecting the angular dependence of the neutron-proton cross-section and use the total cross-section, *i.e.*

$$\sigma_{np}(P;\, \boldsymbol{P}\cdot\boldsymbol{P}') \to (1/4\pi)\sigma_{np}(P); \tag{10.9}$$

consequently, eq. (10.8) becomes

$$\langle\sigma_{np}\rangle = (1/p)(3/16\pi^2\varkappa_F^3)\int|\boldsymbol{p}-\boldsymbol{q}|\sigma_{np}(P)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega\,. \tag{10.10}$$

Let us define the vector

$$\boldsymbol{Q} = \boldsymbol{p} + \boldsymbol{q}\,, \tag{10.11}$$

which, owing to the conservation of the total momentum, turns out to be an axis of cylindrical symmetry. From the relation

$$\boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{q} - \boldsymbol{q}' = \tfrac{1}{2}\boldsymbol{Q} + \boldsymbol{P}' \tag{10.12}$$

one readily obtains, taking into account prescription (10.6),

$$\cos\alpha = (2p'^2 - p^2 - q^2)/|\boldsymbol{p}+\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|; \tag{10.13}$$

then, one has

$$\int\mathrm{d}\Omega = 2\pi\int\sin\alpha\,\mathrm{d}\alpha = \frac{8\pi}{|\boldsymbol{p}+\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\int_{p_0'}^{p_1'}\mathrm{d}(p'^2) = \frac{4\pi(p_1^2 - p_0^2)}{|\boldsymbol{p}+\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,. \tag{10.14}$$

The lower limit of integration is given by

$$p_0 = \varkappa_F; \tag{10.15}$$

the physical meaning of equality (10.15) will be discussed later. The upper limit p_1^2 is determined by the Pauli principle and by the principle of energy conservation: in fact, the latter requires that

$$p'^2 = p^2 + q^2 - q'^2; \tag{10.16}$$

it follows that the maximum of p' is obtained by putting into eq. (10.16) $q' = \varkappa_F$, which is the minimum value of q' compatible with the exclusion principle, *i.e.*

$$p_1^2 = p^2 + q^2 - \varkappa_F^2 . \tag{10.17}$$

The previous considerations ensure that $p_1^2 \geqslant p_0^2$. Equation (10.10) becomes

$$\langle \sigma_{nD} \rangle = (1/p)(3/4\pi\varkappa_F^3) \int \{(p^2 + q^2 - 2\varkappa_F^2)/|\boldsymbol{p} + \boldsymbol{q}|\} \sigma_{np}(P)\, \mathrm{d}\boldsymbol{q} . \tag{10.18}$$

Let $\mathrm{d}\boldsymbol{q} = q^2\, \mathrm{d}q \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varrho$ be the volume element in momentum space. We identify θ with the angle between $\boldsymbol{p}$ and $\boldsymbol{q}$. From the relation

$$Q^2 = |\boldsymbol{p} - \boldsymbol{q}|^2 = p^2 + q^2 - 2pq\cos\theta \tag{10.19}$$

one obtains $\sin\theta\, \mathrm{d}\theta = (Q/pq)\, \mathrm{d}Q$; the analytic expression of the relative momentum (10.6) turns out to be

$$P = \tfrac{1}{2}(2p^2 + 2q^2 - Q^2)^{\frac{1}{2}} . \tag{10.20}$$

Consequently, eq. (10.8) becomes

$$\langle \sigma_{nD} \rangle = (1/p^2)(3/2\varkappa_F^3) \int_{q_0}^{q_1} q(p^2 + q^2 - 2\varkappa_F^2)\, S(p, q)\, \mathrm{d}q , \tag{10.21a}$$

$$S(p, q) = \int_{p-q}^{p+q} \sigma_{np}\{\tfrac{1}{2}(2p^2 + 2q^2 - Q^2)\}\, \mathrm{d}Q , \tag{10.21b}$$

where $q_1 > q_0$ and $p > q$ because $p > \varkappa_F$ and $q \leqslant \varkappa_F$; the limits of integration are

$$\begin{cases} q_0 \equiv (p_0^2 + \varkappa_F^2 - p^2)^{\frac{1}{2}} \leqslant q \leqslant \varkappa_F \equiv q_1, & p^2 \leqslant 2\varkappa_F^2 , \\ q_0 \equiv 0 \leqslant q \leqslant \varkappa_F \equiv q_1 , & p^2 \geqslant 2k_F^2 . \end{cases} \tag{10.22}$$

