Proc. Indian Acad. Sci. (Math. Sci.) Vol. 116, No. 3, August 2006, pp. 337–360.

Sobolev spaces associated to the harmonic oscillator

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MS received 27 September 2005

Abstract. We define the Hermite–Sobolev spaces naturally associated to the harmonic oscillator $H = -\Delta + |x|^2$. Structural properties, relations with the classical Sobolev spaces, boundedness of operators and almost everywhere convergence of solutions of the Schrödinger equation are also considered.

Keywords. Hermite operator; potential spaces; Riesz transforms.

1. Introduction

We consider the second-order differential operator

$$H = -\Delta + |x|^2, \quad x \in \mathbb{R}^d. \tag{1}$$

This operator is self-adjoint on the set of infinitely differentiable functions with compact support C_c^∞, and it can be factorized as

$$H = \frac{1}{2}\sum_{j=1}^{d} A_j A_{-j} + A_{-j} A_j, \tag{2}$$

where

$$A_j = \frac{\partial}{\partial x_j} + x_j \quad \text{and} \quad A_{-j} = A_j^* = -\frac{\partial}{\partial x_j} + x_j, \quad 1 \le j \le d.$$

In the last few years several authors have been concerned with the harmonic analysis associated to the operator H (see for instance [5,10,12]). In this analysis the operators A_j play the role of the partial derivative operators $\partial/\partial x_j$ in the classical Euclidean case. Hence it seems natural to study the spaces of functions in $L^p(\mathbb{R}^d)$ whose *derivatives* also belong to $L^p(\mathbb{R}^d)$. Following this idea, we introduce the Hermite–Sobolev spaces $W^{k,p}$ (Definition 1). These spaces are Banach spaces and the set of linear combinations of Hermite functions is dense in any of them (Proposition 1). The spaces $W^{k,p}$ were previously studied in [14] for $p = 2$ and in [7] for $p \neq 2$.

Once we have the *Laplacian* H, it is also natural to consider the *potential spaces* $\mathfrak{L}_a^p = H^{-a/2}(L^p(\mathbb{R}^d))$ (Definition 3). In other words, the range of the *Hermite fractional integral* operator $H^{-a/2}$ in $L^p(\mathbb{R}^d)$. In order to have a satisfactory description of these potential spaces we need a sharp analysis of the operator H^{-a}, for $a > 0$. Such analysis is contained in Proposition 2 and Lemma 3. It turns out that, for $k \in \mathbb{N}$, the spaces $\mathfrak{L}_k^p$ and $W^{k,p}$ coincide (see Theorem 4). The proof of this theorem uses the boundedness in $L^p(\mathbb{R}^d)$ of the Riesz transforms naturally associated to H. These Riesz transforms were introduced by Thangavelu in [12], and some of their boundedness properties can be found in [5] and [10].

Observe that, in some sense, Theorem 4 allows us to say that the spaces $\mathfrak{L}_a^p$ are the spaces of functions in $L^p(\mathbb{R}^d)$ for which their *derivatives of order a* also belong to $L^p(\mathbb{R}^d)$.

Once we have a satisfactory definition of Hermite–Sobolev (or Hermite *potential*) spaces and hence of *fractional* derivatives, we study their relationship with the corresponding classical Euclidean spaces. We show in Theorem 3 that although the Hermite–Sobolev spaces coincide locally with the Euclidean Sobolev spaces, they are in fact different. In §5, we show that the Hermite–Riesz transforms are bounded on the Hermite–Sobolev spaces while the classical Hilbert transform is not bounded on these spaces. From the careful analysis of the kernel of H^{-a}, we also obtain certain inequalities of *Poincaré type* for the *derivatives* A_j in Theorem 9. Finally, in §7 we give an application to the almost everywhere convergence of the solution of the Schrödinger equation (42) to the initial data.

Our work was heavily inspired in the paper by Thangavelu [14], where the spaces $\mathfrak{L}_a^2$ were defined. These spaces were also considered in [6].

As we said above, in order to develop this work, some nontrivial estimates of the Hermite fractional integral operator (Definition 2) were needed. However, it is not the aim of this paper to make an exhaustive study of this operator. Hence, other natural questions like weak and strong boundedness in the extreme points or BMO-type boundedness of the operator H^{-a} are left aside and they will be the motivation of a forthcoming paper.

2. Hermite–Sobolev spaces

Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and consider the Hermite function of order n,

$$h_n(t) = \frac{(-1)^n}{(2^n n! \pi^{1/2})^{1/2}} H_n(t)\, \mathrm{e}^{-t^2/2}, \quad t \in \mathbb{R},$$

where H_n denotes the Hermite polynomial of degree n (see [12]). Given a multi-index $\alpha = (\alpha_j)_{j=1}^d \in \mathbb{N}_0^d$, we consider the Hermite function, h_α, as

$$h_\alpha(x) = \prod_{j=1}^d h_{\alpha_j}(x_j), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

These functions are eigenvectors of the Hermite operator H defined in (1). In fact

$$H h_\alpha = (2|\alpha| + d)\, h_\alpha,$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$. Moreover, for $1 \le j \le d$,

$$A_j h_\alpha = \sqrt{2\alpha_j}\, h_{\alpha - e_j}, \quad A_{-j} h_\alpha = \sqrt{2(\alpha_j + 1)}\, h_{\alpha + e_j},$$

where e_j is the jth coordinate vector in $\mathbb{N}_0^d$. The operators A_j and A_{-j} are called *annihilation* and *creation* operators respectively.

DEFINITION 1

Given $p \in (1, \infty)$ and $k \in \mathbb{N}$, we define the Hermite–Sobolev space of order k, denoted by $W^{k,p}$, as the set of functions $f \in L^p(\mathbb{R}^d)$ such that

$$A_{j_1} \cdots A_{j_m} f \in L^p(\mathbb{R}^d), \quad 1 \le |j_1|, \ldots, |j_m| \le d, \quad 1 \le m \le k,$$

with the norm

$$\|f\|_{W^{k,p}} = \sum_{1 \le |j_1|, \ldots, |j_m| \le d,\ 1 \le m \le k} \|A_{j_1} \cdots A_{j_m} f\|_p + \|f\|_p.$$

We will show that the set of finite linear combinations of Hermite functions, denoted by $\mathfrak{F}$, is dense in the Hermite–Sobolev spaces. We shall need the following lemmas. Their proofs may be found in [10] and [12], respectively.

Lemma 1. *Let $m \in \mathbb{N}_0$ and $f \in C_c^\infty$. There exists a constant $C_{m,f} > 0$ such that*

$$|\langle f, h_\alpha \rangle| \le C_{m,f}\, (|\alpha| + 1)^m, \quad \alpha \in \mathbb{N}_0^d.$$

Lemma 2. *As $n \to \infty$ the Hermite functions satisfy the estimates*

(i) $\|h_n\|_p \sim n^{\frac{1}{2p} - \frac{1}{4}}, \quad 1 \le p < 4,$

(ii) $\|h_n\|_p \sim n^{-\frac{1}{8}} \log(n), \quad p = 4,$

(iii) $\|h_n\|_p \sim n^{\frac{1}{6p} - \frac{1}{12}}, \quad 4 < p \le \infty.$

PROPOSITION 1

Let p be in the range $1 < p < \infty$ and $k \in \mathbb{N}$. The set W_p^k is a Banach space. Moreover, the sets $\mathfrak{F}$ and C_c^∞ are dense in $W^{k,p}$.

Proof. To see that $W^{1,p}$ is complete, observe that if $\{f_n\}_{n\ge1}$ is a Cauchy sequence in $W^{1,p}$, then

$$\left\{\frac{\partial f_n}{\partial x_j}\right\}_{n\ge1} \quad \text{and} \quad \{x_j f_n\}_{n\ge1}, \quad 1 \le j \le d, \tag{3}$$

are Cauchy sequences in $L^p(\mathbb{R}^d)$. If we call f the limit in $L^p(\mathbb{R}^d)$ of $\{f_n\}_{n\ge1}$, it is easy to see that $\partial f/\partial x_j$ and $x_j f$ are respectively the limits in $L^p(\mathbb{R}^d)$ of (3) (see [8], p. 122).

