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Stability estimates for h-p spectral element methods for elliptic problems

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Abstract. In a series of papers of which this is the first we study how to solve elliptic problems on polygonal domains using spectral methods on parallel computers. To overcome the singularities that arise in a neighborhood of the corners we use a geometrical mesh. With this mesh we seek a solution which minimizes a weighted squared norm of the residuals in the partial differential equation and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity by adding a term which measures the jump in the function and its derivatives at inter-element boundaries, in an appropriate fractional Sobolev norm, to the functional being minimized. Since the second derivatives of the actual solution are not square integrable in a neighborhood of the corners we have to multiply the residuals in the partial differential equation by an appropriate power of r_k, where r_k measures the distance between the point P and the vertex A_k in a sectoral neighborhood of each of these vertices. In each of these sectoral neighborhoods we use a local coordinate system (τ_k, θ_k) where $\tau_k = \ln r_k$ and (r_k, θ_k) are polar coordinates with origin at A_k, as first proposed by Kondratiev. We then derive differentiability estimates with respect to these new variables and a stability estimate for the functional we minimize.

In [6] we will show that we can use the stability estimate to obtain parallel preconditioners and error estimates for the solution of the minimization problem which are nearly optimal as the condition number of the preconditioned system is polylogarithmic in N, the number of processors and the number of degrees of freedom in each variable on each element. Moreover if the data is analytic then the error is exponentially small in N .

Keywords. Corner singularities; geometrical mesh; modified polar coordinates; quasi-uniform mesh; fractional Sobolev norms; stability estimate; polylogarithmic bounds.

1. Introduction

This is the first part of a series of four papers, the other three being, h-p Spectral element methods for Dirichlet problems on parallel computers [6], h-p Spectral element methods for mixed problems on parallel computers [7] and h-p Spectral element methods for elliptic boundary value problems – The general case [8].

Current formulations of spectral methods to solve elliptic problems in nonsmooth domains allow us to recover only algebraic convergence [10]. One method, which yields relatively fast convergence, makes use of a conformal mapping of the form $z = \xi^\alpha$ to smooth out the singularity that occurs at the corner and is referred to as the method of auxiliary mapping. However, 'even though the conformal mapping is an effective way of enhancing convergence, exponential convergence cannot be fully recovered' [10].

A method for obtaining a numerical solution to exponential accuracy for elliptic problems with analytic coefficients posed on curvilinear polygons whose boundary is piecewise analytic with mixed Neumann and Dirichlet boundary conditions, was first proposed by Babuska and Guo [1,2] within the framework of the finite element method. They were able to resolve the singularities which arise at the corners by using a geometrical mesh.

We also use a geometrical mesh to solve the same class of problems to exponential accuracy using h-p spectral element methods but with an important difference. The geometrical mesh becomes geometrically fine in a neighborhood of each of the corners. In a neighborhood of the corner A_k we switch to new variables (τ_k, θ_k) where $\tau_k = \ln r_k$ and (r_k, θ_k) are polar coordinates with origin at A_k. In doing so the geometrical mesh is reduced to a quasi-uniform mesh in a sectoral neighborhood of the corners and so Sobolev's embedding theorems and the trace theorems for Sobolev spaces apply for functions defined on mesh elements in these new variables with a uniform constant. These new variables, which we shall refer to as modified polar coordinates, were first used by Kondratiev in his seminal paper [11]. Away from these sectoral neighborhoods of the corners we retain (x, y) variables for our coordinate system. Thus we also use the auxiliary map $z = \log \xi$ to remove the singularities at the origin and this enables us to obtain the solution with exponential accuracy.

By subtracting an analytic function from the solution if necessary, we may assume that the Dirichlet data vanishes at the corners. We seek an approximate solution which vanishes at the corner-most elements and is a sum of tensor product of polynomials of degree N in τ_k and θ_k in the remaining elements of the sectoral neighborhood of the corners. The remaining quadrilateral elements are mapped to the unit square S and the approximate solution is represented as a sum of tensor products of polynomials of degree N in ξ and η, the transformed variables. If Neumann boundary conditions are imposed on both the sides which meet at the corner, the approximate solution at corner-most elements is represented by a constant, instead of zero.

We now seek a solution as in [4] which minimizes the sum of the squares of a weighted squared norm of the residuals in the partial differential equation and the sum of the squares of the residuals in the Dirichlet boundary conditions in an appropriate Sobolev norm and enforce continuity by adding a term which measures the sum of the squares of the jump in the function and the squares of the jump in its derivatives across inter-element boundaries in appropriate Sobolev norms to the functional being minimized as a penalty term. Since the residuals in the partial differential equation blow up in a neighborhood of the corners, we have to multiply these residuals by an appropriate power of r_k, where r_k measures the distance between the point P and A_k. All these computations are done using modified polar coordinates in a sectoral neighborhood of the corners and a global coordinate system elsewhere.

In [6,7] we restrict ourselves to examining the Poisson's equation with Dirichlet boundary conditions on a polygon. In this paper we obtain differentiability estimates in modified polar coordinates and prove the stability theorem 3.3 on which our method is based. Since the statement of this theorem may appear complicated we try and provide motivation for it by stating the stability theorem 3.1 for a simpler case.

For the Dirichlet problem we use spectral element functions which are nonconforming. To solve the minimization problem we have defined, we need to solve the normal equations for the least-squares problem corresponding to collocating the partial differential equation and boundary conditions at an over-determined set of collocation points and enforcing continuity of the function and its derivatives at the collocation points at inter-element boundaries, suitably weighted. However, we do not need to compute and store any matrices, like the mass and stiffness matrices, to compute the residual in the normal equations [5].

We can precondition the normal equations by using a preconditioner which is of block diagonal form and which allows the solutions for different elements to decouple completely. Moreover, this is nearly optimal as the condition number of the preconditioned system is polylogarithmic in N, which is proportional to the number of processors and the number of degrees of freedom in each variable on each element. Finally we show that the error we commit is exponentially small in N and provide computational results for a model problem.

In [7] we will examine how to solve the Poisson's equation with mixed Dirichlet and Neumann boundary conditions. For the purely Dirichlet problem our spectral element functions were nonconforming and hence there were no common boundary values to solve for. This no longer holds for problems with mixed boundary conditions. Here our spectral element functions are essentially nonconforming except that they are continuous at the vertices of the elements on which they are defined. Hence our set of common boundary values are the values of the function at the vertices of the elements. Thus the cardinality of the set of common boundary values is proportional to the number of elements and is much smaller than the cardinality of common boundary values for the finite element method, which is the set of values of the functions along the edges of their elements.

In order to solve the system of normal equations we need to be able to compute a preconditioner for the Schur complement system corresponding to the common boundary values. Since the dimension of the system is small we can compute an accurate approximation to the Schur complement. This is in contrast to the methodology for the finite element method where complex techniques have to be used to obtain a preconditioner for the Schur complement matrix. Moreover the computational complexity for our scheme is less than for finite element methods. Thus the method we propose can be thought of as a vertex based method. Once again we provide computational results for a model problem.

In [8] we will generalize all our results to elliptic problems with analytic coefficients, posed on curvilinear polygons with piecewise analytic boundaries, which satisfy the Babuska–Brezzi inf-sup conditions. We should mention that once we have obtained our approximate solution consisting of nonconforming spectral element functions we can make a correction to it so that the corrected solution is conforming and is an exponentially accurate approximation to the actual solution in the H^1 norm over the whole domain.

Computational results for a model problem with Dirichlet boundary conditions have been provided in [6]. Again in §5 of [7] computational results have been provided for a model problem with mixed Neumann and Dirichlet boundary conditions.

2. Function spaces and *a priori* estimates

Let Ω be a polygon with vetices $A_1, A_2, \ldots, A_p$ and corresponding sides $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ where Γ_i joins the points A_{i-1} and A_i (figure 1). In addition let the angle subtended at A_j be ω_j. In this paper we shall examine the solution of the problem

$$\Delta u = f \qquad \text{for} \quad (x, y) \in \Omega, \tag{2.1a}$$

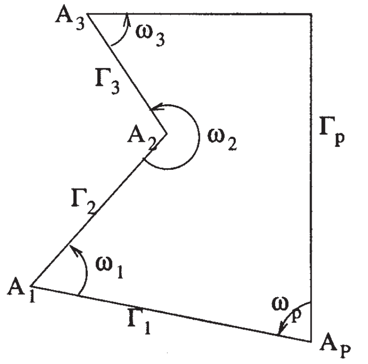

Figure 1.

with Dirichlet boundary conditions

$$u = g_j \qquad \text{for } (x, y) \in \Gamma_j,$$

or

$$u = g^{[0]} \qquad \text{for } (x, y) \in \Gamma^{[0]} = \partial\Omega. \tag{2.1b}$$

Let Z denote the point $Z = (x, y)$. We now need to review a set of *a priori* estimates proved in [1]. Let $H^m(\Omega)$ denote the completion of the space of infinitely differentiable functions with respect to the norm

$$\|v\|^2_{m,\Omega} = \sum_{|\alpha|\leq m} \int\int |D^\alpha v|^2 \mathrm{d}x\,\mathrm{d}y.$$

Let ρ_i denote the Euclidean distance between A_i and Z, i.e., $\rho_i = |Z - A_i|$. We then define $r_i = \min(1, \rho_i)$. We shall let β denote the multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$. Further, we define $\Phi_\beta(Z) = \Pi_{i=1}^p r_i^{\beta_i}(Z)$. By $H_\beta^{m,l}(\Omega)$ we denote the completion of infinitely differentiable functions with respect to the norm

$$\|v\|^2_{H_\beta^{m,l}(\Omega)} = \|v\|^2_{H^{l-1}(\Omega)} + \sum_{k=l,|\alpha|=k}^{m} \left\|D^\alpha v \Phi_{\beta+k-l}\right\|^2_{L^2(\Omega)}, \quad l \geq 1$$

as defined in [1].

Let $H_\beta^{m-1/2,l-1/2}(\Gamma_j)$ be the space of functions ϕ_j such that there exists $f \in H_\beta^{m,l}(\Omega)$ so that $f|_{\Gamma_j} = \phi_j$ and define

$$\|\phi_j\|_{H_\beta^{m-1/2,l-1/2}(\Gamma_j)} = \inf_{f\in H_\beta^{m,l}(\Omega)} \|f\|_{H_\beta^{m,l}(\Omega)}.$$

Let

$$\psi_\beta^l(\Omega) = \{u(Z)\,|u \in H_\beta^{m,l}(\Omega), m \geq l\}$$

and

$$\mathfrak{B}^l{}_\beta(\Omega) = \{u(Z)\,|u \in \psi^l_\beta(\Omega), \| |D^\alpha u|\Phi_{\beta+k-l}\|_{L^2(\Omega)} \le Cd^{k-l}(k-l)!$$
$$\text{for } |\alpha| = k = l, l+1, \ldots, d \ge 1,\ C \text{ independent of } k\}.$$

Let $Q \subseteq \mathbb{R}^2$ be an open set with a piecewise analytic boundary ∂Q and let γ be part or whole of the boundary ∂Q. Finally let $\mathfrak{B}^{l-1/2}_\beta(\gamma)$, $0 \le l \le 2$, be the space of all functions φ for which there exists $f \in \mathfrak{B}^l{}_\beta(Q)$ such that $f = \varphi$ on γ.

We now cite the important regularity theorem 2.1 of [1]. Let $f \in \mathfrak{B}^0_\beta, g^{[0]} \in \mathfrak{B}^{3/2}_\beta\left(\Gamma^{[0]}\right), \beta = \left(\beta_1, \beta_2, \ldots, \beta_p\right), 0 < \beta_i < 1, \beta_i > 1 - \pi/\omega_i$. Then Problem (2.1) has a unique solution in $H^1(\Omega)$ and $u \in \mathfrak{B}^2_\beta(\Omega)$.

Now in §4 of [2] it has been shown that when $g^{[0]}$ is analytic on every closed arc $\overline{\Gamma}_i$ and $g^{[0]}$ is continuous on $\Gamma^{[0]}$ then $g^{[0]} \in \mathfrak{B}^{3/2}_\beta(\Gamma^{[0]})$. Further if f is analytic then it belongs to $\mathfrak{B}^0_\beta$.

Next as in [2] we introduce the space $\mathfrak{C}^2_\beta$:

$$\mathfrak{C}^2_\beta = \{u \in H^{2,2}_\beta(\Omega)\,||D^\alpha u(Z)| \le Cd^k k!(\Phi_{k+\beta-1}(Z))^{-1},$$
$$|\alpha| = k = 1, 2, \ldots, C \ge 1, d \ge 1 \text{ independent of } k\}.$$

The relationship between $\mathfrak{C}^2_\beta$ and $\mathfrak{B}^2_\beta$ is given by Theorem 2.2 of [2] which we state as follows:

$$\mathfrak{B}^2_\beta(\Omega) \subseteq \mathfrak{C}^2_\beta.$$

Finally we need one last result from [2], viz. Lemma 2.1 which is stated below.

Let $u \in H^{2,2}_\beta(\Omega)$. Then u is continuous on $\overline{\Omega}$ and

$$\|u\|_{C(\overline{\Omega})} \le C\,\|u\|_{H^{2,2}_\beta(\Omega)}.$$

Since we are assuming that the data, $g_1, \ldots, g_p$ are analytic and compatible at the vertices the values of u at the vertices $A_1, A_2, \ldots, A_p$ are well-defined. Thus if we subtract from u an analytic function which assumes these values at the vertices then the difference would satisfy (2.1) with a modified set of analytic data and the Dirichlet boundary data would assume the value zero at $A_1, A_2, \ldots, A_p$. Hence without loss of generality we may assume $g_{j-1}\left(A_j\right) = g_j\left(A_j\right) = 0$ for $j > 1$ and $g_p(A_1) = 0$.

We now define one last norm which will be needed in the sequel. Let

$$\|u(\tau_j, \theta_j)\|^2_{m,(-\infty,\ln\mu)\times(\psi^j_l,\psi^j_u)} = \sum_{|\alpha|\le m} \int_{\psi^j_l}^{\psi^j_u} \int_{-\infty}^{\ln\mu} |D^{\alpha_1}_{\tau_j} D^{\alpha_2}_{\theta_j} u|^2 \mathrm{d}\tau_j\, \mathrm{d}\theta_j. \tag{2.2}$$

Let $S^\mu_j = \{(x, y) :\ 0 < r_j < \mu,\ \psi^j_l < \theta_j < \psi^j_u\}$ and let $\widetilde{S}^\mu_j$ denote its image in $\left(\tau_j, \theta_j\right)$ coordinates. We now obtain an asymptotic estimate on $\|u\left(\tau_j, \theta_j\right)\|^2_{m,\widetilde{S}^\mu_j}$ as $\mu \to 0$.