In conclusion, the imaginary part of the optical potential reads

$$\mathfrak{U}_I(E, \varkappa_F) = (5\hbar/16\pi^2)\{v(E)/p^2(E)\} \int_{q_0}^{q_1} q(p^2 + q^2 - 2\varkappa_F^2)\, S(p, q)\, \mathrm{d}q , \tag{10.23}$$

where the equilibrium density of nuclear matter has been expressed by means of the corresponding Fermi momentum,

$$\varrho = A/\Omega = 3/4\pi r_0^3 = (2/3\pi^2)\varkappa_F^3 . \tag{10.24}$$

Although completely unphysical, the assumption that the total neutron-proton cross-section $\sigma_{np}(P)$ is a constant has been widely used in order to calculate $\langle\sigma_{np}\rangle$. In this case one has $S(p, q) = 2\sigma_0 q$ and from eq. (10.21) and prescriptions (10.22) one obtains

$$\langle\sigma_{np}\rangle = \sigma_0\left\{1 - \frac{7}{5}\zeta^2\right\}, \qquad p^2 \geqslant 2\varkappa_F^2, \tag{10.25a}$$

$$\langle\sigma_{np}\rangle = \sigma_0\left\{1 - \frac{7}{5}\zeta^2 + \frac{2}{5}\zeta^2\left(2 - \frac{1}{\zeta^2}\right)^{\frac{5}{2}}\right\}, \qquad p^2 \leqslant 2\varkappa_F^2, \tag{10.25b}$$

where

$$\zeta \equiv \zeta(E) = \varkappa_F/p(E)\,. \tag{10.26}$$

Equation (10.27*a*) has been derived by GOLDBERGER ([55]) following a different procedure and eq. (10.27*b*) has been obtained by YAMAGUCHI ([56]).

A deep insight into equality (10.15) is provided by the following theorem: *the real optical potential, calculated at the average volume energy of nuclear matter, identifies with the single-particle potential energy at the Fermi surface.* To this end we consider the equation

$$\varkappa_F = p(E_0)\,, \tag{10.27}$$

where E_0 is an unknown energy and $\varkappa_F$ is the Fermi momentum corresponding to the minimum of the total energy of infinitely extended nuclear matter. From eq. (10.27) taking into account eq. (10.4), one has

$$E_0 = \varepsilon_F + \mathfrak{V}_R(E_0, \varkappa_F)\,. \tag{10.28}$$

According to eq. (1.19), the optical transform of the single-particle potential energy $v(p, \varkappa_F)$ is

$$\mathfrak{V}_R(E, \varkappa_F) = v\{p(E), \varkappa_F\}; \tag{10.29}$$

it follows that at $E = E_0$ eq. (10.28) becomes

$$E_0 = \varepsilon_F + v(\varkappa_F, \varkappa_F)\,. \tag{10.30}$$

The only physically meaningful value of E_0, satisfying eq. (10.31) consistently with the saturation prescriptions (1.5), is given by the Hugenholtz and Van

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([55]) M. L. GOLDBERGER: *Phys. Rev.*, **74**, 1269 (1948).

([56]) Y. YAMAGUCHI: *Progr. Theor. Phys.*, **5**, 332 (1950); the formula obtained by this author contains some analytical mistakes.

Hove theorem (1.15), *i.e.*

$$E_0 = b_v ; \tag{10.31}$$

in conclusion, the following identity holds:

$$\mathcal{U}_R(b_v, \varkappa_F) \equiv v(\varkappa_F, \varkappa_F) . \tag{10.32}$$

From theorem (10.32) one deduces that the minimum value of $p(E)$ is

$$\varkappa_F = p(b_v) \equiv p_0 . \tag{10.33}$$

It should be clear that the validity of the preceding proof relies entirely on the fact that $v(p, \varkappa)$ is a solution of the differential equation (5.5).

10'2. – In the following discussion we shall consider only the real optical potential (9.40), obtained as optical transform of the factorable single-particle potential energy (7.42): in this way the frivolous model, expressed by eq. (10.23), is logically included in the theoretical description of nuclear matter based on the hyperbolic differential equation (5.5).