Now we shall see that C_c^∞ is a dense set in $W^{1,p}$. Let ψ be a function in C_c^∞ such that $\int_{\mathbb{R}^d} \psi = 1$. For every $\epsilon > 0$, consider

$$\psi_\epsilon(x) = \frac{1}{\epsilon^d}\, \psi\left(\frac{x}{\epsilon}\right).$$

Given f in $W^{1,p}$, define $f_\epsilon = f * \psi_\epsilon$. Following the ideas in p. 123 of [8], we have

$$\|f - f_\epsilon\|_p \to 0 \quad \text{and} \quad \left\| \frac{\partial}{\partial x_j} f_\epsilon - \frac{\partial}{\partial x_j} f \right\|_p \to 0, \quad 1 \le j \le d.$$

On the other hand, for $1 \leq j \leq d$, we call $\phi^j(x) = x_j\, \psi(x)$ and $\phi^j_\epsilon(x) = \frac{1}{\epsilon^d}\, \phi^j\left(\frac{x}{\epsilon}\right)$, then the function $\phi^j \in C_c^\infty$, and for $\epsilon > 0$,

$$\begin{aligned}\|(x_j f) * \psi_\epsilon - x_j f_\epsilon\|_p^p &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y)(y_j - x_j) \frac{1}{\epsilon^d} \psi\left(\frac{x-y}{\epsilon}\right) \mathrm{d}y \right|^p \mathrm{d}x \\ &= \epsilon^p \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) \frac{1}{\epsilon^d} \phi^j\left(\frac{x-y}{\epsilon}\right) \mathrm{d}y \right|^p \mathrm{d}x \\ &= \epsilon^p\, \|f * \phi^j_\epsilon\|_p^p. \end{aligned} \tag{4}$$

Moreover, since $x_j f$ belongs to $L^p(\mathbb{R}^d)$,

$$\|(x_j f) * \psi_\epsilon - x_j f\|_p \to 0. \tag{5}$$

Equations (4) and (5) imply $x_j f_\epsilon \to x_j f$ in $L^p(\mathbb{R}^d)$. Therefore, we conclude that $f_\epsilon \in W^{1,p}$ and $f_\epsilon \to f$ in the $W^{1,p}$-norm.

The functions f_ϵ do not necessarily have compact support, but they can be modified as in the classical case (see [8], p. 123).

It remains to prove that any function in C_c^∞ can be approximated in the $W^{1,p}$-norm by a function in $\mathfrak{F}$. In fact, we will show that any $f \in C_c^\infty$ is the limit, in the $W^{p,1}$-norm, of a subsequence of the partial sums

$$S_N f = \sum_{|\alpha| \leq N} \langle f, h_\alpha \rangle\, h_\alpha.$$

In [10], it is proved that there exists a subsequence of the previous sequence converging to f in the L^p-norm. Hence, it is enough to show that there exists a subsequence of $\{A_j(S_N(f))\}_{N\geq 1} = \{S_N(A_j f)\}_{N\geq 1}$ converging to $A_j f$ in the L^p-norm, for every j with $1 \leq |j| \leq d$.

Let us fix j to be in $1 \leq j \leq d$ (the case $-d \leq j \leq -1$ is similar). The sequence $\{S_N(A_j f)\}_{N\geq 1}$ converges to $A_j f$ in the L^2-norm. Hence we can take a subsequence $\{S_{N_k}(A_j f)\}_{k\geq 1}$ converging to $A_j f$ almost everywhere. Since

$$\begin{aligned} S_N(A_j f) &= \sum_{|\alpha|\leq N} \langle A_j f, h_\alpha \rangle h_\alpha = \sum_{|\alpha|\leq N} \langle f, A_{-j} h_\alpha \rangle h_\alpha \\ &= \sum_{|\alpha|\leq N} \sqrt{2(\alpha_{-j}+1)} \langle f, h_{\alpha+e_j} \rangle h_\alpha, \end{aligned}$$

by Lemma 1 and Hölder's inequality,

$$\begin{aligned} |S_N(A_j f)|^p &\leq C \left(\sum_{|\alpha|\leq N} \sqrt{2(\alpha_{-j}+1)} (|\alpha|+2)^{-M} |h_\alpha| \right)^p \\ &\leq C \left(\sum_{|\alpha|\leq N} (|\alpha|+1)^{-M+1} |h_\alpha| \right)^p \end{aligned}$$

$$\leq C \left(\sum_{|\alpha| \leq N} (|\alpha| + 1)^{-M+1} \right)^{p/p'} \sum_{|\alpha| \leq N} (|\alpha| + 1)^{-M+1} |h_\alpha|^p$$

$$\leq C \sum_{\alpha} (|\alpha| + 1)^{-M+1} |h_\alpha|^p.$$

From Lemma 2, it is easy to see that the function $\sum_\alpha (|\alpha| + 1)^{-M+1} |h_\alpha|^p$ belongs to $L^1(\mathbb{R}^d, \mathrm{d}x)$, and the dominated convergence theorem gives that $\{S_{N_k}(A_j f)\}_{k \geq 1}$ converges to $A_j f$ in the L^p-norm. Now we can repeat the argument for every j, taking a subsequence of the previous subsequence in each step.

For $k > 1$ we leave the details to the reader. □

3. Fractional integral

With the ideas in [8] and [10], we introduce the following operator.

DEFINITION 2

Given $a > 0$, we define for $f \in \mathfrak{F}$, the operator

$$H^{-a} f(x) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-tH} f(x)\, t^a \frac{\mathrm{d}t}{t}, \quad x \in \mathbb{R}^d, \tag{6}$$

where $\{\mathrm{e}^{-tH}\}_{t \geq 0}$ is the heat semi-group associated to H.

Remark 1. If $a > 0$ and $\alpha \in \mathbb{N}_0^d$, by using the Γ function and the fact

$$\mathrm{e}^{-tH} h_\alpha = \mathrm{e}^{-t(2|\alpha|+d)} h_\alpha,$$

we have

$$H^{-a} h_\alpha(x) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-tH} h_\alpha(x)\, t^a \frac{\mathrm{d}t}{t} = (2|\alpha| + d)^{-a} h_\alpha(x), \quad x \in \mathbb{R}^d.$$

PROPOSITION 2

The operator H^{-a} has integral representation

$$H^{-a} f(x) = \int_{\mathbb{R}^d} K_a(x, y)\, f(y)\, \mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{7}$$

for all $f \in \mathfrak{F}$. Moreover, there exist Φ_a in $L^1(\mathbb{R}^d)$ and a constant C such that

$$K_a(x, y) \leq C\, \Phi_a(x - y), \quad \textit{for all } x, y \in \mathbb{R}^d. \tag{8}$$

Proof. If $f \in \mathfrak{F}$, then for $x \in \mathbb{R}^d$,

$$H^{-a} f(x) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-tH} f(x)\, t^a \frac{\mathrm{d}t}{t}$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty \int_{\mathbb{R}^d} G_t(x, y)\, f(y)\, \mathrm{d}y\, t^a \frac{\mathrm{d}t}{t},$$

where

$$G_t(x,y) = (2\pi \sinh 2t)^{-d/2} e^{-\frac{1}{2}|x-y|^2 \coth 2t - x\cdot y \tanh t},$$

for $x, y \in \mathbb{R}^d$, (see [10]). Therefore, if we show that for some constant C,

$$\frac{1}{\Gamma(a)} \int_0^\infty G_t(x,y)\, t^a \,\frac{dt}{t} \le C\, \Phi_a(x-y), \quad \text{for all } x, y \in \mathbb{R}^d,$$

where $\Phi_a \in L^1(\mathbb{R}^d)$, then by Fubini's theorem,

$$H^{-a} f(x) = \int_{\mathbb{R}^d} K_a(x,y)\, f(y)\, dy, \quad x \in \mathbb{R}^d,$$

with

$$K_a(x,y) = \frac{1}{\Gamma(a)} \int_0^\infty G_t(x,y)\, t^a \frac{dt}{t}. \tag{9}$$

We perform the change of variables

$$t = \frac{1}{2} \log\left(\frac{1+s}{1-s}\right),$$

then

$$K_a(x,y) = \frac{1}{\Gamma(a)(4\pi)^{d/2} 2^{a-1}} \int_0^1 \zeta_a(s)\, e^{-\frac{1}{4}\left(s|x+y|^2 + \frac{1}{s}|x-y|^2\right)} \frac{ds}{s}, \tag{10}$$

where

$$\zeta_a(s) = \left(\frac{1-s^2}{s}\right)^{\frac{d}{2}-1} \log\left(\frac{1+s}{1-s}\right)^{a-1}.$$

We split K_a as $K_a = K_{a,0} + K_{a,1}$, where

$$K_{a,0}(x,y) = \frac{1}{\Gamma(a)(4\pi)^{d/2} 2^{a-1}} \int_0^{1/2} \zeta_a(s)\, e^{-\frac{1}{4}\left(s|x+y|^2 + \frac{1}{s}|x-y|^2\right)} \frac{ds}{s}. \tag{11}$$

Since the integral

$$\int_{1/2}^1 \zeta_a(s) \frac{ds}{s} < \infty,$$

we have

$$\begin{aligned} K_{a,1}(x,y) &= \frac{1}{\Gamma(a)(4\pi)^{d/2} 2^{a-1}} \int_{1/2}^1 \zeta_a(s)\, e^{-\frac{1}{4}\left(s|x+y|^2 + \frac{1}{s}|x-y|^2\right)} \frac{ds}{s} \\ &\le C\, e^{-\frac{1}{4}|x-y|^2} e^{-\frac{1}{8}|x+y|^2}. \end{aligned} \tag{12}$$