Theorem 2.1. *There exists a positive constant μ_0 such that for all $0 < \mu \leq \mu_0$ the estimate*

$$\|u(\tau_j,\theta_j)\|^2_{m,\widetilde{S}^{\mu}_j} \leq C\mu^{2(1-\beta_j)}(Cd^{m-2}\,(m-2)!)^2 \tag{2.3}$$

holds, where C and d are constants independent of m.

We estimate the terms in the right-hand side of eq. (2.2) when $|\alpha| \geq 2$.
For

$$\begin{aligned}
&\sum_{2\leq|\alpha|\leq m}\int_{\psi^j_l}^{\psi^j_u}\int_{-\infty}^{\ln\mu}(u_{\tau_j^{\alpha_1}\theta_j^{\alpha_2}})^2\mathrm{d}\tau_j\mathrm{d}\theta_j \\
&\leq \mu^{2(1-\beta_j)}\sum_{2\leq|\alpha|\leq m}\int_{\psi^j_l}^{\psi^j_u}\int_{-\infty}^{\ln\mu}(u_{\tau_j^{\alpha_1}\theta_j^{\alpha_2}})^2\mathrm{e}^{-2(1-\beta_j)\tau_j}\mathrm{d}\tau_j\mathrm{d}\theta_j \\
&\leq \mu^{2(1-\beta_j)}\sum_{2\leq|\alpha|\leq m}\int_{\psi^j_l}^{\psi^j_u}\int_{0}^{\mu}(r_j)^{2\alpha_1}(u_{r_j^{\alpha_1}\theta_j^{\alpha_2}})^2(r_j^{-2+\beta_j})^2 r_j\mathrm{d}r_j\mathrm{d}\theta_j \\
&\leq \mu^{2(1-\beta_j)}(Cd^{m-2}\,(m-2)!)^2.
\end{aligned} \tag{2.4}$$

Here we have used the fact that $u \in \mathfrak{B}^2_\beta\,(\Omega)$ and Theorem 1.1 of [1] to obtain the above result. Next we bound the terms when $|\alpha| = 1$. Since $u \in \mathfrak{C}^2_\beta\,(\Omega)$ we have

$$|D^\alpha u\,(Z)| \leq Cd(\Phi_\beta\,(Z))^{-1} \quad \text{when } |\alpha| = 1. \tag{2.5}$$

Moreover we have

$$\int_{\psi^j_l}^{\psi^j_u}\int_{-\infty}^{\ln\mu}(u^2_{\tau_j}+u^2_{\theta_j})\mathrm{d}\tau_j\mathrm{d}\theta_j = \int_{S^\mu_j}\int(u^2_x+u^2_y)\mathrm{d}x\,\mathrm{d}y.$$

Here

$$S^\mu_j = \{(x,y) : 0 < r_j < \mu,\ \psi^j_l < \theta_j < \psi^j_u\}.$$

Hence by the above relation

$$\begin{aligned}
\int_{S^\mu_j}\int(u^2_x+u^2_y)\mathrm{d}x\mathrm{d}y &\leq 2C^2d^2\int_{\psi^j_l}^{\psi^j_u}\int_0^\mu r_j^{-2\beta_j}r_j\mathrm{d}r_j\mathrm{d}\theta_j \\
&\leq (Kd)^2\,\mu^{2(1-\beta_j)}.
\end{aligned} \tag{2.6}$$

Finally we have to estimate

$$\int_{\psi^j_l}^{\psi^j_u}\int_{-\infty}^{\ln\mu}|u(\tau_j,\theta_j)|^2\mathrm{d}\tau_j\mathrm{d}\theta_j.$$

Since u vanishes at A_j

$$u(\tau_j,\theta_j) = \int_{-\infty}^{\tau_j}u_\eta(\eta,\theta_j)\mathrm{d}\eta.$$

Hence

$$|u(\tau_j,\theta_j)| \leq \left| \int_0^{r_j} u_\rho(\rho,\theta_j)\mathrm{d}\rho \right|.$$

Here $\rho = e^\eta$ and $r_j = e^{\tau_j}$. Since $u \in \mathfrak{C}^2_\beta(\Omega)$ we obtain

$$|u(\tau_j,\theta_j)| \leq C\mathrm{d}r_j^{-\beta_j+1} = Cde^{(1-\beta_j)\tau_j}.$$

And integrating the above with respect to τ_j and θ_j gives

$$\int_{\psi_l^j}^{\psi_u^j} \int_{-\infty}^{\ln\mu} |u(\tau_j,\theta_j)|^2 \mathrm{d}\tau_j \mathrm{d}\theta_j \leq (Kd)^2 \mu^{2(1-\beta_j)}. \tag{2.7}$$

Combining (2.4), (2.6) and (2.7) we get the required estimate

$$\sum_{|\alpha|\leq m} \int_{\psi_l^j}^{\psi_u^j} \int_{-\infty}^{\ln\mu} |u_{\tau_j^{\alpha_1}\theta_j^{\alpha_2}}|^2 \mathrm{d}\tau_j \mathrm{d}\theta_j \leq \mu^{2(1-\beta_j)}(Cd^{m-2}(m-2)!)^2. \;\Box \tag{2.8}$$

Remark. Estimate (2.8) can be proved directly using the relation $0 < 1 - \beta_j < \pi/\omega_j$.

3. Stability estimate

We first need to divide Ω into subdomains. Thus we divide Ω into p subdomains $S^1, S^2, \ldots, S^p$, where S^i denotes a domain which contains the vertex A_i and no other, and on each S^i we define a geometric mesh as has been done in [2].

Let $\mathfrak{S}^k = \{\Omega^k_{i,j},\ j = 1,\ldots,J_k,\ i = 1,\ldots,I_{k,j}\}$ be a partition of S^k and let $\mathfrak{S} = \bigcup_{k=1}^p \mathfrak{S}^k$. Then $\mathfrak{S}$ satisfies the following conditions :

1. $\Omega^k_{i,j}$ are curvilinear quadrilaterals or triangles and the intersection of any two $\Omega^k_{i,j}$ is one common vertex or one entire side or is empty.
2. Let $h^k_{i,j}$ and $\underline{h}^k_{i,j}$ be the maximal and minimal length of the sides of $\Omega^k_{i,j}$. We shall assume there is a constant independent of i, j, k and of the partition such that

$$\frac{h^k_{i,j}}{\underline{h}^k_{i,j}} \leq \lambda. \tag{3.1}$$

3. Let $M = \{M^k_{i,j}, 1 \leq i \leq I_{k,j}, 1 \leq j \leq J_k, 1 \leq k \leq p\}$ in which $M^k_{i,j}$ is a one-to-one mapping of the closed standard master square $S = [0,1] \times [0,1]$ {respectively standard master triangle $T = \{(\xi,\eta)|0 \leq \eta \leq 1-\xi, 0 \leq \xi \leq 1\}$ onto $\overline{\Omega}^k_{i,j}$. Let $P^k_{i,j,l}$ and $\gamma^k_{i,j,l}$ denote the vertices and sides of $\Omega^k_{i,j}$, then $(M^k_{i,j})^{-1}(P^k_{i,j,l})$ and $(M^k_{i,j})^{-1}(\gamma^k_{i,j,l})$ denote the vertices and sides of S (respectively T), $1 \leq l \leq 4$ (respectively $1 \leq l \leq 3$). Moreover if $M^k_{i,j}$ and $M^l_{m,n}$ map the closed standard square S onto elements $\overline{\Omega}^k_{i,j}$ and $\overline{\Omega}^l_{m,n}$ with common side $\gamma = \overline{P_1P_2}$, then for any $P \in \gamma$,

$\text{dist}((M_{i,j}^k)^{-1}(P), (M_{i,j}^k)^{-1}(P_t)) = \text{dist}((M_{m,n}^l)^{-1}(P), (M_{m,n}^l)^{-1}(P_t))$, $1 \leq t \leq 2$. We will assume $M_{i,j}^k$ can be written in the form

$$x = X_{i,j}^k(\xi, \eta), \qquad (\xi, \eta) \in S \text{ (respectively } T) \tag{3.2}$$
$$y = Y_{i,j}^k(\xi, \eta),$$

with $X_{i,j}^k$ and $Y_{i,j}^k$ being analytic functions on S (respectively T). Further we assume that for $|\alpha| \leq 2$

$$|D^\alpha x|, |D^\alpha y| \leq C h_{i,j}^k, \tag{3.3a}$$

and

$$C_1 (h_{i,j}^k)^2 \leq J_{i,j}^k \leq C_2 (h_{i,j}^k)^2 \tag{3.3b}$$

for all i, j and k with constants C, C_1, C_2 independent of i, j and k and $J_{i,j}^k$ being the Jacobian of the mapping $M_{i,j}^k$.

Let $\mu = (\mu_1, \ldots, \mu_p)$ with $0 < \mu_i < 1$. Then $\mathfrak{S}_\mu$ is called a geometrical mesh with ratios $\mu = (\mu_1, \ldots, \mu_p)$ when in addition the following condition is fulfilled.

4. Let $\Omega_{i,j}^k \in \mathfrak{S}$ and $d_{i,j}^k$ denote the distance between $\Omega_{i,j}^k$ and A_k. Then $d_{i,j}^k$ and $h_{i,j}^k$ satisfy

$$C_1 (\mu_k)^{N-j} \leq d_{i,j}^k \leq C_2 (\mu_k)^{N-j}, \quad 1 < j \leq N, 1 \leq i \leq I_{k,j}, \tag{3.4a}$$
$$C_3 \rho \leq d_{i,j}^k \leq C_4 \rho, \quad N < j \leq J_k, \quad 1 \leq i \leq I_{k,j}, \quad 1 \leq k \leq p, \tag{3.4b}$$
$$d_{i,1}^k = 0, \quad 1 \leq i \leq I_{k,1}, \quad 1 \leq k \leq p, \tag{3.4c}$$
$$K_1 d_{i,j}^k \leq \underline{h}_{i,j}^k \leq h_{i,j}^k \leq K_2 d_{i,j}^k \tag{3.4d}$$

for $1 < j \leq J_k, 1 \leq i \leq I_{k,j}, 1 \leq k \leq p$, where C_l for $1 \leq l \leq 4$ and K_l for $1 \leq l \leq 2$ are constants independent of i, j and k. Moreover $J_k = N + O(1)$.

We now put some restrictions on $\mathfrak{S}$. Let (r_k, θ_k) denote polar coordinates with center at A_k. Let $\tau_k = \ln r_k$. We choose ρ so that the sector S_ρ^k with sides Γ_k and Γ_{k+1}, center at A_k and radius ρ satisfies

$$S_\rho^k \subseteq \bigcup_{\Omega_{i,j}^k \in \mathfrak{S}^k} \overline{\Omega}_{i,j}^k.$$

S_ρ^k may be represented as

$$S_\rho^k = \{(x, y) : 0 < r_k < \rho, \psi_l^k < \theta_k < \psi_u^k\}. \tag{3.5a}$$

Let $\{\psi_i^k\}_{i=1,\ldots,I_{k+1}}$ be an increasing sequence of points such that $\psi_1^k = \psi_l^k$ and $\psi_{I_{k+1}}^k = \psi_u^k$. Let $\Delta\psi_i^k = \psi_{i+1}^k - \psi_i^k$. We choose these points so that

$$\max_k(\max_i \Delta\psi_i^k) \leq \lambda \min_k(\min_i \Delta\psi_i^k) \tag{3.5b}$$

for some constant λ. Let

$$\sigma_1^k = 0, \tag{3.5c}$$

and

$$\sigma_j^k = \rho\,(\mu_k)^{N+1-j} \quad \text{for } 2 \le j \le N+1. \tag{3.5d}$$

Finally we define

$$\eta_j^k = \ln \sigma_j^k \quad \text{for } 1 \le j \le N+1. \tag{3.6}$$

Let

$$\Omega_{i,j}^k = \{(x,y) : \sigma_j^k < r_k < \sigma_{j+1}^k, \psi_i^k < \theta_k < \psi_{i+1}^k\} \quad \text{for } 1 \le i \le I_k, \quad 1 \le j \le N. \tag{3.7}$$

We assume there exists a number ν such that

$$\Omega_{i,N+1}^k = \{(x,y) : \rho < r_k < \nu, \psi_i^k < \theta_k < \psi_{i+1}^k\} \quad \text{for } 1 \le i \le I_k. \tag{3.8}$$

In other words $I_{k,j}$ is independent of j for $j \le N+1$. We shall let O^k denote

$$O^k = \{\Omega_{i,j}^k, 1 \le i \le I_k, 1 \le j \le N\} \quad \text{for } 1 \le k \le p. \tag{3.9}$$

Let

$$O^{p+1} = \{\Omega_{i,j}^k : 1 \le i \le I_{k,j}, N+1 \le j \le J_k, 1 \le k \le p\}. \tag{3.10}$$

We shall relabel the elements of O^{p+1} and write

$$O^{p+1} = \{\Omega_l^{p+1}, 1 \le l \le L\}, \tag{3.11}$$

where L denotes the cardinality of O^{p+1}. We shall let Ω^k denote the sector with vertex at A_k given by

$$\Omega^k = \{(x,y) : 0 < r_k < \rho, \psi_l^k < \theta_k < \psi_u^k\} \tag{3.12a}$$

and

$$\Omega^{p+1} = \Omega \setminus \left\{ \bigcup_{k=1}^{p} \overline{\Omega}^k \right\}. \tag{3.12b}$$

Note that all the sets Ω^k are open sets.