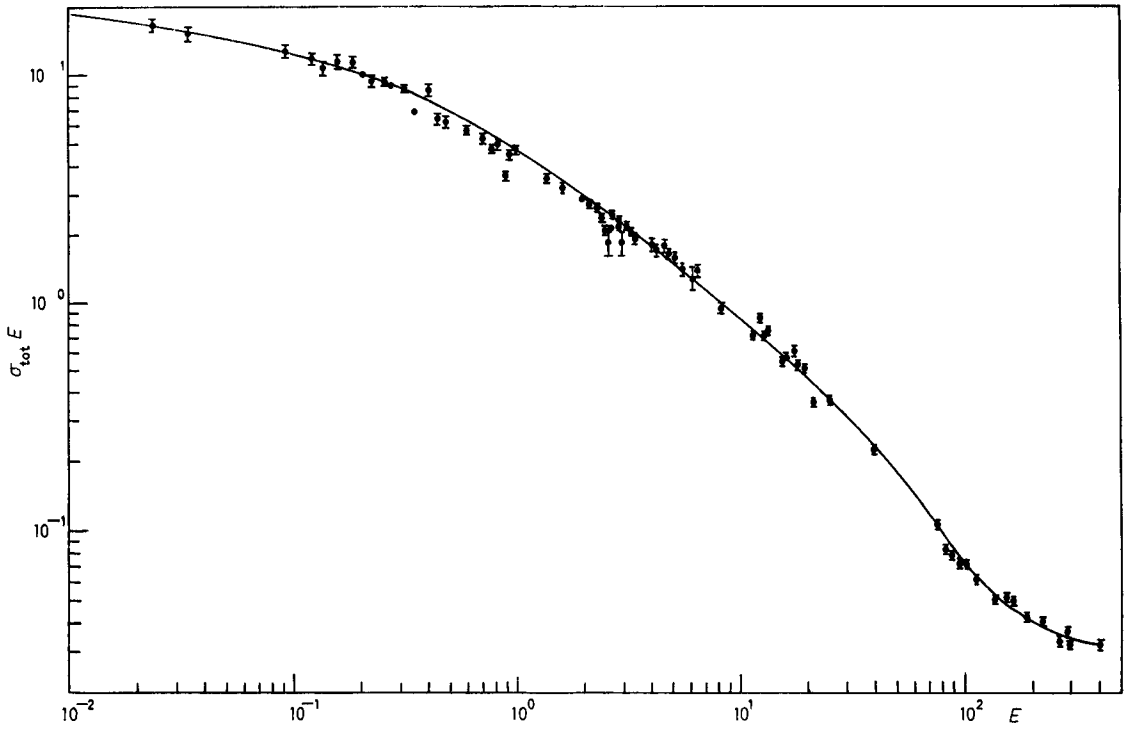

Fig. 6. – Experimental values (in barn) of the total neutron-proton cross-section as functions of energy E (in MeV), measured in the laboratory system of reference.

The calculation of $\mathfrak{V}_1(E, \varkappa_F)$ requires a careful use of the experimental values of the total neutron-proton cross-section in order to obtain reliable numerical values for $\langle \sigma_{np} \rangle$, given in eq. (10.21). For the sake of completeness we list in table XV the values of $\sigma_{np}(E)$ as functions of the energy E in the laboratory system of reference ([57]); the energy dependence of the total neutron-proton cross-section is plotted in fig. 6. We shall now determine the interval of variation of $\sigma_{np}(P)$ in the centre-of-mass system and in nuclear matter as a function of

TABLE XV. – *Experimental values (in* barn*) of the total neutron-proton cross-section as functions of the energy E (in* MeV*), measured in the laboratory system of reference.*

E	$\sigma_{np}(E)$	E	$\sigma_{np}(E)$	E	$\sigma_{np}(E)$
0	—	2.28	2.70 ± 0.06	15.00	0.63
0.024	18.15 ± 0.05	2.40	2.39	16.00	0.66
0.035	16.74 ± 0.41	2.50	1.80 ± 0.40	18.00	0.55
0.095	13.46 ± 0.39	2.60	2.60 ± 0.05	19.50	0.52
0.130	11.85 ± 0.15	2.76	2.40 ± 0.06	19.93	0.504 ± 0.001
0.140	10.50 ± 0.90	2.80	2.17 ± 0.10	21.00	0.41
0.157	11.10	2.90	1.80 ± 0.20	25.00	0.39 ± 0.06
0.160	12.00 ± 0.30	3.00	2.23 ± 0.13	27.00	0.36
0.180	11.30 ± 0.20	3.10	2.18 ± 0.13	39.00	0.223 ± 0.07
0.200	10.0	3.50	2.09 ± 0.09	40.00	0.170
0.220	9.60 ± 0.40	4.00	1.85 ± 0.09	64.50	0.126 ± 0.003
0.245	0.20	4.10	1.73 ± 0.06	90.00	0.082
0.265	9.12 ± 0.24	4.50	1.83 ± 0.10	95.00	0.073 ± 0.0015
0.32	8.70	4.75	1.69 ± 0.06	97.00	0.074 ± 0.010
0.35	7.15 ± 0.24	5.00	1.63 ± 0.05	117.00	0.0616
0.40	8.70 ± 0.90	5.50	1.48 ± 0.06	140.00	0.0485
0.60	5.85 ± 0.25	6.00	1.32 ± 0.12	156.00	0.0493
0.72	5.22 ± 0.12	6.50	1.40 ± 0.11	160.00	0.0512
0.83	5.00 ± 0.10	9.30	0.92 ± 0.08	169.00	0.0492 ± 0.0016
0.90	5.50 ± 1.10	10.60	0.78 ± 0.08	180.00	0.044
1.00	4.16 ± 0.15	12.50	0.69 ± 0.10	220.00	0.0411
1.34	3.64 ± 0.04	12.80	0.83 ± 0.09	260.00	0.035
1.60	3.36 ± 0.08	13.50	0.69 ± 0.019	270.00	0.038 ± 0.0015
2.00	2.96 ± 0.07	14.00	0.71	280.00	0.036
2.14	2.76 ± 0.06	14.80	0.61 ± 0.07	400.00	0.0336