It is easy to see that there exists a constant C_1 which depends on d and a such that

$$\frac{s^{-\frac{d}{2}+a}}{C_1} \le \zeta_a(s) \le C_1 s^{-\frac{d}{2}+a} \quad \text{for } 0 < s < 1/2. \tag{13}$$

Therefore

$$K_{a,0}(x, y) \leq C_1 \int_0^{1/2} s^{-\frac{d}{2}+a} \, \mathrm{e}^{-\frac{1}{4s}|x-y|^2} \, \frac{\mathrm{d}s}{s}.$$

If $|x - y| \geq 1$, the last expression is bounded up to a constant by

$$\mathrm{e}^{-\frac{1}{4}|x-y|^2} \int_0^{1/2} s^{-\frac{d}{2}+a} \, \mathrm{e}^{-\frac{1}{8s}} \, \frac{\mathrm{d}s}{s},$$

hence,

$$K_{a,0}(x, y) \leq C \, \mathrm{e}^{-\frac{1}{4}|x-y|^2}. \tag{14}$$

Let us study now the region $|x - y| < 1$. By a change of variables,

$$\int_0^{1/2} s^{-\frac{d}{2}+a} \, \mathrm{e}^{-\frac{1}{4s}|x-y|^2} \, \frac{\mathrm{d}s}{s} = \frac{1}{|x-y|^{d-2a}} \int_{2|x-y|^2}^{\infty} s^{\frac{d}{2}-a} \, \mathrm{e}^{-\frac{s}{4}} \, \frac{\mathrm{d}s}{s}.$$

In the case $a < (d/2)$, we get

$$K_{a,0}(x, y) \leq \frac{C}{|x-y|^{d-2a}}. \tag{15}$$

For the case $a = (d/2)$,

$$\begin{aligned}
\int_{2|x-y|^2}^{\infty} \mathrm{e}^{-\frac{s}{4}} \, \frac{\mathrm{d}s}{s} &= \int_{2|x-y|^2}^{2} \frac{\mathrm{d}s}{s} + \int_2^{\infty} \mathrm{e}^{-\frac{s}{4}} \, \frac{\mathrm{d}s}{s} \\
&= \log\left(\frac{1}{|x-y|^2}\right) + \int_2^{\infty} \mathrm{e}^{-\frac{s}{4}} \, \frac{\mathrm{d}s}{s} \\
&\leq 2 \log\left(\frac{e}{|x-y|}\right) \left(1 + \int_2^{\infty} \mathrm{e}^{-\frac{s}{4}} \, \frac{\mathrm{d}s}{s}\right).
\end{aligned}$$

Then,

$$K_{a,0}(x, y) \leq C \, \log\left(\frac{e}{|x-y|}\right). \tag{16}$$

Finally, when $a > (d/2)$,

$$\begin{aligned}
\int_{2|x-y|^2}^{\infty} s^{\frac{d}{2}-a} \, \mathrm{e}^{-\frac{s}{4}} \, \frac{\mathrm{d}s}{s} &\leq \int_{2|x-y|^2}^{2} s^{\frac{d}{2}-a} \, \frac{\mathrm{d}s}{s} + \int_2^{\infty} s^{\frac{d}{2}-a} \, \mathrm{e}^{-\frac{s}{4}} \, \frac{\mathrm{d}s}{s} \\
&\leq \frac{1}{2^{2a-d} \left(a - \frac{d}{2}\right) |x-y|^{2a-d}} + \int_2^{\infty} s^{\frac{d}{2}-a} \, \mathrm{e}^{-\frac{s}{4}} \, \frac{\mathrm{d}s}{s}.
\end{aligned}$$

Thus,

$$K_{a,0}(x, y) \leq C. \tag{17}$$

Therefore, from (12), (14)–(17), if we define for $x \in \mathbb{R}^d$,

$$\Phi_a(x) = \begin{cases} \frac{\chi_{\{|x|<1\}}(x)}{|x|^{d-2a}} + \mathrm{e}^{\frac{-|x|^2}{4}} \chi_{\{|x|\geq 1\}}(x), & \text{if } a < \frac{d}{2}, \\ \log\left(\frac{e}{|x|}\right) \chi_{\{|x|<1\}}(x) + \mathrm{e}^{\frac{-|x|^2}{4}} \chi_{\{|x|\geq 1\}}(x), & \text{if } a = \frac{d}{2}, \\ \chi_{\{|x|<1\}}(x) + \mathrm{e}^{\frac{-|x|^2}{4}} \chi_{\{|x|\geq 1\}}(x), & \text{if } a > \frac{d}{2}, \end{cases} \tag{18}$$

we have proved (8). □

Theorem 1. *The operator H^{-a} defined by* (6), *is well-defined and bounded on $L^p(\mathbb{R}^d)$, for all $p \in [1, \infty]$. Moreover, for all f in $L^p(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$, we have*

$$\int_{\mathbb{R}^d} H^{-a} f\, h_\alpha = (2|\alpha| + d)^{-a} \int_{\mathbb{R}^d} f\, h_\alpha. \tag{19}$$

Proof. The boundedness of the operator H^{-a} on $L^p(\mathbb{R}^d)$ is due to the fact that the kernel K_a is bounded by an integrable function. To see (19), let $\alpha \in \mathbb{N}_0^d$. By Proposition 2 and Hölder's inequality,

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |K_a(x, y)|\, |f(y)|\, |h_\alpha(x)|\, \mathrm{d}y\, \mathrm{d}x \\ &\leq C \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\Phi_a(x - y)|\, |f(y)|\, |h_\alpha(x)|\, \mathrm{d}y\, \mathrm{d}x \\ &\leq C\, \|f\|_p \|h_\alpha\|_{p'}, \end{aligned}$$

where the constant C depends on α. Therefore, by Fubini's theorem,

$$\int_{\mathbb{R}^d} H^{-a} f\, h_\alpha = \int_{\mathbb{R}^d} f\, H^{-a} h_\alpha = (2|\alpha| + d)^{-a} \int_{\mathbb{R}^d} f\, h_\alpha.$$

□

Lemma 3. *Let $p \in [1, \infty]$ and $a > 0$, then the operator*

$$|x|^{2a} H^{-a} f \tag{20}$$

is bounded on $L^p(\mathbb{R}^d)$.

Proof. We will see that the kernel of (20) satisfies

$$|x|^{2a} \int_{\mathbb{R}^n} K_a(x, y)\, \mathrm{d}y \;\leq\; C \tag{21}$$

and

$$\int_{\mathbb{R}^n} |x|^{2a} K_a(x, y)\, \mathrm{d}x \;\leq\; C, \tag{22}$$

where the constant C depends only on d and a. Thus, $|x|^{2a} H^{-a} f$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$ (Theorem 6.18 in [4]).

We deal first with (21). From (8) and (18) we see that there exists a constant C such that (21) is valid for all $|x| < 2$.

Assume that $|x| < 2|x - y|$. By using eq. (10) we obtain

$$K_a(x, y) \le \frac{\mathrm{e}^{-\frac{|x-y|^2}{8}}}{\Gamma(a)(4\pi)^{d/2}2^{a-1}} \int_0^1 \zeta_a(s)\, \mathrm{e}^{-\frac{1}{8}\left(s|x+y|^2+\frac{1}{s}|x-y|^2\right)} \frac{\mathrm{d}s}{s}$$
$$= \mathrm{e}^{-\frac{|x-y|^2}{8}} K_a\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right).$$

Therefore, for some constant C,

$$\int_{\{|x|<2|x-y|\}} |x|^{2a} K_a(x, y)\,\mathrm{d}y \le \int_{\mathbb{R}^d} |x-y|^{2a} \mathrm{e}^{-\frac{|x-y|^2}{8}} K_a\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) \mathrm{d}y$$
$$\le C \int_{\mathbb{R}^d} K_a\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) \mathrm{d}y,$$

which is bounded by a constant independent of x.

It remains to consider integral (21) restricted to the set $E_x = \{y\colon |x| > 2|x - y|\}$ when $|x| > 2$. Observe that in this part, due to the identity $|x + y|^2 = 2|x|^2 - |x - y|^2 + 2|y|^2$, we have

$$|x| < |x + y|. \tag{23}$$

As in the proof of Proposition 2, we consider $K_a = K_{a,0} + K_{a,1}$. Then, by using (23),

$$|x|^{2a} \int_{E_x} K_{a,1}(x, y)\mathrm{d}y \le C|x|^{2a}\mathrm{e}^{-\frac{1}{8}|x|^2} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{4}|x-y|^2}\mathrm{d}y \le C,$$

where in the last inequality we have used that for each positive b, there exists a constant C_b such that $|x|^b\mathrm{e}^{-|x|} \le C_b$.