Henceforth to keep our notation simple we will assume that $\Omega_{i,j}^k$ are quadrilaterals for $1 \le k \le p$, $N+1 \le j \le J_k$, $1 \le i \le I_k$. Moreover we assume $I_{k,j} \le I$ for all k and j. Here I is a small integer, and this fact plays a fundamental role in allowing us to use nonconforming spectral elements to solve the Dirichlet problem. Further let $\gamma_{i,j,l}^k$ denote the sides of the quadrilateral $\Omega_{i,j}^k$, $1 \le l \le 4$. Then we assume (figure 2)

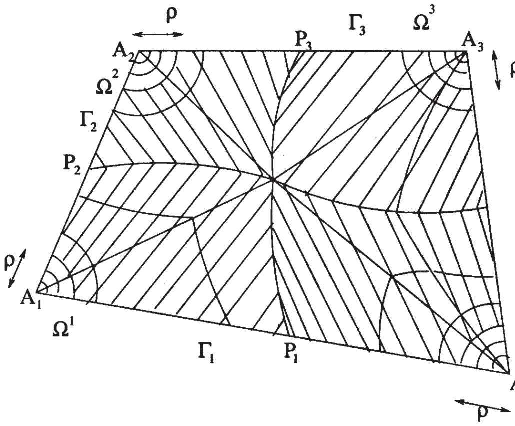

Figure 2.

$$\gamma^k_{i,j,l} : \begin{cases} x = h^k_{i,j}\phi^k_{i,j,l}(\xi), \\ y = h^k_{i,j}\psi^k_{i,j,l}(\xi), \end{cases} \quad 0 \le \xi \le 1, l = 1, 3 \tag{3.13}$$

$$\gamma^k_{i,j,l} : \begin{cases} x = h^k_{i,j}\phi^k_{i,j,l}(\eta), \\ y = h^k_{i,j}\psi^k_{i,j,l}(\eta), \end{cases} \quad 0 \le \eta \le 1, l = 2, 4 \tag{3.14}$$

where $\phi^k_{i,j,l}$, $\psi^k_{i,j,l}$ are analytic functions for all i, j, k and l. We wish to obtain a stability estimate for the function u, given by a nonconforming finite dimensional representation $u^k_{i,j}$ on each domain $\Omega^k_{i,j}$, for the entire polygonal domain Ω. Now, as stated in the introduction we partition the open set Ω into p open sets $S^1, S^2, \ldots, S^p$ such that each S^i contains only the singularity at the vertex A_i. Let S^k be one of these open sets. Then $S^k = \Omega^k \bigcup B^k_\rho \bigcup T^k$ where Ω^k is the open sector with center at A^k and radius ρ, B^k_ρ is the circular arc which bounds Ω^k, and T^k is the open set defined as $T^k = S^k \setminus (\Omega^k \bigcup B^k_\rho)$.

The domain S^k is as shown in figure 3. Two of its sides are the straight lines $\Gamma_{k+1} \bigcap \partial S^k$ and $\Gamma_k \bigcap \partial S^k$. The remaining side ∂S^k_c consists of piecewise analytic arcs. The subscript c in ∂S^k_c denotes curvilinear. $\overline{S}^k$ is partitioned by a set of arcs $\{\gamma_l\}_l$ into subdomains.

Now

$$\Omega^k_{i,j} = \{(x, y) : \sigma^k_j < r_k < \sigma^k_{j+1}, \ \psi^k_i < \theta_k < \psi^k_{i+1}\}$$

for $1 \le i \le I_k$, $1 \le j \le N$.

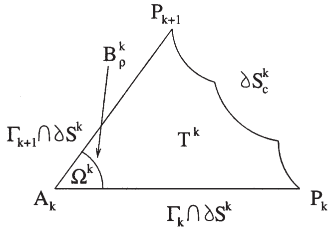

Figure 3.

Let $\tau_k = \ln r_k$. Define

$$\widetilde{\Omega}^k_{i,j} = \{(\tau_k, \theta_k) : \ \eta^k_j < \tau_k < \eta^k_{j+1}, \ \psi^k_i < \theta_k < \psi^k_{i+1}\} \tag{3.15}$$

for $1 \le i \le I_k$, $1 \le j \le N$.

Let ω be a smooth function. Then

$$\omega_{xx} + \omega_{yy} = \frac{1}{r_k^2}\left(r_k \frac{\partial}{\partial r_k} r_k \frac{\partial \omega}{\partial r_k} + \omega_{\theta_k\theta_k}\right) = \mathrm{e}^{-2\tau_k}(\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k}).$$

Hence

$$\int_{\Omega^k_{i,j}}\int r_k^2(\omega_{xx} + \omega_{yy})^2 \mathrm{d}x\mathrm{d}y = \int_{\psi^k_i}^{\psi^k_{i+1}}\int_{\eta^k_j}^{\eta^k_{j+1}} (\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k})^2 \mathrm{d}\tau_k \mathrm{d}\theta_k.$$

Now using integration by parts repeatedly we can show that

$$\begin{aligned}
&\int_{\psi^k_i}^{\psi^k_{i+1}}\int_{\eta^k_j}^{\eta^k_{j+1}} ((\omega_{\tau_k\tau_k})^2 + 2(\omega_{\tau_k\theta_k})^2 + (\omega_{\theta_k\theta_k})^2)\mathrm{d}\tau_k \mathrm{d}\theta_k \qquad (3.16)\\
&= \int_{\psi^k_i}^{\psi^k_{i+1}}\int_{\eta^k_j}^{\eta^k_{j+1}} (\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k})^2 \mathrm{d}\tau_k \mathrm{d}\theta_k + 2\int_{\psi^k_i}^{\psi^k_{i+1}} \omega_{\tau_k\theta_k}\omega_{\theta_k}(\eta^k_{j+1}, \theta_k)\mathrm{d}\theta_k \\
&\quad - 2\int_{\psi^k_i}^{\psi^k_{i+1}} \omega_{\tau_k\theta_k}\omega_{\theta_k}(\eta^k_j, \theta_k)\mathrm{d}\,\theta_k - 2\int_{\eta^k_j}^{\eta^k_{j+1}} \omega_{\tau_k\tau_k}\omega_{\theta_k}(\tau_k, \psi^k_{i+1})\mathrm{d}\tau_k \\
&\quad + 2\int_{\eta^k_j}^{\eta^k_{j+1}} \omega_{\tau_k\tau_k}\omega_{\theta_k}(\tau_k, \psi^k_i)\mathrm{d}\tau_k.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\int_{\Omega^k_{i,j}}\int -\omega(\omega_{xx} + \omega_{yy})\mathrm{d}x\mathrm{d}y \\
&= -\int_{\psi^k_i}^{\psi^k_{i+1}}\int_{\eta^k_j}^{\eta^k_{j+1}} \omega(\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k})\mathrm{d}\tau_k \mathrm{d}\theta_k \\
&= \int_{\psi^k_i}^{\psi^k_{i+1}}\int_{\eta^k_j}^{\eta^k_{j+1}} ((\omega_{\tau_k})^2 + (\omega_{\theta_k})^2)\mathrm{d}\tau_k \mathrm{d}\theta_k - \int_{\psi^k_i}^{\psi^k_{i+1}} \omega\omega_{\tau_k}(\eta^k_{j+1}, \theta_k)\mathrm{d}\theta_k \\
&\quad + \int_{\psi^k_i}^{\psi^k_{i+1}} \omega\omega_{\tau_k}(\eta^k_j, \theta_k)\mathrm{d}\theta_k - \int_{\eta^k_j}^{\eta^k_{j+1}} \omega\omega_{\theta_k}(\tau_k, \psi^k_{i+1})\mathrm{d}\tau_k \\
&\quad + \int_{\eta^k_j}^{\eta^k_{j+1}} \omega\omega_{\theta_k}(\tau_k, \psi^k_i)\mathrm{d}\tau_k.
\end{aligned}$$

And this gives us the following inequality:

$$\int_{\psi^k_i}^{\psi^k_{i+1}}\int_{\eta^k_j}^{\eta^k_{j+1}} ((\omega_{\tau_k})^2 + (\omega_{\theta_k})^2)\mathrm{d}\tau_k \mathrm{d}\theta_k$$

$$
\begin{aligned}
&\leq \frac{K}{2}\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\omega^2 \mathrm{d}\tau_k \mathrm{d}\theta_k + \frac{1}{2K}\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}(\omega_{\tau_k\tau_k}+\omega_{\theta_k\theta_k})^2 \mathrm{d}\tau_k \mathrm{d}\theta_k \\
&\quad + \int_{\psi_i^k}^{\psi_{i+1}^k}\omega\omega_{\tau_k}(\eta_{j+1}^k,\theta_k)\mathrm{d}\,\theta_k - \int_{\psi_i^k}^{\psi_{i+1}^k}\omega\omega_{\tau_k}(\eta_j^k,\theta_k)\mathrm{d}\theta_k \\
&\quad + \int_{\eta_j^k}^{\eta_{j+1}^k}\omega\omega_{\theta_k}(\tau_k,\psi_{i+1}^k)\mathrm{d}\tau_k - \int_{\eta_j^k}^{\eta_{j+1}^k}\omega\omega_{\theta_k}(\tau_k,\psi_i^k)\mathrm{d}\tau_k. \qquad (3.17)
\end{aligned}
$$

Let $u_{i,j}^k$ (τ_k, θ_k) be a set of nonconforming elements, defined on $\widetilde{\Omega}_{i,j}^k$ the image of $\Omega_{i,j}^k$ in (τ_k, θ_k) coordinates, given by

$$
u_{i,j}^k(\tau_k,\theta_k) = \sum_{n=0}^{N}\sum_{m=0}^{N} a_{m,n}\tau_k^m\theta_k^n
$$

for $j > 1$. We shall choose $u_{i,1}^k \equiv 0$ for all k and i.

Let

$$
\begin{aligned}
&[u_{i,j}^k](\eta_{j+1}^k,\theta_k) = (u_{i,j+1}^k - u_{i,j}^k)(\eta_{j+1}^k,\theta_k),\\
&[u_{i,j}^k](\tau_k,\psi_{i+1}^k) = (u_{i+1,j}^k - u_{i,j}^k)(\tau_k,\psi_{i+1}^k),
\end{aligned}
$$

denote the jump in u across inter-element boundaries.

Recollect that B_ρ^k denotes the circular arc with radius ρ and center A_k. Let $\widetilde{B}_\rho^k$ denotes its representation in (τ_k, θ_k) coordinates, i.e., $\widetilde{B}_\rho^k = \{(\tau_k,\theta_k):\ \tau_k = \ln\rho,\ \psi_l^k < \theta_k < \psi_u^k\}$. Similarly let $\widetilde{\Gamma}_k, \widetilde{\Gamma}_{k+1}$ denote the representation of the sides Γ_k and Γ_{k+1} in (τ_k, θ_k) coordinates. Recollect that $\widetilde{\Omega}^k$ is an open set and $\partial\widetilde{\Omega}^k$ denotes its boundary. Let γ_l be a side of $\Omega_{i,j}^k$ for some i and j and let $\widetilde{\gamma}_l$ denotes its representation in (τ_k, θ_k) coordinates.

We now state and prove a stability theorem which will help to motivate the stability theorem 3.3 on which our numerical scheme is based.

Theorem 3.1. *For the sectoral domain $\widetilde{\Omega}^k$ the following stability estimate holds.*

$$
\begin{aligned}
&\sum_{j=2}^{N}\sum_{i=1}^{I_k}\|u_{i,j}^k(\tau_k,\theta_k)\|_{2,\widetilde{\Omega}_{i,j}^k}^2 \\
&\leq C(\ln N)^2\left\{\sum_{j=2}^{N}\sum_{i=1}^{I_k}\|\Delta u_{i,j}^k(\tau_k,\theta_k)\|_{0,\widetilde{\Omega}_{i,j}^k}^2\right.\\
&\quad + \sum_{\widetilde{\gamma}_l\subseteq\widetilde{\Omega}^k}(\|[(u^k)]\|_{0,\widetilde{\gamma}_l}^2 + \|[(u^k)_{\tau_k}]\|_{1/2,\widetilde{\gamma}_l}^2 + \|[(u^k)_{\theta_k}]\|_{1/2,\widetilde{\gamma}_l}^2)\\
&\quad + \sum_{\widetilde{\gamma}_l\subseteq\widetilde{B}_\rho^k}(\|(u^k)\|_{0,\widetilde{\gamma}_l}^2 + \|(u^k)_{\theta_k}\|_{1/2,\widetilde{\gamma}_l}^2)\\
&\quad \left.+ \sum_{m=k}^{k+1}\sum_{\widetilde{\gamma}_l\subseteq\partial\widetilde{\Omega}^k\bigcap\widetilde{\Gamma}_m}(\|(u^k)\|_{0,\widetilde{\gamma}_l}^2 + \|(u^k)_{\tau_k}\|_{1/2,\widetilde{\gamma}_l}^2)\right\}. \qquad (3.18)
\end{aligned}
$$

Here $\|.\|_{s,\widetilde{\gamma}_l}$ denotes the fractional Sobolev norm as defined in [9] *when s is not an integer.*

Adding a weighted combination of (3.16) and (3.17) and summing over i and j gives

$$\sum_{j=1}^{N}\sum_{i=1}^{I_k}\Bigg\{\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}((u_{i,j}^k)_{\tau_k\tau_k}^2+2(u_{i,j}^k)_{\tau_k\theta_k}^2+(u_{i,j}^k)_{\theta_k\theta_k}^2)\mathrm{d}\tau_k\mathrm{d}\theta_k$$
$$+\ R\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}(((u_{i,j}^k)_{\tau_k})^2+\ ((u_{i,j}^k)_{\theta_k})^2)\mathrm{d}\tau_k\mathrm{d}\theta_k\Bigg\}$$
$$\le \text{(I)}+\text{(II)}+\text{(III)}+\text{(IV)}+\text{(V)}+\text{(VI)}+\text{(VII)}+\text{(VIII)}+\text{(IX)}\,. \quad (3.19)$$