([57]) More complete information is given in the review paper by L. BERETTA, C. VILLI and F. FERRARI: *Nuovo Cimento, Suppl.*, **12**, 499 (1954).

the kinetic energy E of the nucleon incident on the target nucleus. The kinetic energy involved in the collision between a nucleon of momentum $p(E) > \varkappa_F$ with a nucleon having in the Fermi sea momentum $q \leqslant \varkappa_F$ is

$$e \equiv e(P) = P^2/M = |\boldsymbol{p} - \boldsymbol{q}|^2/4M \,. \tag{10.34}$$

The interval of maximum variability of the relative momentum is

$$P_0 \equiv \tfrac{1}{2}(p - \varkappa_F) \leqslant P \leqslant \tfrac{1}{2}(p + \varkappa_F) = P_1; \tag{10.35}$$

it follows that $e_0 \leqslant e \leqslant e_1$, where

$$e_0 \equiv e(P_0) = \tfrac{1}{2}\{\sqrt{E - \mathfrak{V}_R(E, \varkappa_F)} - \sqrt{\varepsilon_F}\}^2 \,, \tag{10.36a}$$

$$e_1 \equiv e(P_1) = \tfrac{1}{2}\{\sqrt{E - \mathfrak{V}_R(E, \varkappa_F)} + \sqrt{\varepsilon_F}\}^2 \,. \tag{10.36b}$$

TABLE XVI. – *Auxiliary parameters required for the calculation of the imaginary part of the optical potential given by eq.* (10.23). Quantities E, e_0, e_1 and c_2 are expressed in MeV, and c_1 in b·MeV (see tables XIII and XV; the Fermi energy is assumed to be $\varepsilon_F = 33.68$ MeV).

E	e_0	e_1	c_1	c_2
0	0.82	83.06	4.1651	0.4109
2.5	0.96	84.43	4.1895	0.3792
5.0	1.11	85.81	4.2130	0.3456
10.0	1.45	88.58	4.2468	0.2661
15.0	1.83	01.35	4.2836	0.1814
20.0	2.24	94.15	4.3260	0.0970
25.0	2.69	96.95	4.3530	−0.0008
30.0	3.17	99.77	4.3855	−0.0976
50.0	5.47	111.21	4.5842	0.5994
100.0	13.48	141.09	5.5618	4.8878
150.0	24.48	173.06	6.4567	8.8188
200.0	38.12	206.82	8.3629	45.8345

Energies (10.36) are evaluated in table XVI, using the values of the real optical potential given in table XIII. We simulate the energy dependence of $\sigma_{np}(e)$ by means of the two-parameter formula

$$s(e) = c_1/\{c_2 + e(P)\} \tag{10.37}$$

and minimize the mean square deviation

$$\mathcal{N}(c_1, c_2) = \int_{e_1}^{e_2} \{\sigma_{np}(2e) - s(e)\}^2 \, de \,, \tag{10.38a}$$

$$\partial \mathcal{N}/\partial c_i = 0 \,, \tag{10.38b}$$

where $i = 1, 2$. The values of parameters c_1 and c_2 are listed in table XVI ([58]). If we take into account eqs. (10.20) and (10.34), the empirical formula (10.37) becomes

$$s(p, q; Q) = 4Mc_1/(4Mc_2 + 2p^2 + 2p^2 - Q^2) \,. \tag{10.39}$$

From table XVI it is seen that the approximation $c_2 = 0$ is not correct, except for $E \simeq 25$ MeV ([58]).

The function defined in eq. (10.21*b*) reads

$$S(p, q) = \int_{p-q}^{p+q} s(p, q; Q) \, dQ; \tag{10.40}$$

a straightforward calculation gives

$$S(p, q) = \frac{2Mc_1}{\sqrt{2(2Mc_2 + p^2 + q^2)}} \cdot \ln \frac{4Mc_2 + p^2 + 3q^2 + 2q\sqrt{2(2Mc_2 + p^2 + q^2)}}{4Mc_2 + p^2 + 3q^2 - 2q\sqrt{2(2Mc_2 + p^2 + q^2)}} \,. \tag{10.41}$$

Since in the interval $p - q \leqslant Q \leqslant p + q$ it results that $Q^2 < 2(2Mc_2 + p^2 + q^2)$, from eq. (10.40) one also obtains

$$S(p, q) = \frac{2Mc_1}{\sqrt{2(2Mc_2 + p^2 + q^2)}} \cdot \left[\mathrm{tgh}^{-1} \frac{p + q}{\sqrt{2(2Mc_2 + p^2 + q^2)}} - \mathrm{tgh}^{-1} \frac{p - q}{\sqrt{2(2Mc_2 + p^2 + q^2)}} \right]; \tag{10.42}$$

by means of trivial manipulations eq. (10.42) can be written in the following analytical form, which for numerical computations is more convenient than

([58]) S. HAYAKAWA, M. KAWAI and K. KIKUCHI: *Progr. Theor. Phys.*, **13**, 415 (1955); the approximation $c_2 = 0$ has been adopted also by B. SINHA: *Phys. Rev. C*, **11**, 1546 (1975).