In order to handle $K_{a,0}$, from (13) and (23), after some changes of variables we obtain

$$\int_{E_x} K_{a,0}(x, y)\mathrm{d}y \le C \int_{E_x} \int_0^{1/2} s^{-\frac{d}{2}+a}\mathrm{e}^{-\frac{1}{4}\left(s|x|^2+\frac{1}{s}|x-y|^2\right)} \frac{\mathrm{d}s}{s}\mathrm{d}y$$
$$= C\,|x|^{d-2a} \int_{E_x} \int_0^{|x|^2/2} u^{-\frac{d}{2}+a}\mathrm{e}^{-\frac{1}{4}\left(u+\frac{1}{u}(|x||x-y|)^2\right)} \frac{\mathrm{d}u}{u}\mathrm{d}y$$
$$= C\,|x|^{-2a} \int_0^{|x|^2/2} \int_0^{|x|^2/2} u^{-\frac{d}{2}+a}\mathrm{e}^{-\frac{1}{4}\left(u+\frac{r^2}{u}\right)} \frac{\mathrm{d}u}{u} r^d \frac{\mathrm{d}r}{r}.$$

The last double integral is bounded by

$$\int_0^\infty \int_0^\infty u^{-\frac{d}{2}+a}\mathrm{e}^{-\frac{1}{4}\left(u+\frac{r^2}{u}\right)} \frac{\mathrm{d}u}{u}\, r^d\, \frac{\mathrm{d}r}{r} = \int_0^\infty u^{-\frac{d}{2}+a}\mathrm{e}^{-\frac{u}{4}} \int_0^\infty \mathrm{e}^{-\frac{r^2}{4u}}\, r^d\, \frac{\mathrm{d}r}{r}\, \frac{\mathrm{d}u}{u}$$
$$= \int_0^\infty \mathrm{e}^{-\frac{u}{4}}\, u^a\, \frac{\mathrm{d}u}{u} \int_0^\infty \mathrm{e}^{-\frac{r^2}{4}}\, r^d\, \frac{\mathrm{d}r}{r},$$

and both integrals clearly converge.

Finally we shall prove (22). We split

$$\int_{\mathbb{R}^d} |x|^{2a} K_a(x, y)\,\mathrm{d}x = \left(\int_{\{|y|>2|x-y|\}} + \int_{\{|y|<2|x-y|\}}\right) |x|^{2a} K_a(x, y)\,\mathrm{d}x. \tag{24}$$

Since $|x| \le \frac{3}{2}|y|$ when $|y| > 2|x-y|$ and the kernel K_a is symmetric, the first integral of (24) is less than or equal to

$$\left(\frac{3}{2}\,|y|\right)^{2a} \int_{\{|y|>2|x-y|\}} K_a(y, x)\,\mathrm{d}x,$$

which can be bounded, in the same way as (21), by a constant depending only on d and a. For the second term of (24), we have

$$\int_{\{|y|<2|x-y|<4\}} |x|^{2a} K_a(x, y)\,\mathrm{d}x \ \le\ C \int_{\mathbb{R}^d} K_a(x, y)\,\mathrm{d}x.$$

On the other hand, by estimate (8) and the expression (18),

$$\begin{aligned}\int_{\{|y|<2|x-y|,\ 4<|x-y|\}} |x|^{2a} K_a(x, y)\,\mathrm{d}x &\le C \int_{\{4<|x-y|\}} |x-y|^{2a} \mathrm{e}^{-\frac{|x-y|^2}{8}}\,\mathrm{d}x \\ &\le C \int_{\{4<|x|\}} |x|^{2a} \mathrm{e}^{-\frac{|x|^2}{8}}\,\mathrm{d}x \\ &\le C.\end{aligned}$$

Then, we have proved (22). □

4. Potential spaces

DEFINITION 3

Given $p \in [1, \infty)$ and $a > 0$, we define the space

$$\mathfrak{L}_a^p = H^{-a/2}(L^p(\mathbb{R}^d)),$$

with a norm given by

$$\|f\|_{\mathfrak{L}_a^p} = \|g\|_p,$$

where g is such that $H^{-a/2} g = f$.

Remark 2. The space $\mathfrak{L}_a^p$ is well-defined for $p \in [1, \infty)$ and $a > 0$, since $H^{-a/2}$ is one-to-one. As $\mathfrak{F} = H^{-a/2}(\mathfrak{F})$ then $\mathfrak{F}$ is a dense space of $\mathfrak{L}_a^p$.

Due to the boundedness of the operator H^{-a}, the space $\mathfrak{L}_a^p$ is a subspace of $L^p(\mathbb{R}^d)$. Moreover, we have the following theorem.

Theorem 2. *Let* $0 < a < b$, *then*

(i) $\mathfrak{L}_b^p \subset \mathfrak{L}_a^p \subset L^p$ *and the inclusions are continuous.*
(ii) $\mathfrak{L}_a^p$ *and* $\mathfrak{L}_b^p$ *are isometrically isomorphic.*

Proof. Since $H^{-b/2} = H^{-a/2} \circ H^{-\gamma}$, where $\gamma = (b-a)/2$, we see that (i) easily follows from the definition of $\mathfrak{L}_a^p$ and $\mathfrak{L}_b^p$, and the boundedness of $H^{-\gamma}$. From Definition 3 and the fact that $H^{-\gamma}$ is one-to-one, it is easy to verify that $H^{-\gamma}\colon \mathfrak{L}_a^p \mapsto \mathfrak{L}_b^p$ is an isometric isomorphism, and this gives (ii). □

The members of $\mathfrak{L}_a^p$ have a special decay at infinity, as the following proposition shows.

PROPOSITION 3

If $p \in [1, \infty)$, $a > 0$ *and* $f \in \mathfrak{L}_a^p$, *then* $|x|^a f(x)$ *belongs to* $L^p(\mathbb{R}^d)$.

Proof. By Definition 3, we have $f = H^{-a/2}g$ with $g \in L^p(\mathbb{R}^d)$. Then it is enough to apply Lemma 3. □

We remind the classical Sobolev spaces L_a^p, with p in the range $1 \le p < \infty$ and $a > 0$, defined by

$$L_a^p = (I - \Delta)^{-a/2}(L^p(\mathbb{R}^d)). \tag{25}$$

In this case, the norm for $f \in L_a^p$ is given by

$$\|f\|_{L_a^p} = \|g\|_p,$$

where the function $g \in L^p$ is such that $(I - \Delta)^{-a/2}g = f$ (see [8]). In the following theorem we describe the relation between the spaces L_a^p and $\mathfrak{L}_a^p$.

Theorem 3. *Let* $a > 0$ *and* $p \in (1, \infty)$, *then*

(i) $\mathfrak{L}_a^p \subset L_a^p$.
(ii) $\mathfrak{L}_a^p \neq L_a^p$.
(iii) *If* $f \in L_a^p$ *and has compact support, then* f *belongs to* $\mathfrak{L}_a^p$.

Proof. To see (1) we follow the argument in [14]. Namely, the symbol of

$$(I - \Delta)^{a/2}H^{-a/2}$$

belongs to the class $S_{1,0}^0$ and so it defines a bounded operator on $L^p(\mathbb{R}^d)$ (see [9]). Let $f \in \mathfrak{L}_a^p$, then $h = [(I-\Delta)^{a/2}H^{-a/2}](H^{a/2}f)$ is a function of $L^p(\mathbb{R}^d)$ with $(I-\Delta)^{-a/2}h = f$. Hence f belongs to L_a^p.

In order to see (ii) let

$$g(x) = \frac{1}{(1+|x|)^{1/p+a}} \quad \text{and} \quad f = (I - \Delta)^{-a/2}g.$$

Since $g \in L^p(\mathbb{R}^d)$, then $f \in L^p_a$. We will see that f is not in $\mathfrak{L}^p_a$. From Proposition 3, if f were in $\mathfrak{L}^p_a$ we would have $|x|^a f \in L^p(\mathbb{R}^d)$. However, if $\mathcal{G}_a$ is the kernel of $(I-\Delta)^{-a/2}$,

$$\mathcal{G}_a(x) = \frac{1}{(4\pi)^{a/2}\Gamma(a/2)} \int_0^\infty \mathrm{e}^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} t^{\frac{-d+a}{2}} \frac{\mathrm{d}t}{t},$$

then

$$\begin{aligned} f(x) = (I-\Delta)^{-a/2} g(x) &= \int_{\mathbb{R}^d} \mathcal{G}_a(y) g(x-y)\,\mathrm{d}y \\ &\geq \int_{\{|y|<1\}} \mathcal{G}_a(y)\, g(x-y)\,\mathrm{d}y \geq (2+|x|)^{-1/p-a} \int_{\{|y|<1\}} \mathcal{G}_a(y)\,\mathrm{d}y. \end{aligned}$$

Thus, $|x|^a f$ is not in $L^p(\mathbb{R}^d)$.