Here the terms indicated by roman numerals given above are as follows:

$$\text{(I)}=\sum_{j=1}^{N}\sum_{i=1}^{I_k}\Bigg(\frac{KR}{2}\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{i,j}^k)^2\mathrm{d}\tau_k\mathrm{d}\,\theta_k$$
$$+\left(1+\frac{R}{2K}\right)\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}((u_{i,j}^k)_{\tau_k\tau_k}+(u_{i,j}^k)_{\theta_k\theta_k})^2\mathrm{d}\tau_k\mathrm{d}\,\theta_k\Bigg)\,,$$

$$\text{(II)}=\sum_{j=1}^{N-1}\sum_{i=1}^{I_k}\left(\int_{\psi_i^k}^{\psi_{i+1}^k}-2[(u_{i,j}^k)_{\theta_k}(u_{i,j}^k)_{\tau_k\theta_k}](\eta_{j+1}^k,\theta_k)\mathrm{d}\,\theta_k\right),$$

$$\text{(III)}=\sum_{i=1}^{I_k}2\int_{\psi_i^k}^{\psi_{i+1}^k}(u_{i,N}^k)_{\theta_k}(u_{i,N}^k)_{\tau_k\theta_k}(\ln\rho,\theta_k)\mathrm{d}\,\theta_k,$$

$$\text{(IV)}=\sum_{i=1}^{I_k-1}\sum_{j=2}^{N}\left(\int_{\eta_j^k}^{\eta_{j+1}^k}2[(u_{i,j}^k)_{\theta_k}(u_{i,j}^k)_{\tau_k\tau_k}](\tau_k,\psi_{i+1}^k)\mathrm{d}\tau_k\right),$$

$$\text{(V)}=\sum_{j=1}^{N}\Bigg(2\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{1,j}^k)_{\theta_k}(u_{1,j}^k)_{\tau_k\tau_k}(\tau_k,\psi_1^k)\mathrm{d}\tau_k$$
$$-2\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{I_k,j}^k)_{\theta_k}(u_{I_k,j}^k)_{\tau_k\tau_k}(\tau_k,\psi_{I_k+1}^k)\mathrm{d}\tau_k\Bigg),$$

$$\text{(VI)}=-R\sum_{j=1}^{N-1}\sum_{i=1}^{I_k}\left(\int_{\psi_i^k}^{\psi_{i+1}^k}[(u_{i,j}^k)(u_{i,j}^k)_{\tau_k}](\eta_{j+1}^k,\theta_k)\mathrm{d}\,\theta_k\right),$$

$$\text{(VII)}=R\sum_{i=1}^{I_k}\int_{\psi_i^k}^{\psi_{i+1}^k}(u_{i,N}^k)(u_{i,N}^k)_{\tau_k}(\ln\rho,\theta_k)\mathrm{d}\,\theta_k,$$

$$\text{(VIII)}=R\sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\left(-\int_{\eta_j^k}^{\eta_{j+1}^k}[(u_{i,j}^k)(u_{i,j}^k)_{\theta_k}](\tau_k,\psi_{i+1}^k)\mathrm{d}\tau_k\right),$$

and

$$\text{(IX)}=R\sum_{j=1}^{N}\Bigg(\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{I_k,j}^k)(u_{I_k,j}^k)_{\theta_k}(\tau_k,\psi_{I_k+1}^k)\mathrm{d}\tau_k$$
$$-\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{1,j}^k)(u_{1,j}^k)_{\theta_k}(\tau_k,\psi_1^k)\mathrm{d}\tau_k\Bigg)\,. \quad (3.20)$$

Now using Lemma 4.1 we can conclude that

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{i=1}^{I_k}\int_{\eta_j^k}^{\eta_{j+1}^k}\int_{\psi_i^k}^{\psi_{i+1}^k}(u_{i,j}^k)^2\mathrm{d}\tau_k\mathrm{d}\,\theta_k \\
&\leq C\left(\sum_{j=1}^{N}\sum_{i=1}^{I_k}\int_{\eta_j^k}^{\eta_{j+1}^k}\int_{\psi_i^k}^{\psi_{i+1}^k}((u_{i,j}^k)_{\theta_k})^2\mathrm{d}\tau_k\mathrm{d}\,\theta_k\right. \\
&\quad+\sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\int_{\eta_j^k}^{\eta_{j+1}^k}[(u_{i,j}^k)]^2(\tau_k,\psi_{i+1}^k)\mathrm{d}\tau_k \\
&\quad\left.+\sum_{j=1}^{N}\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{I_k,j}^k)^2(\tau_k,\psi_{I_k+1}^k)\mathrm{d}\tau_k+\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{1,j}^k)^2(\tau_k,\psi_1^k)\mathrm{d}\tau_k\right),
\end{aligned}
\tag{3.21}
$$

where

$$
C=2\max_k\left\{\max\left(\frac{(\psi_{I_k+1}^k-\psi_1^k)^2}{2},(I_k+1)(\psi_{I_k+1}^k-\psi_1^k)\right)\right\}.
$$

Choosing R large enough and adding

$$
\begin{aligned}
&2C\left(\sum_{j=1}^{N}\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{I_k,j}^k)^2(\tau_k,\psi_{I_k+1}^k)\mathrm{d}\tau_k+\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{1,j}^k)^2(\tau_k,\psi_1^k)\mathrm{d}\tau_k\right. \\
&\quad\left.+\sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\int_{\eta_j^k}^{\eta_{j+1}^k}[(u_{i,j}^k)]^2(\tau_k,\psi_{i+1}^k)\mathrm{d}\tau_k\right)
\end{aligned}
$$

to both sides of (3.19) and then applying (3.21) and choosing K small enough we get the inequality

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{i=1}^{I_k}\left\{\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}(((u_{i,j}^k)_{\tau_k\tau_k})^2+2((u_{i,j}^k)_{\tau_k\theta_k})^2\right. \\
&\quad+((u_{i,j}^k)_{\theta_k\theta_k})^2)\mathrm{d}\tau_k\mathrm{d}\,\theta_k \\
&\quad\left.+\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}((u_{i,j}^k)_{\tau_k}^2+(u_{i,j}^k)_{\theta_k}^2)\mathrm{d}\tau_k\mathrm{d}\,\theta_k\right\} \\
&\quad+\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{i,j}^k)^2\mathrm{d}\tau_k\mathrm{d}\,\theta_k \\
&\leq T\sum_{j=1}^{N}\sum_{i=1}^{I_k}\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}((u_{i,j}^k)_{\tau_k\tau_k}+(u_{i,j}^k)_{\theta_k\theta_k})^2\mathrm{d}\tau_k\mathrm{d}\,\theta_k
\end{aligned}
$$

$$
\begin{aligned}
&+ 2C\left(\sum_{j=1}^{N}\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{I_k,j}^k)^2(\tau_k,\psi_{I_k+1}^k)\mathrm{d}\tau_k\right.\\
&+\int_{\eta_j^k}^{\eta_{j+1}^k}(u_{1,j}^k)^2(\tau_k,\psi_1^k)\mathrm{d}\tau_k\\
&\left.+\sum_{i=1}^{I_k-1}\sum_{j=1}^{N}\int_{\eta_j^k}^{\eta_{j+1}^k}[(u_{i,j}^k)]^2(\tau_k,\psi_{i+1}^k)\mathrm{d}\tau_k\right)\\
&+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V})+(\mathrm{VI})+(\mathrm{VII})+(\mathrm{VIII})+(\mathrm{IX})\,. \qquad (3.22)
\end{aligned}
$$

We shall now estimate the terms indicated by roman numerals on the right hand side of (3.22) using Theorem 4.1. We begin by estimating

$$
\begin{aligned}
|(\mathrm{II})| = \Bigg|\sum_{j=1}^{N-1}\sum_{i=1}^{I_k}\Bigg(&-\int_{\psi_i^k}^{\psi_{i+1}^k}2[(u_{i,j}^k)_{\theta_k}][(u_{i,j}^k)_{\tau_k\theta_k}](\eta_{j+1}^k,\theta_k)\mathrm{d}\theta_k\\
&-\int_{\psi_i^k}^{\psi_{i+1}^k}2[(u_{i,j}^k)_{\theta_k}](u_{i,j}^k)_{\tau_k\theta_k}(\eta_{j+1}^k,\theta_k)\mathrm{d}\theta_k\\
&-\int_{\psi_i^k}^{\psi_{i+1}^k}2(u_{i,j}^k)_{\theta_k}[(u_{i,j}^k)_{\tau_k\theta_k}](\eta_{j+1}^k,\theta_k)\mathrm{d}\theta_k\Bigg)\Bigg|\,.
\end{aligned}
$$

Now by Theorem 4.1

$$
\begin{aligned}
&\left|\int_{\psi_i^k}^{\psi_{i+1}^k}2[(u_{i,j}^k)_{\theta_k}](u_{i,j}^k)_{\tau_k\theta_k}(\eta_{j+1}^k,\theta_k)\mathrm{d}\theta_k\right| \qquad (3.23)\\
&\le 2C\,(\ln N)\,\|[(u_{i,j}^k)_{\theta_k}](\eta_{j+1}^k,\theta_k)\|_{1/2,(\psi_i^k,\psi_{i+1}^k)}\\
&\quad\times\|(u_{i,j}^k)_{\tau_k}(\eta_{j+1}^k,\theta_k)\|_{1/2,(\psi_i^k,\psi_{i+1}^k)}\\
&\le\frac{(C\ln N)^2}{K}\|[(u_{i,j}^k)_{\theta_k}](\eta_{j+1}^k,\theta_k)\|^2_{1/2,(\psi_i^k,\psi_{i+1}^k)}\\
&\quad+K\|(u_{i,j}^k)_{\tau_k}(\eta_{j+1}^k,\theta_k)\|^2_{1/2,(\psi_i^k,\psi_{i+1}^k)}
\end{aligned}
$$

for any postive K.

By the trace theorem for Sobolev spaces there exists a constant M, such that

$$
\|(u_{i,j}^k)_{\tau_k}(\eta_{j+1}^k,\theta_k)\|^2_{1/2,(\psi_i^k,\psi_{i+1}^k)}\le M\|(u_{i,j}^k)(\tau_k,\theta_k)\|^2_{2,\widetilde{\Omega}_{i,j}^k}\,.
$$

Choosing $K=1/32M$ we have

$$
\begin{aligned}
&\left|\int_{\psi_i^k}^{\psi_{i+1}^k}2[(u_{i,j}^k)_{\theta_k}](u_{i,j}^k)_{\tau_k\theta_k}(\eta_{j+1}^k,\theta_k)\mathrm{d}\theta_k\right|\\
&\le T\,(\ln N)^2\,\|[(u_{i,j}^k)_{\theta_k}](\eta_{j+1}^k,\theta_k)\|^2_{1/2,(\psi_i^k,\psi_{i+1}^k)}\\
&\quad+\frac{1}{32}\|(u_{i,j}^k)(\tau_k,\theta_k)\|^2_{2,\widetilde{\Omega}_{i,j}^k}\,. \qquad (3.24)
\end{aligned}
$$

And so we conclude that

$$
\begin{aligned}
|(\mathrm{II})| \leq C\,(\ln N)^2 \sum_{j=1}^{N-1} \sum_{i=1}^{I_k} \Big(&\| [(u^k_{i,j})_{\tau_k}](\eta^k_{j+1}, \theta_k) \|^2_{1/2,(\psi^k_i,\psi^k_{i+1})} \\
&+ \| [(u^k_{i,j})_{\theta_k}](\eta^k_{j+1}, \theta_k) \|^2_{1/2,(\psi^k_i,\psi^k_{i+1})} \Big) \\
&+ \sum_{j=1}^{N-1} \sum_{i=1}^{I_k} \frac{1}{16} \| (u^k_{i,j})(\tau_k, \theta_k) \|^2_{2,\widetilde{\Omega}^k_{i,j}} .
\end{aligned} \tag{3.25}
$$

Similarly we have

$$
\begin{aligned}
|(\mathrm{V})| \leq C\,(\ln N)^2 \sum_{j=1}^{N} \Big(&\| (u^k_{1,j})_{\tau_k}(\tau_k, \psi^k_1) \|^2_{1/2,(\eta^k_j,\eta^k_{j+1})} \\
&+ \| (u^k_{I_k,j})_{\tau_k}(\tau_k, \psi^k_{I_k+1}) \|^2_{1/2,(\eta^k_j,\eta^k_{j+1})} \Big) \\
&+ \sum_{j=1}^{N} \frac{1}{16} \Big(\| (u^k_{1,j})(\tau_k, \theta_k) \|^2_{2,\widetilde{\Omega}^k_{1,j}} + \| (u^k_{I_k,j})(\tau_k, \theta_k) \|^2_{2,\widetilde{\Omega}^k_{I_k,j}} \Big).
\end{aligned} \tag{3.26}
$$

We can estimate the terms (IV), (VI), (VIII) and (IX) in the right-hand side of (3.20) in a similar manner. Putting all these estimates together we can write (3.22) in the form

$$
\begin{aligned}
&\frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{I_k} \| (u^k_{i,j})(\tau_k, \theta_k) \|^2_{2,\widetilde{\Omega}^k_{i,j}} \\
&\leq C\,(\ln N)^2 \Bigg\{ \sum_{j=1}^{N} \sum_{i=1}^{I_k} \int_{\psi^k_i}^{\psi^k_{i+1}} \int_{\eta^k_j}^{\eta^k_{j+1}} (\Delta u^k_{i,j})^2 \mathrm{d}\tau_k \mathrm{d}\theta_k \\
&\quad + \sum_{j=1}^{N-1} \sum_{i=1}^{I_k} \Big(\| [(u^k_{i,j})_{\tau_k}](\eta^k_{j+1}, \theta_k) \|^2_{1/2,(\psi^k_i,\psi^k_{i+1})} \\
&\quad + \| [(u^k_{i,j})](\eta^k_{j+1}, \theta_k) \|^2_{3/2,(\psi^k_i,\psi^k_{i+1})} \Big) \\
&\quad + \sum_{j=1}^{N} \sum_{i=1}^{I_k-1} \Big(\| [(u^k_{i,j})](\tau_k, \psi^k_{i+1}) \|^2_{3/2,(\eta^k_j,\eta^k_{j+1})} \\
&\quad + \| [(u^k_{i,j})_{\theta_k}](\tau_k, \psi^k_{i+1}) \|^2_{1/2,(\eta^k_j,\eta^k_{j+1})} \Big) \\
&\quad + \sum_{j=1}^{N} \Big(\| (u^k_{1,j})(\tau_k, \psi^k_1) \|^2_{3/2,(\eta^k_j,\eta^k_{j+1})} \\
&\quad + \| (u^k_{I_k,j})(\tau_k, \psi^k_{I_k+1}) \|^2_{3/2,(\eta^k_j,\eta^k_{j+1})} \Big) \Bigg\} \\
&\quad + \sum_{i=1}^{I_k} (R \int_{\psi^k_i}^{\psi^k_{i+1}} (u^k_{i,N})(u^k_{i,N})_{\tau_k}(\ln \rho, \theta_k) \mathrm{d}\theta_k \\
&\quad + 2 \int_{\psi^k_i}^{\psi^k_{i+1}} (u^k_{i,N})_{\theta_k} (u^k_{i,N})_{\tau_k \theta_k}(\ln \rho, \theta_k) \mathrm{d}\theta_k).
\end{aligned} \tag{3.27}
$$

Estimating the last two terms in the same way we get the result. □

Remark. The *a priori* stability estimate for sectoral domains (3.18), in a sense, replicates the estimates, known as the shift theorems, for elliptic problems on smooth domains. These estimates are valid, however, only in modified polar coordinates and only for spectral element functions which are polynomials of degree N in each variable separately. Moreover, the constant multiplying the right-hand side is not independent of N but grows slowly with N like $(\ln N)^2$. Thus, Theorem 3.1 states that the sum of the squares of the H^2 norms of the spectral element functions depends continuously on a quadratic form, which consists of the sum of the squares of the L^2 norms of the differential operators in their respective elements plus the sum of the squares of the tangential derivatives in the $H^{3/2}$ norm along the image of the sectoral boundary plus a penalty term. The penalty term consists of the sum of the squares of the L^2 norms of the jumps in the the function elements plus the sum of the squares of $H^{1/2}$ norms of the jumps in the derivatives across inter-element boundaries.