that expressed by eq. (10.41), *i.e.*

$$(10.43) \qquad S(p, q) = \frac{2Mc_1}{\sqrt{2(2Mc_2 + p^2 + q^2)}} \operatorname{tgh}^{-1} \frac{2q\sqrt{2(2Mc_2 + p^2 + q^2)}}{4Mc_2 + p^2 + 3q^2} .$$

In conclusion, the imaginary optical potential (10.23) turns out to have the following analytical form:

$$(10.44a) \qquad \mathfrak{V}_I(E, \varkappa_F) = C_I\{E - \mathfrak{V}_R(E, \varkappa_F)\} g(E, \varkappa_F) ,$$

$$(10.44b) \qquad C_I \equiv C_I(E) = (5/4\pi^2)(M/\hbar^2) c_1(E) ,$$

$$(10.44c) \qquad g(E, \varkappa_F) = \int_{z_0}^{z_1} \frac{z(z^2 - \mu)}{\sqrt{2z^2 + \nu)}} \operatorname{tgh}^{-1}\left(\frac{2z\sqrt{2z^2 + \nu}}{3z^2 + \nu - 1}\right) dz ,$$

$$(10.44d) \qquad \mu \equiv \mu(E, \varkappa_F) = 2\zeta^2 - 1 , \qquad \nu \equiv \nu(E, \varkappa_F) = 2\{1 + (c_2/c_F)\zeta^2\};$$

the limits of integration are

$$(10.44e) \qquad \mu \geqslant 0: \quad z_0 \equiv \sqrt{\mu} \leqslant z \leqslant \zeta \equiv z_1 , \qquad \mu \leqslant 0: \quad z_0 \equiv 0 \leqslant z \leqslant \zeta \equiv z_1 ,$$

ζ being the energy-dependent parameter defined in (10.26): *model* (10.38) *is consistent with the saturation prescriptions of nuclear matter, provided* $\mathfrak{V}_R(E, \varkappa_F)$ *is the optical transform of a single-particle solution of the differential equation* (5.5). The numerical values of the parameters μ, ν, z_0 and z_1 as functions of the incident energy E are given in table XVII, assuming that the nuclear matter is

TABLE XVII. – *Energy dependence of the parameters involved by the imaginary optical potential* (10.44). Energy E is expressed in MeV. (See tables XIII and XV.)

E	μ	ν	z_0	z_1	C_I
0	0.3418	2.0164	0.5847	0.8191	1.2762
2.5	0.3021	2.0147	0.5497	0.8069	1.2833
5.0	0.2649	2.0129	0.5147	0.7953	1.2909
10.0	0.1954	2.0094	0.4420	0.7731	1.3012
15.0	0.1321	2.0060	0.3634	0.7523	1.3125
20.0	0.0741	2.0030	0.2723	0.7328	1.3255
25.0	0.0212	2.0000	0.1457	0.7146	1.3337
30.0	−0.0276	1.9972	0	0.6972	1.3437
50.0	−0.1884	2.0144	0	0.6370	1.4046
100.0	−0.4428	2.0808	0	0.5278	1.7041
150.0	−0.5889	2.1076	0	0.4534	1.9783
200.0	−0.6811	2.4339	0	0.3993	2.5624

characterized by the parameters (9.45) and using the energy dependence of the real part of the optical potential given in table XIII. Taking into account eq. (10.27), one has

$$z_0(b_v) = z_1(b_v) = 1 , \tag{10.45}$$

and, consequently,

$$\mathfrak{V}_1(b_v, \varkappa_F) = 0 ; \tag{10.46}$$

property (10.46), *possessed by the imaginary potential* (10.44), *is consistent with the shell model basic assumption that the nucleons in the nucleus should be embedded in a real potential well.*

10'3. – The imaginary potential (10.44) cannot be evaluated analytically because of the complicated integral $g(E, \varkappa_F)$. For our purposes it is preferable to perform the numerical integration by using the Gauss method ([59]), which is as precise as other methods, but entails much less work. Let us re-write eq. (10.44*c*)

$$g(E, \varkappa_F) = \int_{z_0}^{z_1} f(z)\,\mathrm{d}z , \tag{10.47a}$$

$$f(z) = \frac{z(z^2 - \mu)}{2z^2 + \nu} \operatorname{tgh}^{-1} \frac{2z\sqrt{2z^2 + \nu}}{3z^2 + \nu - 1} ; \tag{10.47b}$$

the usefulness of the Gauss method is due to the fact that in the considered case the interval of integration is very limited and the positive function $f(z)$ is well behaved. In applying the method, it is convenient to change the limits of integral (10.47*a*) by making the substitution

$$z = z_0 + (z_1 - z_0)\tau ; \tag{10.48}$$

consequently, one has

$$g(E, \varkappa_F) = (z_1 - z_0) \int_0^1 F(\tau)\,\mathrm{d}\tau , \tag{10.49}$$

where we have put $f\{z_0 + (z_1 - z_0)\tau\} \equiv F(\tau)$. The final result obtained by Gauss' method is

$$g(E, \varkappa_F) \to G(E, \varkappa_F) = (z_1 - z_0) \sum_{m=0}^{n} a_m F_m , \tag{10.50}$$