Finally, (iii) is a direct consequence of the following inequality,

$$\int_{\mathbb{R}^d} |H^a(\psi f)|^p \leq C(\psi) \int_{\mathbb{R}^d} |(-\Delta + I)^a f|^p,$$

where $\psi \in C_c^\infty$ and $C(\psi)$ is a constant depending on ψ. This inequality was proved in [14] for $p = 2$ and the same proof works for p in the range $1 < p < \infty$. $\square$

Theorem 4. *Let $k \in \mathbb{N}$ and $p \in (1,\infty)$, then*

$$W^{k,p} = \mathfrak{L}^p_k$$

and the norms $\|\cdot\|_{W^{k,p}}$ and $\|\cdot\|_{\mathfrak{L}^p_k}$ are equivalent.

We first present some technical results that we shall need for the proof of this theorem.

Lemma 4. *Let $b \in \mathbb{R}$, then for all f in $\mathfrak{F}$, we have*

$$A_j H^b f = (H+2)^b A_j f, \quad 1 \leq j \leq d, \tag{26}$$

$$A_j H^b f = (H-2)^b A_j f, \quad -d \leq j \leq -1, \tag{27}$$

$$H^b A_j f = A_j (H-2)^b f, \quad 1 \leq j \leq d, \tag{28}$$

and

$$H^b A_j f = A_j (H+2)^b f, \quad -d \leq j \leq -1, \tag{29}$$

where $H^b h_\alpha = (2|\alpha| + d)^b h_\alpha$ and $(H+2)^b h_\alpha = (2|\alpha| + d + 2)^b h_\alpha$, for all $\alpha \in \mathbb{N}_0^d$, and $(H-2)^b h_\alpha = (2|\alpha| + d - 2)^b h_\alpha$, for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \geq 1$.

Proof. Let $1 \leq j \leq d$ and $\alpha \in \mathbb{N}_0^d$, then

$$\begin{aligned} A_j H^b h_\alpha &= (2|\alpha| + d)^b A_j h_\alpha = \sqrt{2\alpha_j}\,(2|\alpha| + d)^b\, h_{\alpha - e_j} \\ &= \sqrt{2\alpha_j}\,(2(|\alpha| - 1) + d + 2)^b\, h_{\alpha - e_j} = \sqrt{2\alpha_j}\,(H+2)^b h_{\alpha - e_j} \\ &= (H+2)^b A_j h_\alpha, \end{aligned}$$

and this gives (26) by linearity. In the same way we obtain (27) and (29). We are assuming in (28) that f is a linear combination of Hermite functions with order $|\alpha| \geq 1$. $\square$

The Hermite–Riesz transforms associated to H are defined as

$$R_j = A_j H^{-1/2}, \quad 1 \leq |j| \leq d$$

and the Hermite–Riesz transform vector

$$R = (R_{-d}, R_{-d+1}, \ldots, R_{-1}, R_1, \ldots, R_{d-1}, R_d).$$

These operators were introduced by Thangavelu in [12] (see also [10]). He proved that they are bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, and of weak type $(1, 1)$. Given $m \in \mathbb{N}$ the Hermite–Riesz of order m is defined as

$$R_{j_1, j_2, \ldots, j_m} = A_{j_1} A_{j_2} \cdots A_{j_m} H^{-m/2}, \tag{30}$$

where $1 \leq |j_n| \leq d$, for every $1 \leq n \leq m$.

For further references, we enounce the following crystallized theorem of known facts about the Hermite–Riesz transforms.

Theorem 5.

(a) *The Hermite–Riesz transforms R_j, $1 \leq |j| \leq d$, are pseudo-differential operators whose symbols belong to $S^0_{1,0}$. In particular, they are bounded in the classical Sobolev spaces* (25).

(b) *Let $m \in \mathbb{N}$ and $p \in (1, \infty)$. Then there exists a constant $C_{p,m}$ not depending on the dimension d, such that*

$$\left\| \left(\sum_{1 \leq |j_1|, \ldots, |j_m| \leq d} |R_{j_1, \ldots, j_m} f|^2 \right)^{1/2} \right\|_p \leq C_{p,m} \|f\|_p.$$

Proof. For the proof of (a) see [13]. For (b) see [5] (see also [10] and [12] for the case $m = 1$). □

Now we have the following proposition.

PROPOSITION 4

For f and g in $\mathfrak{F}$, we have

$$\int_{\mathbb{R}^d} fg = 2 \int_{\mathbb{R}^d} \sum_{1 \leq |j| \leq d} R_j f \, R_j g.$$

Let $p \in (1, \infty)$. Then there exists a constant C such that for all f in $\mathfrak{F}$, we have

$$\|f\|_p \leq C \left\| \left(\sum_{1 \leq |j| \leq d} |R_j f|^2 \right)^{1/2} \right\|_p. \tag{31}$$

Proof. For $1 \leq |j| \leq d$, we have

$$R_j^* R_j = H^{-1/2} A_j^* A_j H^{-1/2} = H^{-1/2} A_{-j} A_j H^{-1/2}.$$

Then by formula (2),

$$\sum_{1\le|j|\le d} R_j^* R_j = H^{-1/2}\left(\sum_{1\le|j|\le d} A_{-j}A_j\right)H^{-1/2} = 2I.$$

Therefore, if f and g are in $\mathfrak{F}$,

$$\sum_{1\le|j|\le d}\int_{\mathbb{R}^d} R_j f\, R_j g = \sum_{1\le|j|\le d}\int_{\mathbb{R}^d} R_j^* R_j f g = \int_{\mathbb{R}^d}\left(\sum_{1\le|j|\le d} R_j^* R_j f\right) g$$
$$= 2\int_{\mathbb{R}^d} fg.$$

In order to prove (31), by Hölder's inequality, we get

$$\begin{aligned}\|f\|_p &= \sup_{\{g\in\mathfrak{F}:\ \|g\|_{p'}=1\}}\int_{\mathbb{R}^d} fg = \frac{1}{2}\sup_{\{g\in\mathfrak{F}:\ \|g\|_{p'}=1\}}\sum_{1\le|j|\le d}\int_{\mathbb{R}^d} R_j(f)R_j(g)\\ &\le \frac{1}{2}\sup_{\{g\in\mathfrak{F}:\ \|g\|_{p'}=1\}}\left\|\left(\sum_{1\le|j|\le d}|R_j(f)|^2\right)^{1/2}\right\|_p\left\|\left(\sum_{1\le|j|\le d}|R_j(g)|^2\right)^{1/2}\right\|_{p'}\\ &\le C\left\|\left(\sum_{1\le|j|\le d}|R_j(f)|^2\right)^{1/2}\right\|_p,\end{aligned}$$

where in the last inequality we have used Theorem 5(b). □

Proof of Theorem 4. Since $\mathfrak{F}$ is dense in both spaces it is enough to show the equivalence of the norm for functions in $\mathfrak{F}$. Let $f\in\mathfrak{F}$, and let $f = H^{-k/2}g$. Then by Theorem 5(b) and Theorem 1, we obtain

$$\begin{aligned}\|f\|_{W^{k,p}} &= \sum_{1\le|j_1|,\dots,|j_k|\le d,\ 1\le m\le k}\|R_{j_1,\dots,j_m}H^{-(k-m)/2}g\|_p + \|H^{-k/2}g\|_p\\ &\le C_{k,d}\,\|g\|_p = C_{k,d}\,\|f\|_{\mathfrak{L}^p_k}.\end{aligned}$$

To prove the converse inequality, we first consider the case $k=1$. By Proposition 4, we get

$$\|f\|_{\mathfrak{L}^p_1} = \|H^{1/2}f\|_p \le C\sum_{1\le|j|\le d}\|A_j f\|_p \le C\,\|f\|_{W^{1,p}}.$$

Now we shall use an inductive argument. Suppose we have

$$\|f\|_{\mathfrak{L}^p_m} = \|H^{m/2}f\|_p \le C_m\|f\|_{W^{m,p}},$$

for $f\in\mathfrak{F}$, with $m<k$. Since for some constants $c_1, c_2, \dots, c_{k-1}$,

$$\sum_{1\le|j_1|,\dots,|j_k|\le d} A_{j_k}^*\cdots A_{j_1}^*\,A_{j_1}\cdots A_{j_k} = 2^k H^k + \sum_{m=1}^{k-1} c_m\,H^m,$$

and, since H is autoadjoint, for all $f, g \in \mathfrak{F}$,

$$\|f\|_{\mathfrak{L}_k^p} = \|H^{k/2} f\|_p = \sup_{\{g\in\mathfrak{F}:\ \|g\|_{p'}=1\}} \int_{\mathbb{R}^d} (H^{k/2} f)\, g$$
$$= \sup_{\{g\in\mathfrak{F}:\ \|g\|_{p'}=1\}} \int_{\mathbb{R}^d} (H^k f)\,(H^{-k/2} g).$$