Consider the function $\{u^k_{i,j}\}_{i,j}$ defined on $\widetilde{\Omega}^k_{i,j} \subseteq \widetilde{\Omega}^k$. Then the inequality

$$\begin{aligned}
&\frac{1}{2}\sum_{j=2}^{N}\sum_{i=1}^{I_k} \|(u^k_{i,j})(\tau_k,\theta_k)\|^2_{2,\widetilde{\Omega}^k_{i,j}} \\
&\leq C\,(\ln N)^2 \Bigg\{ \sum_{j=2}^{N}\sum_{i=1}^{I_k} \|\Delta u^k_{i,j}(\tau_k,\theta_k)\|^2_{0,\widetilde{\Omega}^k_{i,j}} \\
&\quad + \sum_{m=k}^{k+1} \sum_{\widetilde{\gamma}_i \subseteq \widetilde{\Gamma}_m \bigcap \partial\widetilde{\Omega}^k} (\|u^k\|^2_{0,\widetilde{\gamma}_i} + \|u^k_{\tau_k}\|^2_{1/2,\widetilde{\gamma}_i}) \\
&\quad + \sum_{\widetilde{\gamma}_i \subseteq \widetilde{\Omega}^k} (\|[u^k]\|^2_{0,\widetilde{\gamma}_i} + \|[u^k_{\tau_k}]\|^2_{1/2,\widetilde{\gamma}_i} + \|[u^k_{\theta_k}]\|^2_{1/2,\widetilde{\gamma}_i}) \Bigg\} \\
&\quad + \sum_{i=1}^{I_k} \Bigg(R\int_{\psi^k_i}^{\psi^k_{i+1}} (u^k_{i,N})(u^k_{i,N})_{\tau_k}(\ln\rho,\theta_k)\mathrm{d}\theta_k \\
&\quad + 2\int_{\psi^k_i}^{\psi^k_{i+1}} (u^k_{i,N})_{\theta_k}(u^k_{i,N})_{\tau_k\theta_k}(\ln\rho,\theta_k)\mathrm{d}\theta_k \Bigg)
\end{aligned} \tag{3.28}$$

is valid.

The estimate (3.28) follows immediately from (3.27). The reader may now directly proceed to Theorem 3.3 which is a generalization of Theorem 3.1 and refer back to the proof, which is quite involved, later.

We need to obtain a similar estimate for T^k. On each $\overline{\Omega}^k_{i,j} \subseteq \overline{T}^k$ a function $u^k_{i,j}\,(x,y)$ is defined. Now, this $\Omega^k_{i,j} \subseteq \Omega^{p+1}$ and hence $\Omega^k_{i,j} = \Omega^{p+1}_l$ for some l. We shall use $\Omega^k_{i,j}$ and Ω^{p+1}_l interchangeably in what follows.

We now need to obtain an energy inequality similar to Theorem 3.1 on T^k. Integrating by parts repeatedly we get

$$\rho^2 \int\int_{\mathcal{O}} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 \mathrm{d}x\mathrm{d}y$$
$$= \rho^2 \left(\int\int_{\mathcal{O}} \left(\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right) \mathrm{d}x\mathrm{d}y \right.$$
$$\left. +2\int_{\partial\mathcal{O}} \frac{\partial w}{\partial y}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial w}{\partial x}\right)\mathrm{d}s\right). \tag{3.29}$$

Here s denotes arc length along $\partial\mathcal{O}$ measured from some point on it and the line integral is evaluated in the clockwise direction and n denotes the outward normal to $\partial\mathcal{O}$, the boundary of $\mathcal{O}$. Hence

$$\rho^2 \int_{\Omega^k_{i,N+1}}\int \left(\frac{\partial^2 u^k_{i,N+1}}{\partial x^2} + \frac{\partial^2 u^k_{i,N+1}}{\partial y^2}\right)^2 \mathrm{d}x\mathrm{d}y$$
$$= \rho^2 \int_{\Omega^k_{i,N+1}}\int \left(\left(\frac{\partial^2 u^k_{i,N+1}}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u^k_{i,N+1}}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 u^k_{i,N+1}}{\partial y^2}\right)^2\right)\mathrm{d}x\mathrm{d}y$$
$$+ 2\rho^2\left(\int_{\partial\Omega^k_{i,N+1}\bigcap\left(B^k_\rho\right)^c} + \int_{\partial\Omega^k_{i,N+1}\bigcap\left(B^k_\rho\right)}\right)\frac{\partial u^k_{i,N+1}}{\partial y}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u^k_{i,N+1}}{\partial x}\right)\mathrm{d}s.$$

Here B^k_ρ denotes the boundary of the circle of radius ρ with center at A_k. Now a simple calculation yields

$$2\rho^2 \int_{\partial\Omega^k_{i,N+1}\bigcap B^k_\rho} \frac{\partial u^k_{i,N+1}}{\partial y}\frac{\mathrm{d}}{\mathrm{ds}}\left(\frac{\partial u^k_{i,N+1}}{\partial x}\right)\mathrm{d}s$$
$$= 2\int_{\psi^k_i}^{\psi^k_{i+1}} (u^k_{i,N+1})_{\theta_k}(u^k_{i,N+1})_{\theta_k\tau_k}(\ln\rho,\theta_k)\mathrm{d}\theta_k$$
$$- \int_{\psi^k_i}^{\psi^k_{i+1}} ((u^k_{i,N+1})^2_{\tau_k} + (u^k_{i,N+1})^2_{\theta_k})(\ln\rho,\theta_k)\mathrm{d}\theta_k$$
$$+ \rho^2 B(\theta_k, (u^k_{i,N+1})_{r_k}, (u^k_{i,N+1})_{n_k})(\ln\rho,\theta_k)\Big|^{\psi^k_{i+1}}_{\psi^k_i}. \tag{3.30}$$

Here

$$B(\theta, a, b) = \left(a^2\frac{\sin 2\theta}{2} - b^2\frac{\sin 2\theta}{2} - 2ab\sin^2\theta\right) \tag{3.31}$$

and

$$\frac{\partial}{\partial n_k} = \frac{1}{\rho}\frac{\partial}{\partial\theta_k}.$$

Next suppose Ω^{p+1}_l is such that $\partial\Omega^{p+1}_l \bigcap \Gamma_m \neq \phi$. Then $\partial\Omega^{p+1}_l \bigcap \Gamma_m$ is the straight line joining the points D^m_j and D^m_{j+1} for $m \in \{k, k+1\}$ and for some $1 \leq j \leq M_m$ as shown

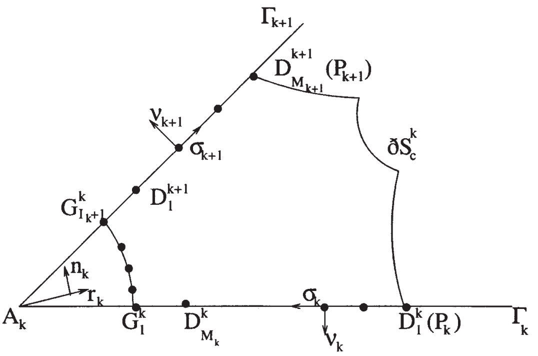

Figure 4.

in figure 4. Here $M_m = J_m - N$. Now we can show that

$$\begin{aligned}
&2\rho^2 \int_{\partial\Omega_l^{p+1} \bigcap \Gamma_m} \frac{\partial u_l^{p+1}}{\partial y} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\partial u_l^{p+1}}{\partial x} \right) \mathrm{d}s \\
&= 2\rho^2 \int_{\partial\Omega_l^{p+1} \bigcap \Gamma_m} (u_l^{p+1})_{\nu_m} (u_l^{p+1})_{\sigma_m\sigma_m} \mathrm{d}\sigma_m \\
&\quad + \rho^2 B(\psi_1^m, (u_l^{p+1})_{\sigma_m} (u_l^{p+1})_{\nu_m}) \Big|_{D_j^m}^{D_{j+1}^m}. \qquad (3.32)
\end{aligned}$$

Here $\mathrm{d}/\mathrm{d}\,\sigma_m$ denotes the tangential derivative and $\mathrm{d}/\mathrm{d}\,\nu_m$ the normal derivative along Γ_m.

Now

$$\begin{aligned}
&R \int_{\Omega_{i,N+1}^k} \int \left(\left(\frac{\partial u_{i,N+1}^k}{\partial x} \right)^2 + \left(\frac{\partial u_{i,N+1}^k}{\partial y} \right)^2 \right) \mathrm{d}x\mathrm{d}y \\
&\leq \frac{R}{2K} \int_{\Omega_{i,N+1}^k} \int \left(\frac{\partial^2 u_{i,N+1}^k}{\partial x^2} + \frac{\partial^2 u_{i,N+1}^k}{\partial y^2} \right)^2 \mathrm{d}x\mathrm{d}y \\
&\quad + \frac{KR}{2} \int_{\Omega_{i,N+1}^k} \int (u_{i,N+1}^k)^2 \mathrm{d}x\mathrm{d}y \\
&\quad - R \int_{\psi_i^k}^{\psi_{i+1}^k} (u_{i,N+1}^k)(u_{i,N+1}^k)_{\tau_k} (\ln\rho, \theta_k) \mathrm{d}\theta_k \\
&\quad + R \int_{\partial\Omega_{i,N+1}^k \bigcap (B_\rho^k)^c} (u_{i,N+1}^k) \frac{\partial (u_{i,N+1}^k)}{\partial n} \mathrm{d}s. \qquad (3.33)
\end{aligned}$$

Similarly for $\Omega_l^{p+1} = \Omega_{i,j}^k$ with $j > N+1$ we get

$$R \int_{\Omega_l^{p+1}} \int \left(\left(\frac{\partial u_l^{p+1}}{\partial x} \right)^2 + \left(\frac{\partial u_l^{p+1}}{\partial y} \right)^2 \right) \mathrm{d}x\mathrm{d}y$$
$$\leq \frac{R}{2K} \int_{\Omega_l^{p+1}} \int \left(\frac{\partial^2 u_l^{p+1}}{\partial x^2} + \frac{\partial^2 u_l^{p+1}}{\partial y^2} \right)^2 \mathrm{d}x\mathrm{d}y$$
$$+ \frac{KR}{2} \int_{\Omega_l^{p+1}} \int (u_l^{p+1})^2 \mathrm{d}x\mathrm{d}y + R \int_{\partial\Omega_l^{p+1}} u_l^{p+1} \frac{\partial u_l^{p+1}}{\partial n} \mathrm{d}s. \tag{3.34}$$

Let $L^k = \left\{ l : \Omega_l^{p+1} \subseteq T^k \right\}$. We can now prove the following lemma:

Lemma 3.1

$$\sum_{l \in L^k} \left(\rho^2 \int_{\Omega_l^{p+1}} \int \left(\left(\frac{\partial^2 u_l^{p+1}}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 u_l^{p+1}}{\partial x \partial y} \right)^2 \right.\right.$$
$$\left.\left. + \left(\frac{\partial^2 u_l^{p+1}}{\partial y^2} \right)^2 \right) \mathrm{d}x\mathrm{d}y + R \int_{\Omega_l^{p+1}} \int \left(\left(\frac{\partial u_l^{p+1}}{\partial x} \right)^2 + \left(\frac{\partial u_l^{p+1}}{\partial y} \right)^2 \right) \mathrm{d}x\mathrm{d}y \right)$$
$$\leq (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}) + (\mathrm{VI}) + (\mathit{VII}) + (\mathrm{VIII}) + (\mathrm{IX}) + (\mathrm{X}). \tag{3.35}$$

The terms indicated by roman numerals are as follows:

$$(\mathrm{I}) = \left(\rho^2 + \frac{R}{2K} \right) \sum_{l \in L^k} \int_{\Omega_l^{p+1}} \int (\Delta u_l^{p+1}(x, y))^2 \mathrm{d}\, x\mathrm{d}y,$$
$$(\mathrm{II}) = \frac{KR}{2} \sum_{l \in L^k} \int_{\Omega_l^{p+1}} \int (u_l^{p+1}(x, y))^2 \mathrm{d}x\mathrm{d}y,$$
$$(\mathrm{III}) = \sum_{i=1}^{I_k} \left\{ -2 \int_{\psi_i^k}^{\psi_{i+1}^k} (u_{i,N+1}^k)_{\theta_k} (u_{i,N+1}^k)_{\tau_k\theta_k} (\ln\rho, \theta_k) \mathrm{d}\theta_k \right.$$
$$\left. - R \int_{\psi_i^k}^{\psi_{i+1}^k} (u_{i,N+1}^k)(u_{i,N+1}^k)_{\tau_k} (\ln\rho, \theta_k) \mathrm{d}\theta_k \right\},$$
$$(\mathrm{IV}) = \rho \sum_{i=1}^{I_k} \int_{\partial\Omega_{i,N+1}^k \bigcap B_\rho^k} \int ((u_{i,N+1}^k)_x^2 + (u_{i,N+1}^k)_y^2) \mathrm{d}s,$$
$$(\mathrm{V}) = R \sum_{l \in L^k} \left\{ \left(\sum_{\gamma_s \subseteq \partial\Omega_l^{p+1} \bigcap T^k} + \sum_{\gamma_s \subseteq \partial\Omega_l^{p+1} \bigcap \partial S_c^k} \right) \int_{\gamma_s} u_l^{p+1} \frac{\partial u_l^{p+1}}{\partial n} \mathrm{d}s \right.$$
$$\left. + \sum_{m=k}^{k+1} \sum_{\gamma_s \subseteq \partial\Omega_l^{p+1} \bigcap \Gamma_m} \int_{\gamma_s} (u_l^{p+1})(u_l^{p+1})_{\nu_m} \mathrm{d}\sigma_m \right\},$$

$$(\mathrm{VI}) = -2\rho^2 \sum_{l\in L^k} \left\{ \left(\sum_{\gamma_s \subseteq \partial\Omega_l^{p+1} \bigcap T^k} + \sum_{\gamma_s \subseteq \partial\Omega_l^{p+1} \bigcap \partial S_c^k} \right) \right.$$

$$\times \int_{\gamma_s} \frac{\partial u_l^{p+1}}{\partial y} \frac{\mathrm{d}}{\mathrm{ds}} \left(\frac{\partial u_l^{p+1}}{\partial x} \right) \mathrm{d}s.$$

$$\left. + \sum_{m=k}^{k+1} \sum_{\gamma_s \subseteq \partial\Omega_l^{p+1} \bigcap \Gamma_m} \int_{\gamma_s} (u_l^{p+1})_{\nu_m} (u_l^{p+1})_{\sigma_m\sigma_m} \mathrm{d}\sigma_m \right\},$$

$$(\mathrm{VII}) = -\sum_{i=1}^{I_k} \rho^2 B(\theta_k, (u_{i,N+1}^k)_{r_k}, (u_{i,N+1}^k)_{n_k})(\ln\rho, \theta_k)\Big|_{\psi_i^k}^{\psi_{i+1}^k},$$

$$(\mathrm{VIII}) = -\sum_{m=k}^{k+1} \sum_{i=1}^{M_m-1} \sum_{l, \partial\Omega_l^{p+1} \bigcap \Gamma_{m,i} \neq \phi} \rho^2 B(\psi_1^m, (u_l^{p+1})_{\sigma_m},$$

$$(u_l^{p+1})_{\nu_m})\Big|_{D_i^m}^{D_{i+1}^m},$$

$$(\mathrm{IX}) = -\rho^2 B(\psi_1^k, (u_{1,N+1}^k)_{\sigma_k}, (u_{1,N+1}^k)_{\nu_k})\Big|_{D_{M_k}^k}^{G_1^k},$$

and $$(\mathrm{X}) = -\rho^2 B(\psi_1^{k+1}, (u_{I_k,N+1}^k)_{\sigma_{k+1}}, (u_{I_k,N+1}^k)_{\nu_{k+1}})\Big|_{G_{I_k+1}^k}^{D_1^{k+1}}. \tag{3.36}$$

By γ_s we denote an arc which is a side of $\partial\Omega_l^{p+1}$ for $l \in L^k$. Here $\partial/\partial r_k$ denotes the radial derivative and $\partial/\partial n_k$ the tangential derivative to the circle with center at A_k and radius ρ, i.e., $\partial/\partial n_k = (1/\rho)(\partial/\partial\theta_k)$. Moreover, $\partial/\partial\sigma_k$ denotes the tangential derivative and $\partial/\partial\nu_k$ the normal derivative to the side Γ_k (figure 4). Finally $\Gamma_{m,i}$ is the open subset of the straight line Γ_m between the points D_i^m and D_{i+1}^m and $u_{I_{m+1},N+1}^{m+1}$ denotes $u_{I_1,N+1}^1$ if $m = p$.