([59]) H. MARGENAU and G. M. MURPHY: *The Mathematics of Physics and Chemistry* (New York, N. Y., 1943).

where F_m means the numerical value of $F(\tau)$ calculated for a value τ_m of the new variable τ; values τ_m and coefficients a_m are determined in such a way that the difference between $g(E, \varkappa_F)$ and $G(E, \varkappa_F)$ be a minimum. For the reader's convenience we report in table XVIII the values of τ_m and a_m corresponding to the approximations of order $n = 2, 3, 4$ and 5.

TABLE XVIII. – *Parameters required for performing numerical integrations using the Gauss method in $n = 2, 3, 4$ and 5 approximation.*

$n = 2$			$n = 3$		
m	τ_m	a_m	m	τ_m	a_m
0	0.112 702	0.277 778	0	0.069 432	0.173 927
1	0.5	0.444 444	1	0.330 009	0.326 072
2	0.887 298	0.277 778	2	0.669 990	0.326 072
			3	0.930 568	0.173 927

$n = 4$			$n = 5$		
m	τ_m	a_m	m	τ_m	a_m
0	0.046 910	0.118 463	0	0.033 765	0.085 662
1	0.230 765	0.239 314	1	0.169 395	0.180 381
2	0.5	0.284 444	2	0.380 690	0.233 957
3	0.769 235	0.239 314	3	0.619 309	0.233 957
4	0.953 090	0.118 463	4	0.830 605	0.180 381
			5	0.966 235	0.085 662

The numerical values of $\mathfrak{V}_I(E, \varkappa_F)$ expressed by eqs. (10.44), calculated using the $n = 5$ approximation of Gauss' method, are given in table XIX; the energy behaviour is shown in fig. 7. The zero-energy limit of $\mathfrak{V}_I(E, \varkappa_F)$ turns out to be (with a very high degree of approximation) a linear function

TABLE XIX. – *Energy dependence of the imaginary part of the optical potential, expressed by eq.* (10.44). The ordinate of the plot and the energy E are given in MeV. (See table XVI.)

E	$\mathfrak{V}_I(E, \varkappa_F)$	E	$\mathfrak{V}_I(E, \varkappa_F)$
0	2.33	25.0	5.66
2.5	2.66	30.0	6.28
5.0	3.01	50.0	8.01
10.0	3.68	100.0	10.71
15.0	4.36	150.0	12.33
20.0	5.06	200.0	11.85

of the zero-energy limit of the real part of the potential: as pointed out in sect. **9**, the value of $\mathcal{U}_R(0, \varkappa_F)$ is a very crucial one in order to extract reliable information on the nuclear-matter parameters (b_v, K, r_0) from the elastic-scattering data of nucleons from nuclei. The maximum of $\mathcal{U}_I(E, \varkappa_F)$ occurs at an incident energy included in the interval (100÷200) MeV: it arises mainly from the competing effects due to the decrease, with increasing incident energy,

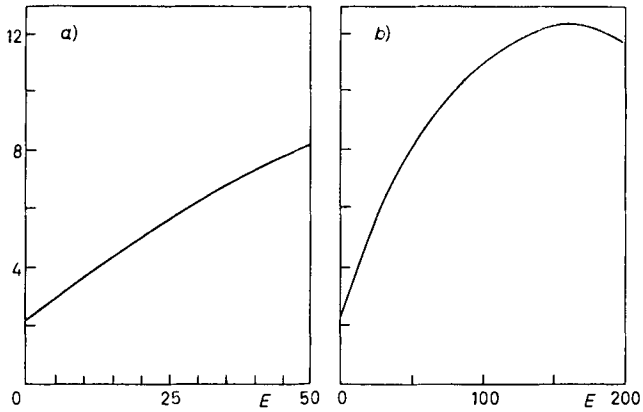

Fig. 7. – Energy dependence of the imaginary part of the optical potential. The ordinate is expressed in MeV.

of the restrictions brought about by the Pauli principle (which tend to augment $\mathcal{U}_I(E, \varkappa_F)$) and to the decrease of the neutron-proton total cross-section (which tends to reduce the value of $\mathcal{U}_I(E, \varkappa_F)$). The position of the maximum is influenced by the energy dependence of the real optical potential: it is gratifying to note that the outlined version of the Clementel and Villi model, although conceptually and numerically more refined, leads to values of $-\mathcal{U}_I(E, \varkappa_F)$ which are in substantial agreement with those obtained on the basis of the older model ([52]). The predicted energy dependence of $\mathcal{U}_I(E, \varkappa_F)$, although consistent (except at low energies) with the behaviour of the real optical potential obtained by GIANNINI, RICCO and ZUCCHIATTI (see subsect. **9**·2) deviates markedly from the imaginary part of the potential given by these authors ([60]): the occurrence of a maximum between $E = 20$ MeV and $E = 60$ MeV together with the predicted high-energy behaviour cannot be reproduced by our model. It would be interesting to compare the results given in table XIV with those obtained (or obtainable) using the models based on