Now by using formula (4) and the definition (30) of the Hermite–Riesz transform of higher order, we have

$$2^k \int_{\mathbb{R}^d} H^k f\, H^{-k/2} g$$
$$= \int_{\mathbb{R}^d} \left(\sum_{1\le|j_1|,\dots,|j_k|\le d} A_{j_k}^* \cdots A_{j_1}^* A_{j_1} \cdots A_{j_k} - \sum_{m=1}^{k-1} c_m\, H^m \right) f\, H^{-k/2} g$$
$$= \sum_{1\le|j_1|,\dots,|j_k|\le d} \int_{\mathbb{R}^d} A_{j_1} \cdots A_{j_k} f\, R_{j_1\dots,j_k} g - \sum_{m=1}^{k-1} c_m \int_{\mathbb{R}^d} H^{m/2} f\, H^{-\frac{(k-m)}{2}} g.$$

Thus, by Hölder's inequality the last expression is bounded by

$$\sum_{1\le|j_1|,\dots,|j_k|\le d} \|A_{j_1} \cdots A_{j_k} f\|_p \|R_{j_1\dots,j_k} g\|_{p'}$$
$$+ \sum_{m=1}^{k-1} |c_m|\, \|H^{m/2} f\|_p \|H^{-\frac{(k-m)}{2}} g\|_{p'}. \tag{32}$$

From Theorem 5(b), Theorem 1, the induction hypotheses and Definition 1, there exists a constant C such that (32) is bounded by

$$\left(C \sum_{m=1}^{k-1} |c_m| \right) \|f\|_{W^{k,p}}.$$

5. Boundedness of some operators on $\mathfrak{L}_a^p$

Theorem 6. *Let $p \in (1, \infty)$, $a > 1$ and $1 \le |j| \le d$. Then A_j is bounded from $\mathfrak{L}_a^p$ into $\mathfrak{L}_{a-1}^p$.*

Proof. Let $1 \le j \le d$ (the case $-d \le j \le -1$ is similar). If $f \in \mathfrak{F}$, by Lemma 4,

$$A_j f = H^{-(a-1)/2} \left(\frac{H}{H+2} \right)^{(a-1)/2} R_j\, H^{a/2} f. \tag{33}$$

As the function $m(\lambda) = (\lambda/(\lambda+2))^{(a-1)/2}$ satisfies the hypotheses of Theorem 4.2.1 in [12], the operator $(H/(H+2))^{(a-1)/2}$ is bounded on $L^p(\mathbb{R}^d)$. Hence, by Theorem 5(a) and Theorem 1, the operator

$$H^{-(a-1)/2} \left(\frac{H}{H+2} \right)^{(a-1)/2} R_j \tag{34}$$

is bounded on $L^p(\mathbb{R}^d)$. If $f \in \mathfrak{L}^p_a$, then $A_j f = R_j H^{-(a-1)/2} H^{a/2} f$. Since operators (34) and $R_j H^{-(a-1)/2}$ coincide in $\mathfrak{F}$, both are bounded on $L^p(\mathbb{R}^d)$ and $\mathfrak{F}$ is dense in $\mathfrak{L}^p_a$, formula (33) also works for all $f \in \mathfrak{L}^p_a$.

Therefore, if $f \in \mathfrak{L}^p_a$, the function $h = (H/(H+2))^{(a-1)/2} R_j H^{a/2} f$ belongs to $L^p(\mathbb{R}^d)$ and by (33), we have

$$\|A_j f\|_{\mathfrak{L}^p_{a-1}} = \|h\|_p \leq C\|H^{a/2} f\|_p = C\|f\|_{\mathfrak{L}^p_a}.$$

□

Theorem 7. *If $p \in (1,\infty)$, $a > 0$ and $1 \leq |j| \leq d$, then the operators R_j are bounded on $\mathfrak{L}^p_a$.*

Proof. By Theorem 2, $H^{-1/2}$ is bounded from $\mathfrak{L}^p_a$ into $\mathfrak{L}^p_{a+1}$. Hence, Theorem 6 gives the desired result. □

Remark 3. As the Hermite–Riesz transforms are pseudo-differential operators with symbols in the class $S^0_{1,0}$ (see Theorem 5(a)), they map the classical Sobolev spaces L^p_a into themselves for all $p \in (1,\infty)$ and $a > 0$. However, the classical Riesz transforms are not bounded on $\mathfrak{L}^p_a$ for any $a > (1/p')$ as the following proposition shows.

PROPOSITION 5

Let p be in the range $1 < p < \infty$. The Hilbert transform on $\mathbb{R}$ is not bounded in $\mathfrak{L}^p_a$ for any $a > (1/p')$.

Proof. Let $\mathcal{H}$ be the Hilbert transform on the line, that is

$$\mathcal{H}f(x) = \lim_{\epsilon\to 0}\int_{|y|>\epsilon} \frac{f(x-y)}{y}\,\mathrm{d}y.$$

Consider the function f in $\mathfrak{L}^p_a$ given by

$$f(x) = \exp(-|x|^2).$$

Given $\epsilon > 0$ and $x > 2$, we have

$$\int_{|y|>\epsilon} \frac{f(x-y)}{y}\,\mathrm{d}y = \left(\int_{\epsilon<|y|<1} + \int_{-\infty}^{-1} + \int_1^{\infty}\right)\frac{f(x-y)}{y}\,\mathrm{d}y. \tag{35}$$

By the mean value theorem, the first integral of the last expression can be written as

$$\begin{aligned}\int_{\epsilon<|y|<1} \frac{\mathrm{e}^{-|x-y|^2}}{y}\,\mathrm{d}y &= -\int_{\epsilon<|y|<1} \frac{\mathrm{e}^{-|x|^2} - \mathrm{e}^{-|x-y|^2}}{x-(x-y)}\,\mathrm{d}y\\ &= -2\int_{\epsilon<|y|<1} \theta(x,y)\,\mathrm{e}^{-\theta(x,y)^2}\,\mathrm{d}y,\end{aligned}$$

where $x-1 < x-y < \theta(x,y) < x$ if $\epsilon < y < 1$, and $x < \theta(x,y) < x-y < x+1$ if $-1 < y < -\epsilon$. Thus

$$\begin{aligned}\left|\int_{\epsilon<|y|<1} \frac{\mathrm{e}^{-|x-y|^2}}{y}\,\mathrm{d}y\right| &< 2\left|\int_{\epsilon<|y|<1} \theta(x,y)\,\mathrm{e}^{-\theta(x,y)^2}\,\mathrm{d}y\right|\\ &\leq\ 4(x+1)\,\mathrm{e}^{-|x-1|^2}. \end{aligned} \tag{36}$$

For the second integral of (35),

$$\left|\int_{-\infty}^{-1} \frac{e^{-|x-y|^2}}{y}\,dy\right| \le e^{-|x|^2}\int_{-\infty}^{-1}\frac{e^{-|y|^2}}{|y|}\,dy. \tag{37}$$

Finally, since $x > 2$,

$$\left|\int_{1}^{\infty} \frac{e^{-|x-y|^2}}{y}\,dy\right| \ge \frac{1}{x}\int_{1}^{x} e^{-|x-y|^2}\,dy \ge \frac{1}{x}\int_{0}^{1} e^{-|u|^2}\,du. \tag{38}$$

Therefore, from (36)–(38), there exist constants C and M independent of ϵ, such that

$$\left|\int_{|y|>\epsilon} \frac{f(x-y)}{y}\,dy\right| \ge \frac{C}{x} \quad \text{for all } x > M.$$

Thus, for any $a > (1/p')$, $|x|^a\,\mathcal{H}f$ is not in $L^p(\mathbb{R})$, and by Proposition 3, $\mathcal{H}f$ is not in $\mathfrak{L}_a^p$. □

6. Poincaré inequalities

This section is devoted to some Poincaré-type inqualities.

Remark 4. Observe that by formula (18), if $a < d$ and we take a positive function f, then

$$H^{-a/2}f(x) \le C\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{d-a}}dy.$$

Therefore, in the case $a < d$ the operator $H^{-a/2}$ inherits the boundedness properties of the fractional integral. In particular, $H^{-a/2}$ is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ for $\frac{1}{q} = \frac{1}{p} - \frac{a}{d}$, where $1 < p < q < \infty$.

Next theorem gives the behavior of $H^{-a/2}$ on $L^p(\mathbb{R}^d)$, for $p \ge 1$. Inequality (21) allows us to obtain some boundedness of the operator $H^{-a/2}$ that has a different flavor from the boundedness of the classical fractional integral.

Theorem 8. *Let* a, d *such that* $0 < a < d$, *then*

(i) *There exists a constant* C, *such that*

$$\|H^{-a/2}f\|_q \le C\|f\|_1,$$

for all $f \in L^1(\mathbb{R}^d)$ *if and only if* $1 \le q < d/(d-a)$.

(ii) *There exists a constant* C, *such that*

$$\|H^{-a/2}f\|_\infty \le C\|f\|_p,$$

for all f *in* $L^p(\mathbb{R}^d)$ *if and only if* $p > \frac{d}{a}$.