Using the estimates (3.30)–(3.34) we obtain (3.35). □

Recall from (3.2) that there exists a mapping M_l^{p+1} from the unit square S to $\overline{\Omega}_l^{p+1}$ given by

$$x = X_l^{p+1}(\xi, \eta)$$

$$y = Y_l^{p+1}(\xi, \eta).$$

Similarly there exists a mapping $M_{i,N+1}^k$ from S to $\overline{\Omega}_{i,N+1}^k$.

We now define another semi-norm in terms of the transformed variables ξ and η:

$$|v(\xi, \eta)|_{m,S}^2 = \sum_{|\alpha|=m} \int_S \int |D_\xi^{\alpha_1} D_\eta^{\alpha_2} v(X_l^{p+1}(\xi, \eta), Y_l^{p+1}(\xi, \eta))|^2 \mathrm{d}\xi \mathrm{d}\eta.$$

Let

$$\left|M_l^{p+1}\right|_{m,\infty,S} = \text{ess} \sup_{(\xi,\eta)\in S} \Big(\max\Big(\max_{|\alpha|\le m} (|D^\alpha X_l^{p+1}|), \max_{|\alpha|\le m} (|D^\alpha Y_l^{p+1}|)\Big)\Big). \tag{3.37a}$$

Then we have the following results [3]

$$|u(\xi,\eta)|_{0,S}^2 \le \frac{C}{|J_{M_l^{p+1}}|} |u|^2_{0,\Omega_l^{p+1}} \le C\,|u|^2_{0,\Omega_l^{p+1}}, \tag{3.37b}$$

$$|u(\xi,\eta)|_{1,S}^2 \le C\frac{|M_l^{p+1}|^2_{1,\infty,S}}{|J_{M_l^{p+1}}|} |u|^2_{1,\Omega_l^{p+1}} \le C\,|u|^2_{1,\Omega_l^{p+1}}, \tag{3.37c}$$

and

$$\begin{aligned}|u(\xi,\eta)|_{2,S}^2 &\le \frac{C}{|J_{M_l^{p+1}}|}\big(|M_l^{p+1}|^4_{1,\infty,S}\,|u|^2_{2,\Omega_l^{p+1}} + |M_l^{p+1}|^2_{2,\infty,S}\,|u|^2_{1,\Omega_l^{p+1}}\big)\\ &\le C\big(|u|^2_{2,\Omega_l^{p+1}} + |u|^2_{1,\Omega_l^{p+1}}\big).\end{aligned} \tag{3.37d}$$

Here $J_{M_l^{p+1}}$ denotes the Jacobian of the transformation M_l^{p+1} as defined in (3.3a) and (3.3b) and $|J_{M_l^{p+1}}| = \min_{(\xi,\eta)\in S} |J_{M_l^{p+1}}(\xi,\eta)|$; note that we have used the bound given in (3.3b) to arrive at the above results. Consider the point D_i^k in figure 4. Then there exist two domains Ω_l^{p+1} and Ω_m^{p+1} on whose boundary D_i^k lies. Let

$$[w^{p+1}](D_i^k) = w_m^{p+1}(D_i^k) - w_l^{p+1}(D_i^k),$$

where $\partial\Omega_m^{p+1} \bigcap \Gamma_k$ is traversed first if we travel along Γ_k from D_1^k to $D_{M_k}^k$.

Moreover let

$$[w^{p+1}](G_i^k) = w_{i,N+1}^k(G_i^k) - w_{i-1,N+1}^k(G_i^k).$$

We now prove the following lemma.

Lemma 3.2

$$\begin{aligned}&|(\text{VII}) + (\text{VIII}) + (\text{IX}) + (\text{X})| \le (\text{XI})\\ &\quad + \frac{6\rho^2}{32}\sum_{l\in L^k}\Big(|u_l^{p+1}|^2_{1,\Omega_l^{p+1}} + |u_l^{p+1}|^2_{2,\Omega_l^{p+1}}\Big),\end{aligned} \tag{3.38}$$

where

$$
\begin{aligned}
(\mathrm{XI}) = C \ln N \Bigg(& \sum_{i=2}^{I_k} |[(u^{p+1})_x](G_i^k)|^2 + |[(u^{p+1})_y](G_i^k)|^2 \\
& + \sum_{m=k}^{k+1} \sum_{i=2-\delta_{m,k+1}}^{M_m-\delta_{m,k+1}} (([(u^{p+1})_x](D_i^m))^2 \\
& + ([(u^{p+1})_y](D_i^m))^2) \Bigg) \\
& + \sum_{m=k}^{k+1} (-1)^{m+k-1} \rho^2 \{B(\psi_1^m, u_{\nu_m}, u_{\sigma_m})\}(P_m). \qquad (3.39)
\end{aligned}
$$

Here P_m *is* D_1^k *if* $m = k$ *and* P_m *is* $D_{M_{k+1}}^{k+1}$ *if* $m = k + 1$.

We first estimate one of the terms in the right-hand side of (3.38). Now

$$
\begin{aligned}
& |\rho^2 \sin^2(\psi_i^k)\{(u_{i-1,N+1}^k)_{r_k}(u_{i-1,N+1}^k)_{n_k} \\
& \quad - (u_{i,N+1}^k)_{r_k}(u_{i,N+1}^k)_{n_k}\}(G_i^k)| \\
& \leq \rho^2(|(u_{i,N+1}^k)_{r_k}(G_i^k)||[(u_{i,N+1}^k)_{n_k}](G_i^k)| \\
& \quad + |(u_{i-1,N+1}^k)_{n_k}(G_i^k)||[(u_{i,N+1}^k)_{r_k}](G_i^k)|). \qquad (3.40)
\end{aligned}
$$

Now by Corollary 4.80 of [12] we have that if a and b are real numbers such that $a^2+b^2 = 1$ and w is a smooth function defined on Ω_l^{p+1} such that

$$
w(X_l^{p+1}(\xi, \eta), Y_l^{p+1}(\xi, \eta)) = \sum_{n=0}^{N} \sum_{m=0}^{N} a_{m,n} \xi^m \eta^n
$$

then

$$
\left| \left(a w_x + b w_y \right)(P) \right|^2 \leq C (\ln N) \left(|w|_{1,\Omega_l^{p+1}}^2 + |w|_{2,\Omega_l^{p+1}}^2 \right). \qquad (3.41)
$$

Using (3.41) we obtain

$$
\begin{aligned}
& |\rho^2 \sin^2(\psi_i^k)\{(u_{i-1,N+1}^k)_{r_k}(u_{i-1,N+1}^k)_{n_k} \\
& \quad - (u_{i,N+1}^k)_{r_k}(u_{i,N+1}^k)_{n_k}\}(G_i^k)| \\
& \leq C \ln N \{([(u_{i,N+1}^k)_{r_k}](G_i^k))^2 + ([(u_{i,N+1}^k)_{n_k}](G_i^k))^2\} \\
& \quad + \frac{\rho^2}{32} \Big(|u_{i,N+1}^k|_{1,\Omega_{i,N+1}^k}^2 + |u_{i,N+1}^k|_{2,\Omega_{i,N+1}^k}^2 + |u_{i-1,N+1}^k|_{1,\Omega_{i-1,N+1}^k}^2 \\
& \quad + |u_{i-1,N+1}^k|_{2,\Omega_{i-1,N+1}^k}^2 \Big).
\end{aligned}
$$

Treating the other terms in the same way we obtain the result. □

We now estimate the term (IV) in (3.36). Let w be a smooth function on $\Omega^k_{i,N+1}$. Then

$$\begin{aligned}\int_{\partial\Omega^k_{i,N+1}\bigcap B^k_\rho} w^2 \mathrm{d}s &= \int_{\psi^k_i}^{\psi^k_{i+1}} w^2(\rho,\theta_k)\,\rho \mathrm{d}\theta_k \\ &= \int_{\psi^k_i}^{\psi^k_{i+1}}\int_\rho^\nu -\rho\frac{\partial}{\partial r_k}\left(\frac{\nu-r_k}{\nu-\rho}w^2\right)\mathrm{d}r_k\mathrm{d}\theta_k \\ &= \int_{\psi^k_i}^{\psi^k_{i+1}}\int_\rho^\nu \frac{\rho}{\nu-\rho}w^2\mathrm{d}r_k\mathrm{d}\theta_k + \int_{\psi^k_i}^{\psi^k_{i+1}}\int_\rho^\nu -2\rho\left(\frac{\nu-r_k}{\nu-\rho}\right)ww_{r_k}\mathrm{d}r_k\mathrm{d}\theta_k \\ &\le \frac{1}{\nu-\rho}\int_{\psi^k_i}^{\psi^k_{i+1}}\int_\rho^\nu w^2 r_k\mathrm{d}r_k\mathrm{d}\theta_k + 2\int_{\psi^k_i}^{\psi^k_{i+1}}\int_\rho^\nu |ww_{r_k}|r_k\mathrm{d}r_k\mathrm{d}\theta_k.\end{aligned}$$

And so we obtain

$$\begin{aligned}\int_{B^k_\rho\bigcap\partial\Omega^k_{i,N+1}} w^2\mathrm{d}s &\le \left(\frac{1}{\nu-\rho}+\alpha\right)\int_{\psi^k_i}^{\psi^k_{i+1}}\int_\rho^\nu w^2 r_k\mathrm{d}r_k\mathrm{d}\theta_k \\ &+ \frac{1}{\alpha}\int_{\psi^k_i}^{\psi^k_{i+1}}\int_\rho^\nu (w_{r_k})^2 r_k\mathrm{d}r_k\mathrm{d}\theta_k.\end{aligned}\tag{3.42}$$

for any $\alpha > 0$.

Hence using (3.42) we get

$$\begin{aligned}(\mathrm{IV}) &= \rho\int_{\partial\Omega^k_{i,N+1}\bigcap B^k_\rho}((u^k_{i,N+1})^2_x + (u^k_{i,N+1})^2_y)\mathrm{d}s \\ &\le \left(\frac{\rho}{\nu-\rho}+\alpha\rho\right)\int_{\Omega^k_{i,N+1}}\int((u^k_{i,N+1})^2_x + (u^k_{i,N+1})^2_y)\mathrm{d}x\mathrm{d}y \\ &\quad + \frac{\rho}{\alpha}\int_{\Omega^k_{i,N+1}}\int((u^k_{i,N+1})^2_{xx} + 2(u^k_{i,N+1})^2_{xy} + (u^k_{i,N+1})^2_{yy})\mathrm{d}x\mathrm{d}y.\end{aligned}\tag{3.43}$$

Choose α so large that $(\rho/\alpha) \le (\rho^2/32)$ and choose $R > [\rho/(\nu-\rho)] + \alpha\rho + (\rho^2/2)$. Then combining (3.43) with Lemma 3.2, we have the result

$$\begin{aligned}&\sum_{l\in L^k}\frac{25}{32}\rho^2|u_l^{p+1}|^2_{2,\Omega_l^{p+1}} + \sum_{l\in L^k}\left(R-\frac{\rho}{\nu-\rho}-\alpha\rho\right)|u_l^{p+1}|^2_{1,\Omega_l^{p+1}} \\ &\le ((\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{V})+(\mathrm{VI}))+(\mathrm{XI})\,.\end{aligned}\tag{3.44}$$

We now obtain an estimate for the term (VI) as defined in (3.36).