([60]) M. M. GIANNINI, G. RICCO and A. ZUCCHIATTI: *Ann. Phys.* (*N.Y.*), **124**, 208 (1980), fig. 11. For a comparison with earlier calculations of the imaginary optical potential we refer to P. E. HODGSON: *The Optical Model of Elastic Scattering* (Oxford, 1963), fig. 10.5, p. 180. See also E. A. GLASSGOLD, W. B. CHESTON, M. L. STEIN, S. B. SHOULDT and G. W. ERICSON: *Phys. Rev.*, **106**, 1207 (1957).

the assumption that the absorption arises entirely from the forward elastic nucleon-nucleon scattering amplitudes: unfortunately, these models are reported in the literature in a rather elusive theoretical manner and with scanty numerical details ([61]). Finally, it is worthwhile to point out that the model, outlined in this subsection consistently with the saturation prescriptions of infinitely extended nuclear matter, can easily be re-formulated with the purpose of calculating the imaginary part of the optical potential related to the scattering of pions and kaons from nuclei ([62]).

11. – Concluding remarks.

Our program of extracting the maximum of theoretical information on the overall behaviour of infinite nuclear matter from the saturation prescriptions and the Hugenholtz and Van Hove theorem with a minimum of physical assumptions has been accomplished. The discovery of the differential equation (5.5) has allowed us to develop a mathematical scheme entirely free from any detailed description of the nucleon-nucleon interaction. This approach makes realistically irrelevant the very many heuristic shadows arising from the following paralysing doubt: since scattering experiments are all carried out on the energy shell, do they give all the information needed in the nuclear-matter problem, where matrix elements off the energy shell play a crucial role? Arguments have been proposed to support Bethe's opinion that two-body potentials with a hard core are not compatible with a consistent description of nuclear matter: they contribute to so-called re-arrangement energy, which in our scheme is exactly equal to zero. In conclusion, one might be forgiven for regarding the variety of nucleon-nucleon forces introduced into the nuclear-matter problem as a trick for opening a Pandora's box. The solutions of the differential equation of nuclear matter provide an unconventional answer to the self-consistent field problem of finding the single-particle potential energy, which is the major point in Brueckner's theory. Many distinguished physicists who have dealt with this crucial aspect of the theory, have devised and so far unsuccessfully, though unwillingly, played a sort of « nuclear-matter game », whose rules are well summarized by DAY ([63]) « ... the single-particle potential

([61]) The relations existing between nucleon-nucleus and nucleon-nucleon scattering are reviewed by W. B. RIESENFELD and K. M. WATSON: *Phys. Rev.*, **102**, 1157 (1956). It has to be noted that according to this kind of model the real part of the potential can be constructed as the optical transform of the single-particle potential energy (2.1), the incertitudes arising from the use of the nucleon-nucleon asymptotic phase shifts have been pointed out in subsect. 7·2.

([62]) R. M. FRANK, J. L. GAMMEL and K. M. WATSON: *Phys. Rev.*, **101**, 891 (1956); R. M. STERNHEIMER: *Phys. Rev.*, **106**, 1027 (1957).

([63]) B. D. DAY: *Rev. Mod. Phys.*, **39**, 719 (1967), p. 738. The function indicated by this author with the symbol $U(k)$ corresponds to our $v(p, \varkappa)$.

energy is at our disposal; it is to be chosen with a view towards making the summation of the Brueckner-Goldston series as easy as possible. Certain diagrams contain U interactions and, if $U(k)$ is appropriately defined, these diagrams may cancel other diagrams. This cancellation reduces the number of diagrams that must be explicitly evaluated. This is the basic idea underlying any definition of $U(k)$. How one chooses $U(k)$, therefore, depends on which diagrams one decides to cancel by this choice. In making this decision, one should keep two points in mind. First, only certain types of diagrams are conveniently cancelled by diagrams involving $U(k)$. Second, choosing U to cancel the maximum number of diagrams is not necessarily the best procedure. One should have an idea of which diagrams are appreciable and which are negligible. Then, U should be defined so as to cancel as many of the *important* diagrams as possible. We, therefore, see that the choice of U is closely related to the question of the convergence of the Brueckner-Goldstone expansion. » Thus the theoretical game looks more like a conceptual gamble! This paper, which was first conceived with a pragmatic view, has been aiming *in itinere* more and more at finding a way out of such an astonishing approach to the nuclear-matter problem.