(iii) *There exists a constant* C, *such that*

$$\|H^{-a/2}f\|_q \le C\|f\|_\infty,$$

for all $f \in L^\infty(\mathbb{R}^d)$ *if and only if* $q > (d/a)$.

(iv) *There exists a constant* C, *such that*

$$\|H^{-a/2}f\|_1 \leq C\|f\|_p,$$

for all $f \in L^p(\mathbb{R}^d)$ *if and only if* $1 \leq p < d/(d-a)$.

(v) *If* $1 < p < \infty$, $1 < q < \infty$ *and* $\frac{1}{p} - \frac{a}{d} \leq \frac{1}{q} < \frac{1}{p} + \frac{a}{d}$, *then there exists a constant* C, *such that*

$$\|H^{-a/2}f\|_q \leq C\|f\|_p,$$

for all $f \in L^p(\mathbb{R}^d)$.

Proof. By Minkowski's integral inequality,

$$\begin{aligned}\int_{\mathbb{R}^d} (H^{-a/2}f)^q &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K_{a/2}(x,y)f(y)\,\mathrm{d}y\right)^q \mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} K_{a/2}(x,y)^q\,\mathrm{d}x\right)^{1/q} \mathrm{d}y\right)^q,\end{aligned}$$

for all $f \in L^1(\mathbb{R}^d)$. From inequalities (8) and (18), we get

$$\int_{\mathbb{R}^d} K_{a/2}(x,y)^q\,\mathrm{d}x \leq C \left(\int_{\{|x|<2\}} \frac{\mathrm{d}x}{|x|^{q(d-a)}} + \int_{\{|x|>2\}} \mathrm{e}^{-q\frac{|x|^2}{4}}\,\mathrm{d}x\right),$$

and this integral is finite if $1 \leq q < d/(d-a)$, proving (i). Conversely, by (13) and a change of variables, if $|x-y| < 1$,

$$\begin{aligned}K_{a/2}(x,y) &\geq \frac{\mathrm{e}^{-|x+y|^2}}{C_1\Gamma(a/2)(4\pi)^{d/2}2^{a/2-1}} \int_0^{1/2} s^{-\frac{d-a}{2}}\,\mathrm{e}^{-\frac{1}{4s}|x-y|^2}\,\frac{\mathrm{d}s}{s} \\ &\geq \frac{\int_0^{\frac{1}{2|x-y|^2}} s^{-\frac{d-a}{2}}\mathrm{e}^{-\frac{1}{4s}}\frac{\mathrm{d}s}{s}}{C_1\Gamma(a/2)(4\pi)^{d/2}2^{\frac{a}{2}-1}}\,\frac{\mathrm{e}^{-|x+y|^2}}{|x-y|^{d-a}} \\ &\geq \frac{\int_0^{\frac{1}{2}} s^{-\frac{d-a}{2}}\mathrm{e}^{-\frac{1}{4s}}\frac{\mathrm{d}s}{s}}{C_1\Gamma(a/2)(4\pi)^{d/2}2^{\frac{a}{2}-1}}\,\frac{\mathrm{e}^{-|x+y|^2}}{|x-y|^{d-a}}.\end{aligned} \tag{39}$$

Now, let f_n, $n \geq 0$, be an approximation to the identity. Suppose that inequality (i) holds for all $f \in L^1(\mathbb{R}^d)$, then by inequality (39) there exists a constant C, such that

$$C \geq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{\chi_{\{|x-y|<1\}}(y)\mathrm{e}^{-|x+y|^2}}{|x-y|^{d-a}} f_n(y)\,\mathrm{d}y\right)^q \mathrm{d}x,$$

for all $n \geq 0$, so that

$$C \geq \int_{\{|x|<1\}} \frac{\mathrm{e}^{-q|x|^2}}{|x|^{q(d-a)}}\,\mathrm{d}x,$$

but this is false when $q(d-a) \geq d$.

To obtain inequality (ii), by Hölder's inequality,

$$\left|\int_{\mathbb{R}^d} K_{a/2}(x,y)\, f(y)\,\mathrm{d}y\right| \le \|f\|_p \left(\int_{\mathbb{R}^d} K_{a/2}(x,y)^{p'}\,\mathrm{d}y\right)^{1/p'},$$

and in the same manner as we dealt with (i), the last integral is finite when $p > (d/a)$. To see that $p > (d/a)$ is necessary, let $\epsilon > 0$ and

$$f(x) = \begin{cases} |x|^{-a}\left(\log \frac{1}{|x|}\right)^{-(a/d)(1+\epsilon)} & \text{if } |x| \le 1/2, \\ 0 & \text{if } |x| > 1/2. \end{cases}$$

Then $f \in L^p(\mathbb{R}^d)$ for all $p \le (d/a)$. However, $H^{-a}f$ is essentially unbounded as by estimate (39),

$$H^{-a/2} f(0) \ge C \int_{|x|\le 1/2} |x|^{-d}\left(\log\frac{1}{|x|}\right)^{-(a/d)(1+\epsilon)} = \infty,$$

when ϵ is small enough.

To see (iii), let f in $L^\infty(\mathbb{R}^d)$, then

$$\begin{aligned}\int_{\mathbb{R}^d} |H^{-a/2}f|^q &\le \|f\|_\infty^q \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} K_{a/2}(x,y)\,\mathrm{d}y\right)^q \mathrm{d}x \\ &\le \|f\|_\infty^q \left[\int_{\{|x|\le 1\}} + \int_{\{|x|>1\}} \left(\int_{\mathbb{R}^d} K_{a/2}(x,y)\,\mathrm{d}y\right)^q \mathrm{d}x\right].\end{aligned}$$

The first integral of the last expression is finite due to estimate (8). By inequality (21), the second integral is

$$\int_{\{|x|>1\}}\left(\int_{\mathbb{R}^d} K_{a/2}(x,y)\,\mathrm{d}y\right)^q \mathrm{d}x \le C^q \int_{\{|x|>1\}} \frac{\mathrm{d}x}{|x|^{qa}},$$

which is finite when $q > (d/a)$.

To see the converse, let $f \equiv 1$. Then,

$$\int_{\mathbb{R}^d} |H^{-a/2}f|^q = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} K_{a/2}(x,y)\,\mathrm{d}y\right)^q \mathrm{d}x, \tag{40}$$

and we will see that there exists a constant c such that

$$\int_{\mathbb{R}^d} K_{a/2}(x,y)\,\mathrm{d}y \ge c\,|x|^{-a}, \quad \text{for all } |x| > 2. \tag{41}$$

Thus, integral (40) is infinite when $q \le (d/a)$.

To see (41), let $x \in \mathbb{R}^d$ with $|x| > 2$. If $|x-y| < 1/|x|$, then $|x+y| < \frac{5}{2}|x|$, and

$$\mathrm{e}^{-\frac{1}{4}\left(s|x+y|^2 + \frac{1}{s}|x-y|^2\right)} \ge \mathrm{e}^{-2\left(s|x|^2 + \frac{1}{s}|x-y|^2\right)}.$$

Therefore, from (13) and a change of variables, we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} K_a(x, y)\,\mathrm{d}y &\geq \int_{\{|x-y|<\frac{1}{|x|}\}} K_{a,0}(x, y)\,\mathrm{d}y \\
&\geq \frac{1}{C_1} \int_{\{|x-y|<\frac{1}{|x|}\}} \int_0^{1/2} s^{-\frac{d}{2}+a}\, \mathrm{e}^{-2\left(s|x|^2+\frac{1}{s}|x-y|^2\right)} \frac{\mathrm{d}s}{s}\,\mathrm{d}y \\
&= C \int_0^{1/|x|} \int_0^{1/2} s^{-\frac{d}{2}+a}\, \mathrm{e}^{-2\left(s|x|^2+\frac{r^2}{s}\right)} \frac{\mathrm{d}s}{s}\, r^d \frac{\mathrm{d}r}{r} \\
&= C\, |x|^{-2a} \int_0^1 \left(\int_0^{|x|^2/2} u^{-\frac{d}{2}+a}\, \mathrm{e}^{-2\left(u+\frac{t^2}{u}\right)} \frac{\mathrm{d}u}{u} \right) t^d \frac{\mathrm{d}t}{t} \\
&\geq C\, |x|^{-2a} \int_0^1 \left(\int_0^1 u^{-\frac{d}{2}+a}\, \mathrm{e}^{-2\left(u+\frac{t^2}{u}\right)} \frac{\mathrm{d}u}{u} \right) t^d \frac{\mathrm{d}t}{t}.
\end{aligned}
$$

Thus, we have proved (41).