We shall estimate the first term in (VI). Now

$$\begin{aligned}&\left|2\rho^2\sum_{l\in L^k}\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap T^k}\int_{\gamma_s}\frac{\partial u_l^{p+1}}{\partial y}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u_l^{p+1}}{\partial x}\right)\mathrm{d}s\right| \\ &\le 2\rho^2\sum_{\gamma_s\subseteq T^k}\left|\int_{\gamma_s}\left[\frac{\partial u_l^{p+1}}{\partial y}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u_l^{p+1}}{\partial x}\right)\right]\mathrm{d}s\right|.\end{aligned}\tag{3.45}$$

Let us get an upper bound on a typical element in the sum, in the right-hand side of the above inequality, which is of the form

$$2\rho^2 \left| \int_{\gamma_s} \left(\left(\frac{\partial u_m^{p+1}}{\partial y} \right) \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\partial u_m^{p+1}}{\partial x} \right) - \left(\frac{\partial u_n^{p+1}}{\partial y} \right) \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\partial u_n^{p+1}}{\partial x} \right) \right) \mathrm{d}s \right|. \quad (3.46)$$

We shall assume, to be specific, that $\partial\Omega_{m,l}^{p+1}$ is the image of the side $\xi = 1$ of the square S under the mapping M_m^{p+1} and $\partial\Omega_{n,j}^{p+1}$ is the image of the side $\xi = 0$ of S under the mapping M_n^{p+1}. Recall from (3.10) and (3.11) that there exists i, j and k such that $\Omega_l^{p+1} = \Omega_{i,j}^k$ for some i, j with $j > N$. Hence the mapping M_m^{p+1} is the mapping $M_{i,j}^k$ from S to $\overline{\Omega}_{i,j}^k$ as in (3.2). This representation is needed only for $1 \le k \le p$, $N < j \le J_k$ and $1 \le i \le I_{k,j}$. Now $J_k = N + O(1)$ and $I_{k,j} \le I$ and hence there are a fixed number of $\Omega_{i,j}^k$ for which this representation is needed even if we let $N \to \infty$. As such we may assume

$$\max_{i,j,k,j>N} |M_{i,j}^k|_{m,\infty,S} \le C_m \quad (3.47)$$

where the norm has been defined in (3.37a). Note that C_m is independent of N. We shall impose further restrictions on C_m in the second part of this paper where we shall examine the accuracy of our numerical scheme. Here we shall only establish the stability of our scheme and for that an estimate of the type (3.47) is adequate. Now

$$\frac{\partial u_m^{p+1}}{\partial x} = (u_m^{p+1})_\xi \xi_x + (u_m^{p+1})_\eta \eta_x, \quad (3.48a)$$

and

$$\frac{\partial u_m^{p+1}}{\partial y} = (u_m^{p+1})_\xi \xi_y + (u_m^{p+1})_\eta \eta_y. \quad (3.48b)$$

We have that

$$\begin{cases} x = X_m^{p+1}(\xi, \eta) \\ y = Y_m^{p+1}(\xi, \eta) \end{cases}, \quad 0 \le \xi \le 1, 0 \le \eta \le 1,$$

Let $\widehat{\xi_x}(\xi, \eta)$, $\widehat{\eta_x}(\xi, \eta)$, $\widehat{\xi_y}(\xi, \eta)$ and $\widehat{\eta_y}(\xi, \eta)$ be the unique polynomials in ξ and η which are the orthogonal projections of $\xi_x(\xi, \eta)$, $\eta_x(\xi, \eta)$, $\xi_y(\xi, \eta)$ and $\eta_y(\xi, \eta)$ into the space of polynomials of degree $(N - 1)$ in each variable separately with respect to the usual inner product in $H^2\left((0, 1)^2\right)$, as defined in [12]. We now define approximations to the derivatives $\partial u_m^{p+1}/\partial x$ and $\partial u_m^{p+1}/\partial y$ as follows. Let

$$\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a = (u_m^{p+1})_\xi \widehat{\xi_x} + (u_m^{p+1})_\eta \widehat{\eta_x}, \quad (3.49a)$$

and

$$\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a = (u_m^{p+1})_\xi \widehat{\xi_y} + (u_m^{p+1})_\eta \widehat{\eta_y}. \quad (3.49b)$$

Using the approximation results in [12] we have that

$$|\xi_x - \widehat{\xi}_x|_{1,\infty,S} \leq K_m N^{6-m} \, \|\xi_x\|_{m,S} \, . \tag{3.50}$$

Now $\xi_x = (Y_m^{p+1})_\eta / J_m^{p+1}$. Moreover by (3.47)

$$|M_{i,j}^k|_{m,\infty,S} \leq C_m \quad \text{for all } j > N,$$

and by (3.3b)

$$A_1 \rho^2 \leq |J_{i,j}^k| \leq A_2 \rho^2 \quad \text{for all } j > N.$$

So it is easy to see that

$$|\xi_x - \widehat{\xi}_x|_{1,\infty,S} \leq C N^{-4} \tag{3.51}$$

for all $M_{i,j}^k$ with $j > N$, and N large enough. A similar result holds for ξ_y, η_x and η_y.

We are now in a position to prove the following lemma.

Lemma 3.3. *Let γ_s be contained in T^k. Then*

$$\begin{aligned} 2\rho^2 &\left| \int_{\gamma_s} \left[\frac{\partial u^{p+1}}{\partial y} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\partial u^{p+1}}{\partial x} \right) \right] \mathrm{d}s \right| \\ &\leq C \, (\ln N)^2 \left(\left\| \left[\left(\frac{\partial u^{p+1}}{\partial x} \right)^a \right] \right\|_{1/2,\gamma_s}^2 + \left\| \left[\left(\frac{\partial u^{p+1}}{\partial y} \right)^a \right] \right\|_{1/2,\gamma_s}^2 \right) \\ &\quad + \frac{\rho^2}{16} \sum_{l=1}^{2} (|u_m^{p+1}|_{l,\Omega_m^{p+1}}^2 + |u_n^{p+1}|_{l,\Omega_n^{p+1}}^2). \end{aligned} \tag{3.52a}$$

Here

$$\left\| \left[\left(\frac{\partial u^{p+1}}{\partial x} \right)^a \right] \right\|_{1/2,\gamma_s}^2 = \left\| \left(\frac{\partial u_m^{p+1}}{\partial x} \right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial x} \right)^a (0,\eta) \right\|_{1/2,(0,1)}^2 , \tag{3.52b}$$

and

$$\left\| \left[\left(\frac{\partial u^{p+1}}{\partial y} \right)^a \right] \right\|_{1/2,\gamma_s}^2 = \left\| \left(\frac{\partial u_m^{p+1}}{\partial y} \right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial y} \right)^a (0,\eta) \right\|_{1/2,(0,1)}^2 . \tag{3.52c}$$

It is easy to see that

$$\int_{\gamma_s} \frac{\partial u_m^{p+1}}{\partial y} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u_m^{p+1}}{\partial x}\right) \mathrm{d}s - \int_{\gamma_s} \left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a \mathrm{d}s$$
$$= \int_0^1 \Big\{((u_m^{p+1})_\xi(\xi_y - \widehat{\xi}_y) + (u_m^{p+1})_\eta(\eta_y - \widehat{\eta}_y))$$
$$\times \frac{\mathrm{d}}{\mathrm{d}\eta}\,((u_m^{p+1})_\xi \xi_x + (u_m^{p+1})_\eta \eta_x)\Big\}(1,\eta)\mathrm{d}\eta$$
$$+ \int_0^1 \Big\{((u_m^{p+1})_\xi(\widehat{\xi}_y) + (u_m^{p+1})_\eta(\widehat{\eta}_y))$$
$$\times \frac{\mathrm{d}}{\mathrm{d}\eta}\,((u_m^{p+1})_\xi(\xi_x - \widehat{\xi}_x) + (u_m^{p+1})_\eta(\eta_x - \widehat{\eta}_x))\Big\}(1,\eta)\mathrm{d}\eta. \tag{3.53}$$

Hence

$$2\rho^2 \left| \int_{\gamma_s} \frac{\partial u_m^{p+1}}{\partial y} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u_m^{p+1}}{\partial x}\right) \mathrm{d}s - \int_0^1 \Big\{((u_m^{p+1})_\xi \widehat{\xi}_y + (u_m^{p+1})_\eta \widehat{\eta}_y) \right.$$
$$\left. \times \frac{\mathrm{d}}{\mathrm{d}\eta}((u_m^{p+1})_\xi \widehat{\xi}_x + (u_m^{p+1})_\eta \widehat{\eta}_x)\Big\}(1,\eta)\mathrm{d}\eta \right|$$
$$\leq \frac{C}{N^4} \sum_{l=1}^{2} |u_m^{p+1}|^2_{l,\partial S} \leq \frac{C}{N^4}(\|(u_m^{p+1})_\xi\|^2_{3/2,S} + \|(u_m^{p+1})_\eta\|^2_{3/2,S})$$
$$\leq \frac{C}{N^2}\Big(\sum_{l=1}^{2} |u_m^{p+1}|^2_{l,\Omega_m^{p+1}}\Big) \tag{3.54}$$

by the trace theorem and an inequality for fractional Sobolev spaces we obtain below along with (3.37a)–(3.37d). The inequality is as follows.

Let

$$w(\xi,\eta) = \sum_{m=0}^{N} \sum_{n=0}^{N} a_{m,n} \xi^m \eta^n$$

defined on S. Then

$$\|w\|^2_{1/2,S} \leq C\,\|w\|_{0,S}\,\|w\|_{1,S} \leq CN^2\,\|w\|^2_{0,S}$$

by the interpolation inequality and the inverse inequality for differentiation in [12]. Thus for N large enough

$$2\rho^2 \left| \int_{\gamma_s} \frac{\partial u_m^{p+1}}{\partial y} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u_m^{p+1}}{\partial x}\right) \mathrm{d}s - \int_{\gamma_s} \left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a \mathrm{d}s \right|$$
$$\leq \frac{\rho^2}{32} \sum_{l=1}^{2} |u_m^{p+1}|^2_{l,\Omega_m^{p+1}}. \tag{3.55}$$

Now

$$\begin{aligned}
&\left|\int_0^1 \left\{\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a\right\}(1,\eta)\,\mathrm{d}\eta\right.\\
&\quad\left.-\int_0^1 \left\{\left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a\right\}(0,\eta)\,\mathrm{d}\eta\right|\\
&\leq \left|\int_0^1 \left\{\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta)\right\}\right.\\
&\quad\left.\times\ \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)\right)\mathrm{d}\eta\right|\\
&\quad+\left|\int_0^1 \left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta)\left\{\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)\right.\right.\right.\\
&\quad\left.\left.\left.-\left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a (0,\eta)\right)\right\}\mathrm{d}\eta\right|.
\end{aligned}$$

Clearly $(\partial u_m^{p+1}/\partial x)^a(1,\eta)$, $(\partial u_m^{p+1}/\partial y)^a(1,\eta)$, $(\partial u_n^{p+1}/\partial x)^a(0,\eta)$ and $(\partial u_n^{p+1}/\partial y)^a\ (0,\eta)$ are polynomials in η of degree at most $2N$. Hence by Theorem 4.1

$$\begin{aligned}
&2\rho^2\left|\int_0^1 \left\{\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a\right\}(1,\eta)\,\mathrm{d}\eta\right.\\
&\quad\left.-\int_0^1 \left\{\left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a\right\}(0,\eta)\,\mathrm{d}\eta\right|\\
&\leq \frac{C}{K}(\ln N)^2\left\{\left\|\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta)\right\|^2_{1/2,(0,1)}\right.\\
&\quad\left.+\left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a (0,\eta)\right\|^2_{1/2,(0,1)}\right\}\\
&\quad+K\left\{\left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)\right\|^2_{1/2,(0,1)}+\left\|\left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta)\right\|^2_{1/2,(0,1)}\right\},
\end{aligned}\tag{3.56}$$

for any $K > 0$.

Now

$$\begin{aligned}
\left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)\right\|_{1/2,(0,1)} &\leq C_1(\|\widehat{\xi}_x\|_{1,\infty,(0,1)}\|(u_m^{p+1})_\xi(1,\eta)\|_{1/2,(0,1)}\\
&\quad+\|\widehat{\eta}_x\|_{1,\infty,(0,1)}\|(u_m^{p+1})_\eta\|_{1/2,(0,1)}).
\end{aligned}$$

Using the above estimate, the trace theorem and (3.37a)–(3.37d) we get

$$\left\| \left(\frac{\partial u_m^{p+1}}{\partial x} \right)^a (1, \eta) \right\|_{1/2,(0,1)}^2 \leq C \sum_{l=1}^{2} |u_m^{p+1}|_{l,\Omega_m^{p+1}}^2 . \tag{3.57}$$

Substituting the above estimates into (3.56) and choosing K small enough we can conclude that

$$\begin{aligned} &2\rho^2 \left| \int_0^1 \left\{ \left(\frac{\partial u_m^{p+1}}{\partial y} \right)^a \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\frac{\partial u_m^{p+1}}{\partial x} \right)^a \right\} (1, \eta) \, \mathrm{d}\eta \right. \\ &\quad \left. - \int_0^1 \left\{ \left(\frac{\partial u_n^{p+1}}{\partial y} \right)^a \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\frac{\partial u_n^{p+1}}{\partial x} \right)^a \right\} (0, \eta) \, \mathrm{d}\eta \right| \\ &\leq C \, (\ln N)^2 \left\{ \left\| \left(\frac{\partial u_m^{p+1}}{\partial y} \right)^a (1, \eta) - \left(\frac{\partial u_n^{p+1}}{\partial y} \right)^a (0, \eta) \right\|_{1/2,(0,1)}^2 \right. \\ &\quad \left. + \left\| \left(\frac{\partial u_m^{p+1}}{\partial x} \right)^a (1, \eta) - \left(\frac{\partial u_n^{p+1}}{\partial x} \right)^a (0, \eta) \right\|_{1/2,(0,1)}^2 \right\} \\ &\quad + \frac{\rho^2}{32} \left(\sum_{l=1}^{2} |u_m^{p+1}|_{l,\Omega_m^{p+1}}^2 + |u_n^{p+1}|_{l,\Omega_n^{p+1}}^2 \right) . \end{aligned} \tag{3.58}$$

Hence we get the required result. □

Finally

$$\begin{aligned} &2\rho^2 \left| \int_{\partial\Omega_l^{p+1} \bigcap \Gamma_m} (u_l^{p+1})_{\nu_m} (u_l^{p+1})_{\sigma_m \sigma_m} \mathrm{d}\sigma_m \right| \\ &= 2\rho^2 \left| \int \left\{ (u_l^{p+1})_{\nu_m} \frac{\mathrm{d}}{\mathrm{d}\eta} ((u_l^{p+1})_{\sigma_m}) \right\} (1, \eta) \, \mathrm{d}\eta \right| \end{aligned}$$

or a similar expression.