Our basic idea underlying the definition of $U(k) \equiv v(p, \varkappa)$ is that such a function has to be a particular integral of the differential equation of nuclear matter: such a definition, however approximate it may be, is nevertheless well grounded, because the Hugenholtz and Van Hove theorem, concealed within the mathematical structure of eq. (5.5), turns out to be exactly fulfilled by any single-particle potential energy consistent with the saturation requirements of infinitely extended nuclear matter. The choice of the physically significant $v(p, \varkappa)$ is determined by the energy dependence of the real optical potential. The criterion developed in sect. 7 for the simultaneous determination of the average volume energy, nuclear radius and compressibility modulus from the phenomenological evidence extracted from the nucleon-nucleus scattering experiments provides, if adroitly handled, a valuable heuristic tool so far completely ignored in the literature. It represents an attempt to patch up several fragmentary aspects of the nuclear-matter problem which have contributed to making it appear rather chaotic. The real and imaginary parts of the optical potential have been constructed consistently with eq. (5.5): in performing the related calculations we have put aside all formal theories together with their theoretical subtleties which generally prevent one from obtaining clear-cut numerical results. The extension of the differential equation of nuclear matter to finite nuclei is promising: a detailed account will be published in a forthcoming paper ([64]).

([64]) The differential equation (5.5) can be used to obtain the single-particle potential energy in a strongly degenerated nucleon gas at a nuclear temperature $T = 0$, provided the limiting momentum $\varkappa$ is assumed to be temperature dependent, *i.e.* $\varkappa \equiv \varkappa(T)$.

Recalling to mind the many years I have been engaged in this research, through the various events of my life, I feel I have somehow fared like biblical king Saul. Saul while looking for some donkeys found a kingdom; I merely wanted to perform some numerical calculations and came across the differential equation of nuclear matter: *unicuique suum*!

* * *

I thank Dr. L. SALVADORI for continuous assistance through the preparation of this paper and for taking care of its publication.

I am also grateful to Mrs. A. SPALLA, for carefully and patiently typing and re-typing the manuscript, and to Mrs. M. EVANS PROSPERI FLAVIANI, who helped to solve some linguistic subtleties.

From the normalization condition it is found, up to terms in T^2 (I. E. MAYER and M. G. MAYER: *Statistical Mechanics* (New York, 1950))

$$\varkappa_T(T) = \varkappa_F\{1 - (\pi^2/24)(T/\varepsilon_F)^2\},$$

where $\varkappa_F \equiv \varkappa(0)$ is the Fermi momentum consistent with nuclear saturation. The total energy of infinite nuclear matter reads

$$W(T, \varkappa_F) = \frac{3A}{\varkappa_F^3}\int_0^\infty p^2\left[\frac{p^2}{2M} + \frac{1}{2}v(p, \varkappa(T))\right]\varphi(e, T)\,\mathrm{d}p,$$

where $\varphi(e, T)$ is the Fermi-Dirac distribution function and $e = p^2/2M$. For illustrative purposes we assume the effective-mass approximation (7.4). The excitation energy, up to terms in T^2, is found to be

$$E(T, \varkappa_F) = W(T, \varkappa_F) - W(0, \varkappa_F) = b_F T^2,$$

$$b_F = a_F\{1 - (\varkappa_F^3/4\varepsilon_F)C_1^{(1)} + (\varkappa_F^5/10\varepsilon_F)C_3^{(1)}\},$$

where $a_F = \pi^2 A/4\varepsilon_F$ is the level density parameter of the conventional statistical theory; using eqs. (7.7) one also has

$$b_F = \{(\varepsilon_F - b_v)/2\varepsilon_F\}a_F < a_F.$$

The inequality $b_F < a_F$ has an important role in the thermodynamical description of the bulk properties of excited heavy nuclei; its physical meaning is that the momentum dependence of the single-particle potential energy reduces the specific heat of the excited Fermi sea as compared with that of a strongly degenerated nucleon gas moving in a constant potential well: consequently, the degrees of freedom of the nuclear-matter system as a whole are also reduced. It has been ascertained that such a correlated model, constructed on the basis of the factorable single-particle potential energy (6.21), provides a systematic explanation of the empirical adjustements of the nuclear-matter parameters required for fitting several experimental data (principal resonances in γ-ray absorption, rotational levels, average spacing of excited levels, de-excitation processes, evaporation spectra, etc.).

● RIASSUNTO

Scopo di questa nota è di delineare uno schema matematico per la descrizione del comportamento globale della materia nucleare infinita. Esso è stato concepito con lo scopo pragmatico di fornire un semplice strumento analitico per eseguire calcoli numerici attendibili. A questo scopo sono stati anche riesaminati alcuni aspetti finora insufficientemente esplorati di ben note teorie che sembrano essere del tutto ignorate dai piú recenti sviluppi teorici.

Математический подход к проблеме ядерного вещества.

Резюме (*). — Цель этой статьи — развитие математической схемы для описания поведения бесконечного ядерного вещества. Предложенный подход развит с прагматической целью получения простого метода для проведения реалистичных численных вычислений.

(*) *Переведено редакцией.*