In order to show (iv), let $f \in L^p(\mathbb{R}^d)$ with $1 \leq p < \frac{d}{d-\alpha}$. By Fubini's theorem, (18) and Hölder's inequality,

$$
\begin{aligned}
\int_{\mathbb{R}^d} |H^{-a/2} f| &\leq \int_{\{|x|\leq 1\}} \int_{\mathbb{R}^d} K_{a/2}(y, x)\,\mathrm{d}y\, |f(x)|\,\mathrm{d}x \\
&\quad + \int_{\{|x|>1\}} \int_{\mathbb{R}^d} K_{a/2}(y, x)\,\mathrm{d}y\, |f(x)|\,\mathrm{d}x \\
&\leq C\, \|f\|_p + \|f\|_p \int_{\{|x|>1\}} \left(\int_{\mathbb{R}^d} K_{a/2}(y, x)\,\mathrm{d}y \right)^{p'} \mathrm{d}x.
\end{aligned}
$$

By the symmetry of $K_{a/2}$, inequality (21) and $p < d/(d-a)$, the last integral is finite. On the other hand, if $p \geq d/(d-a)$ and

$$
f(y) = \begin{cases} |y|^{-d+a}(\log|y|)^{-1} & \text{for } |y| > 2, \\ 0 & \text{otherwise,} \end{cases}
$$

then f belongs to $L^p(\mathbb{R}^d)$ but, from (41),

$$
\begin{aligned}
\int_{\mathbb{R}^d} |H^{-a/2} f| &\geq \int_{\{|y|>1\}} \int_{\mathbb{R}^d} K_{a/2}(x, y)\,\mathrm{d}x\, |f(y)|\,\mathrm{d}y \\
&\geq \int_{\{|y|>1\}} |y|^{-d}(\log|y|)^{-1}\,\mathrm{d}y = \infty.
\end{aligned}
$$

Finally, (v) is a consequence of Remark 4, (i), (ii), (iii), (iv) and the Riesz–Torin interpolation theorem. □

As a consequence of the last theorem we have the following Poincaré-type inequalities.

Theorem 9. *Let $d > 1$. Define the Hermite gradient as*

$$
\nabla_H f = (A_{-d} f, \ldots, A_{-1} f, A_1 f, \ldots, A_d f).
$$

Let p, q in the range $1 < p,\ q < \infty$ such that $\frac{1}{p} - \frac{1}{d} \leq \frac{1}{q} < \frac{1}{p} + \frac{1}{d}$. Then

$$\|f\|_q \leq C\, \|\nabla_H f\|_{L^p_{\mathbb{R}^{2d}}},$$

for all $f \in \mathfrak{L}^p_1$.

Proof. It is enough to prove the result for $f \in \mathfrak{F}$. From inequality (31), we see that

$$\|f\|_q \leq C \sum_{1\leq|j|\leq d} \|R_j(f)\|_q,$$

so that, by Lemma 4,

$$R_j f = \left(\frac{H}{H+2}\right)^{1/2} H^{-1/2} A_j f,$$

where $1 \leq j \leq d$.

We have already seen in the proof of Theorem 6, that the operator $(H/(H+2))^{1/2}$ is bounded on $L^q(\mathbb{R}^d)$. Hence, by using Theorem 8(v),

$$\|R_j f\|_q \leq C\, \|A_j f\|_p,$$

where $\frac{1}{p} - \frac{1}{d} \leq \frac{1}{q} < \frac{1}{p} + \frac{1}{d}$. □

7. Some applications to Schrödinger solutions

In this section we deal with the unidimensional Schrödinger equation

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} = Hu(x,t) \quad x, t \in \mathbb{R} \\ u(x,0) = f(x) \end{cases} \tag{42}$$

for some initial data f.

We are interested in where we have to pick the function f in order to have almost everywhere convergence of the solution

$$u(x,t) = \mathrm{e}^{itH} f(x)$$

of (42) to f as t tends to 0.

In [1] and [3] the problem with the classical Laplacian is considered. In [2] the problem for a more general operator is studied. From that work, it can be derived, for H as a particular case, that if f belongs to $\mathfrak{L}^p_a$ with $a > 1$, then we have almost everywhere convergence. Next theorem gives convergence for orders of differentiability greater than $1/2$.

Theorem 10. *If $a > 1/2$ and f belongs to $\mathfrak{L}^2_a(\mathbb{R})$, then $\mathrm{e}^{itH}f$ converges to f almost everywhere as t tends to* 0.

Proof. If f is a finite linear combination of Hermite functions,

$$\lim_{t\to 0} \mathrm{e}^{itH} f(x) = f(x)$$

everywhere. Since this kind of functions are dense in $\mathfrak{L}^2_a$, it is enough to prove that the maximal function

$$T^* f = \sup_{t>0} |\mathrm{e}^{itH} f|$$

satisfies the inequality

$$\int_I T^* f \leq C \, \|f\|_{\mathfrak{L}^2_a},$$

for all compact interval I of the real line not containing the origin, and C a constant that may depend on the interval I but not on f.

In order to see this property, we will use the following estimate of Hermite functions that can be found in [11] (Theorem 8.91.3, p. 236).

If I is a bounded interval and does not contain the origin, there exist constants C and k_0 such that

$$|h_k(x)| \leq \frac{C}{k^{1/4}} \tag{43}$$

for all $x \in I$ and $k \geq k_0$.

Let f be in $\mathfrak{L}^2_a$. As f belongs to $L^2(\mathbb{R})$ it can be written as

$$f(x) = \sum_{k=0}^{\infty} a_k \, h_k(x).$$

By Tonelli's theorem, estimate (43) and Hölder's inequality, we get

$$\begin{aligned}
\int_I |T^* f(x)| \, \mathrm{d}x &\leq \int_I \sup_{t>0} \left| \sum_{k=0}^{\infty} a_k \, \mathrm{e}^{it(2k+1)} \, h_k(x) \right| \mathrm{d}x \\
&\leq \sum_{k=0}^{\infty} |a_k| \int_I |h_k(x)| \, \mathrm{d}x \\
&\leq C \left(C + \sum_{k=k_0}^{\infty} \frac{1}{k^{1/2}(2k+1)^a} \right)^{1/2} \left(\sum_{k=0}^{\infty} a_k^2 \, (2k+1)^a \right)^{1/2} \\
&\leq C \left(C + \sum_{k=k_0}^{\infty} \frac{1}{k^{1/2+a}} \right)^{1/2} \|f\|_{\mathfrak{L}^2_a}.
\end{aligned}$$

Since $a > 1/2$, we have $1/2 + a > 1$ and the last series is convergent. □

Theorem 11. *If $a < 1/4$, then there exists a function f in $\mathfrak{L}^2_a(\mathbb{R})$ such that*

$$\lim_{t \to 0} \mathrm{e}^{-itH} f(x) = \infty$$

for almost every $x \in \mathbb{R}$.

Proof. For $a < 1/4$, in [3] the authors find an f belonging to the classical Sobolev space L_a^2 and compactly supported so that

$$\liminf_{t\to 0} |\mathrm{e}^{-it\Delta} f(x)| = \infty. \tag{44}$$

Since f is compactly supported, it follows from Theorem 3(iii) that f belongs to $\mathfrak{L}_a^2$. Then, it is sufficient to compare the kernels of $\mathrm{e}^{-it\Delta}$ and e^{-itH} for small values of t. In fact,

$$\mathrm{e}^{-it\Delta} f(x) = \int_{\mathbb{R}} W_{it}(x-y)\, f(y)\,\mathrm{d}y,$$

with

$$W_z(x) = \frac{1}{\sqrt{4\pi z}} \exp\left(\frac{-|x|}{4z}\right), \quad z \in \mathbb{C}$$

and

$$\mathrm{e}^{-itH} f(x) = \int_{\mathbb{R}} G_{it}(x,y)\, f(y)\,\mathrm{d}y,$$

where

$$G_z(x,y) = \frac{1}{\sqrt{2\pi \sinh(2z)}} \exp\left(-\frac{1}{2}|x-y|^2 \coth(2z) - x\cdot y \tanh(z)\right), \quad z \in \mathbb{C}.$$

Then, for a fixed $x \in \mathbb{R}$, we have

$$\lim_{t\to 0} |W_{it}(x-y) - G_{it}(x,y)| = \frac{1}{\sqrt{2\pi}} \lim_{t\to 0} \left| \frac{\mathrm{e}^{i\frac{|x-y|^2}{2\tan(2t)} - ix\cdot y\tan(t)}}{\sqrt{\sin(2t)}} - \frac{\mathrm{e}^{i\frac{|x-y|^2}{4t}}}{\sqrt{2t}} \right| = 0$$

uniformly for y in a compact subset of $\mathbb{R}$. Thus, by (44) we also have

$$\liminf_{t\to 0} |\mathrm{e}^{-itH} f(x)| = \infty.$$

□

Acknowledgement

The first author is grateful to the Department of Mathematics at Universidad Autónoma de Madrid for its hospitality during the period of this research. The authors were supported by MCYT 2002-04013-C02-02 and HARP network HPRN-CT-2001-00273 of the European Commission, and CONICET (Argentina).

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