Now

$$(u_l^{p+1})_{\sigma_m} (1, \eta) = A\,(\eta)\, (u_l^{p+1})_{\eta}\, (1, \eta)\, ,$$

and

$$(u_l^{p+1})_{\nu_m} (1, \eta) = B(\eta) (u_l^{p+1})_{\xi}\, (1, \eta) + C\,(\eta)\, (u_l^{p+1})_{\eta}\, (1, \eta)\, .$$

The form of the expressions $B\,(\eta)$ and $C\,(\eta)$ do not matter except that they are analytic functions of η involving X_l^{p+1}, Y_l^{p+1} and their derivatives at $(1, \eta)$. Hence we can bound the derivatives of B and C as in (3.48a)–(3.51). Let $\widehat{A}\,(\eta)$ be the unique polynomial that is the orthogonal projection of $A\,(\eta)$ into the space of polynomials of degree $N - 1$ with respect to the usual norm defined on $H^2\,(0, 1)$. We now define

$$(u_l^{p+1})_{\sigma_m}^a (1, \eta) = \widehat{A}(\eta) (u_l^{p+1})_{\eta} (1, \eta),$$

and

$$(u_l^{p+1})^a_{\nu_m}(1,\eta) = \widehat{B}(\eta)(u_l^{p+1})_\xi(1,\eta) + \widehat{C}(\eta)(u_l^{p+1})_\eta(1,\eta).$$

It is easy to prove as we did the estimate (3.52a)–(3.52c) that

$$\begin{aligned} &2\rho^2 \left| \int_{\gamma_s} \{(u_l^{p+1})_{\nu_m}(u_l^{p+1})_{\sigma_m\sigma_m}\} d\sigma_m \right| \\ &\leq C\,(\ln N)^2 \, \|(u_l^{p+1})^a_{\sigma_m}\|^2_{1/2,\gamma_s} + \frac{\rho^2}{32}\left(\sum_{i=1}^{2} |u_l^{p+1}|^2_{i,\Omega_l^{p+1}}\right), \end{aligned} \tag{3.59a}$$

where $\gamma_s \subseteq \Gamma_m \bigcap \partial\Omega_l^{p+1}$ for some $m \in \{k, k+1\}$.

Here

$$\|(u_l^{p+1})^a_{\sigma_m}\|^2_{1/2,\gamma_s} = \|(u_l^{p+1})^a_{\sigma_m}(1,\eta)\|^2_{1/2,(0,1)}. \tag{3.59b}$$

Hence using the estimates (3.52a)–(3.52c) and (3.59a) and (3.59b) we obtain

$$|(\mathrm{VI})| \leq (\mathrm{XII}) + \sum_{l\in L^k} \frac{\rho^2}{8} \sum_{i=1}^{2} |u_l^{p+1}(x,y)|^2_{i,\Omega_l^{p+1}} \tag{3.60a}$$

where

$$\begin{aligned} (\mathrm{XII}) = C\,(\ln N)^2 &\Bigg\{ \sum_{\gamma_s\subseteq T^k} (\|[(u^{p+1})^a_x]\|^2_{1/2,\gamma_s} + \|[(u^{p+1})^a_y]\|^2_{1/2,\gamma_s}) \\ &+ \left(\sum_{m=k}^{k+1} \sum_{l\in L^k} \sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\Gamma_m} \|(u_l^{p+1})^a_{\sigma_m}\|^2_{\gamma_s} \right) \Bigg\} \\ &- 2\rho^2 \sum_{l\in L^k} \sum_{\gamma_s\subseteq\partial S_c^k\bigcap\partial\Omega_l^{p+1}} \int_{\gamma_s} \frac{\partial u_l^{p+1}}{\partial y} \frac{d}{ds}\left(\frac{\partial u_l^{p+1}}{\partial x}\right) ds. \end{aligned} \tag{3.60b}$$

Now using Lemma 4.1 and (3.3a) and (3.3b) we get

$$\begin{aligned} \sum_{l\in L^k} |u_l^{p+1}(x,y)|^2_{0,\Omega_l^{p+1}} \leq T &\Bigg\{ \sum_{\gamma_s\subseteq T^k} |[u^{p+1}]|^2_{0,\gamma_s} \\ &+ \left(\sum_{m=k}^{k+1} \sum_{l\in L^k} \sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\Gamma_m} |u_l^{p+1}|^2_{0,\gamma_s} \right) + \sum_{l\in L^k} |u_l^{p+1}\,(x,y)\,|^2_{1,\Omega_l^{p+1}} \Bigg\}. \end{aligned} \tag{3.61}$$

Here the constant T is independent of N.

Choose

$$R > \frac{\rho}{\nu-\rho} + \alpha\rho + (T+1)\,\rho^2.$$

Adding

$$T\rho^2\left\{\sum_{\gamma_s\subseteq T^k}|[u^{p+1}]\|^2_{0,\gamma_s}+\left(\sum_{m=k}^{k+1}\sum_{l\in L^k}\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\Gamma_m}\left|u_l^{p+1}\right|^2_{0,\gamma_s}\right)\right\}$$

to both sides of (3.44) we obtain

$$\begin{aligned}
&\sum_{l\in L^k}\Bigg(\rho^2|u_l^{p+1}(x,y)|^2_{0,\Omega_l^{p+1}}\\
&\quad+\left(R-\frac{\rho}{\nu-\rho}-\alpha\rho-T\rho^2\right)|u_l^{p+1}(x,y)|^2_{1,\Omega_l^{p+1}}\\
&\quad+\frac{21}{32}\rho^2|u_l^{p+1}(x,y)|^2_{2,\Omega_l^{p+1}}\Bigg)\\
&\leq((\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{V}))+((\mathrm{XI})+(\mathrm{XII}))\\
&\quad+T\rho^2\left\{\sum_{\gamma_s\subseteq T^k}\|[u^{p+1}]\|^2_{0,\gamma_s}\right.\\
&\quad\left.+\left(\sum_{m=k}^{k+1}\sum_{l\in L^k}\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\Gamma_m}\|u_l^{p+1}\|^2_{0,\gamma_s}\right)\right\}.
\end{aligned}\tag{3.62}$$

We now have to approximate

$$|\Delta u_l^{p+1}|^2_{0,\Omega_l^{p+1}}=\int_{\Omega_l^{p+1}}\int(\Delta u_l^{p+1})^2\mathrm{d}x\;\mathrm{d}y.$$

Now

$$\begin{aligned}
\Delta u_l^{p+1}&=a_l^{p+1}(u_l^{p+1})_{\xi\xi}+2b_l^{p+1}(u_l^{p+1})_{\xi\eta}+c_l^{p+1}(u_l^{p+1})_{\eta\eta}\\
&\quad+d_l^{p+1}(u_l^{p+1})_{\xi}+e_l^{p+1}(u_l^{p+1})_{\eta}.
\end{aligned}$$

Hence

$$\int_{\Omega_l^{p+1}}\int(\Delta u_l^{p+1})^2\mathrm{d}x\;\mathrm{d}y=\int_{(0,1)\times(0,1)}\int(L_l^{p+1}u_l^{p+1})^2\mathrm{d}\xi\;\mathrm{d}\eta,$$

where

$$L_l^{p+1}w=A_l^{p+1}w_{\xi\xi}+2B_l^{p+1}w_{\xi\eta}+C_l^{p+1}w_{\eta\eta}+D_l^{p+1}w_{\xi}+E_l^{p+1}w_{\eta},$$

and $A_l^{p+1}=a_l^{p+1}\sqrt{J_l^{p+1}}$, etc. Let $\widehat{A}_l^{p+1}$ denote the unique polynomial which is the orthogonal projection of A_l^{p+1} into the space of polynomials of degree $N-1$ in ξ and η with respect to the usual inner product in $H^2(S)$. We define $\widehat{B}_l^{p+1}$, $\widehat{C}_l^{p+1}$, $\widehat{D}_l^{p+1}$ and $\widehat{E}_l^{p+1}$ in the same way.

Let

$$(L_l^{p+1})^a w = \widehat{A}_l^{p+1} w_{\xi\xi} + 2\widehat{B}_l^{p+1} w_{\xi\eta} + \cdots .$$

Then it is easy to prove that for N large enough

$$\begin{aligned}(\mathrm{I}) &= \sum_{l\in L^k}\left(\rho^2+\frac{R}{2K}\right)\int_{\Omega_l^{p+1}}\int(\Delta u_l^{p+1})^2 \mathrm{d}x\,\mathrm{d}y\\ &\le \sum_{l\in L^k} 2\left(\rho^2+\frac{R}{2K}\right)\int_S\int((L_l^{p+1})^a u_l^{p+1})^2 \mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad+\frac{\rho^2}{16}\|u_l^{p+1}\|_{2,S}^2. \end{aligned} \tag{3.63}$$

Substituting $K = \rho^2/2R$ in (II) and estimating the term (V) as before along with the above estimates we obtain

$$\begin{aligned}&\sum_{l\in L^k}\frac{\rho^2}{2}\|u_l^{p+1}\|_{2,S}^2\\ &\le C\,(\ln N)^2\left(\left(\sum_{l\in L^k}\|(L_l^{p+1})^a u_l^{p+1}(\xi,\eta)\|_{0,S}^2\right)\right.\\ &\quad+\sum_{\gamma_s\subseteq T^k}(\|[u^{p+1}]\|_{0,\gamma_s}^2+\|[(u^{p+1})_x^a]\|_{1/2,\gamma_s}^2+\|[(u^{p+1})_y^a]\|_{1/2,\gamma_s}^2)\\ &\quad\left.+\sum_{m=k}^{k+1}\sum_{\gamma_s\subseteq\partial T^k\bigcap\Gamma_m}(\|u^{p+1}\|_{0,\gamma_s}^2+\|(u^{p+1})_{\sigma_m}^a\|_{1/2,\gamma_s}^2)\right)\\ &\quad+\sum_{\gamma_s\subseteq\partial S_c^k}\left(\int_{\gamma_s}Ru^{p+1}\frac{\partial u^{p+1}}{\partial n}\mathrm{d}s-2\rho^2\int_{\gamma_s}\frac{\partial u^{p+1}}{\partial y}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u^{p+1}}{\partial x}\right)\mathrm{d}s\right)\\ &\quad+(\mathrm{III})+(\mathrm{XI}). \end{aligned} \tag{3.64}$$

Here (III) is as defined in (3.36) and (XI) is as defined in (3.39).

We are now in a position to prove an energy inequality for the subdomain S^k which we state in the following theorem.

Theorem 3.2. *Consider the subdomain S^k. Then for N large enough*

$$\begin{aligned}&\frac{1}{8}\sum_{j=1}^{N}\sum_{i=1}^{I_k}\|u_{i,j}^k(\tau_k,\theta_k)\|_{2,\widetilde{\Omega}_{i,j}^k}^2+\frac{\alpha}{8}\sum_{l\in L^k}\|u_l^{p+1}(\xi,\eta)\|_{2,\Omega_l^{p+1}}^2\\ &\le\{(1)+(2)+(3)+(4)\}, \end{aligned} \tag{3.65}$$

where

$$
\begin{aligned}
(1) = C\,(\ln N)^2 \Bigg(& \sum_{j=2}^{N} \sum_{i=1}^{I_k} \| \Delta u_{i,j}^k(\tau_k, \theta_k) \|_{0,\widetilde{\Omega}_{i,j}^k}^2 \\
& + \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Omega}^k} (\| [u^k] \|_{0,\widetilde{\gamma}_l}^2 + \| [u_{\tau_k}^k] \|_{1/2,\widetilde{\gamma}_l}^2 + \| [u_{\theta_k}^k] \|_{1/2,\widetilde{\gamma}_l}^2) \\
& + \sum_{m=k}^{k+1} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Gamma}_m \bigcap \partial \widetilde{\Omega}^k} (\| u^k \|_{0,\widetilde{\gamma}_l}^2 + \| u_{\tau_k}^k \|_{1/2,\widetilde{\gamma}_l}^2) \Bigg), \\
(2) = C\,(\ln N)^2 \Bigg(& \sum_{\widetilde{\gamma}_l \subseteq \widetilde{B}_\rho^k} (\| [u^k] \|_{0,\widetilde{\gamma}_l}^2 + \| [(u^k)_{\tau_k}^a] \|_{1/2,\widetilde{\gamma}_l}^2 \\
& + \| [(u^k)_{\theta_k}^a] \|_{1/2,\widetilde{\gamma}_l}^2) \Bigg), \\
(3) = C\,(\ln N)^2 \Bigg(& \sum_{l \in L^k} \| (L_l^{p+1})^a u_l^{p+1}(\xi, \eta) \|_{0,S}^2 \\
& + \sum_{\gamma_s \subseteq T^k} (\| [u^{p+1}] \|_{0,\gamma_s}^2 + \| [(u^{p+1})_x^a] \|_{1/2,\gamma_s}^2 + \| [(u^{p+1})_y^a] \|_{1/2,\gamma_s}^2) \\
& + \sum_{m=k}^{k+1} \sum_{\gamma_s \subseteq \partial T^k \bigcap \Gamma_m} (\| u^{p+1} \|_{0,\gamma_s}^2 + \| (u^{p+1})_{\sigma_m}^a \|_{1/2,\gamma_s}^2) \Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
(4) = & \sum_{m=k}^{k+1} (-1)^{m-k+1} (\rho^2 B(\psi_1^m, u_{\nu_m}^k, u_{\sigma_m}^k))(P_m) \\
& + \sum_{\gamma_s \subseteq \partial S_c^k} \int_{\gamma_s} \left(R u^k \frac{\partial u^k}{\partial n} - 2\rho^2 \frac{\partial u^k}{\partial y} \frac{\mathrm{d}}{\mathrm{ds}} \left(\frac{\partial u^k}{\partial x} \right) \right) \mathrm{d}s.
\end{aligned}
$$

By (3.37a)–(3.37d) there exists a positive constant α such that

$$
\alpha \| u_l^{p+1}(\xi, \eta) \|_{2,S}^2 \le \rho^2 \| u_l^{p+1}(x, y) \|_{2,\Omega_l^{p+1}}^2 \tag{3.66}
$$

for all $\Omega_l^{p+1} \subseteq T^k$ and for all k.

Then combining (3.27) and (3.64) and using (3.66) we obtain

$$
\begin{aligned}
& \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{I_k} \| u_{i,j}^k(\tau_k, \theta_k) \|_{2,\widetilde{\Omega}_{i,j}^k}^2 + \frac{\alpha}{2} \sum_{l \in L^k} \| u_l^{p+1}(\xi, \eta) \|_{2,S}^2 \\
& \quad \le \{(1) + (3) + (4)\} + (\mathrm{XIII}) + (\mathrm{XIV}) + (\mathrm{XV}).
\end{aligned} \tag{3.67}
$$

Here

$$(\text{XIII}) = C\ln N\Bigg(\sum_{i=2}^{I_k}(|[(u^k)_{\tau_k}](G_i^k)|^2 + |[(u^k)_{\theta_k}](G_i^k)|^2) + \sum_{m=k}^{k+1}\sum_{i=2-\delta_{m,k+1}}^{M_m-\delta_{m,k+1}}(|[(u^{p+1})_x](D_i^m)|^2 + |[(u^{p+1})_y](D_i^m)|^2)\Bigg).$$

The remaining two terms are

$$(\text{XIV}) = R\left(\sum_{i=1}^{I_k}\int_{\psi_i^k}^{\psi_{i+1}^k}((u_{i,N}^k)(u_{i,N}^k)_{\tau_k} - (u_{i,N+1}^k)(u_{i,N+1}^k)_{\tau_k})(\ln\rho,\theta_k)\,\mathrm{d}\,\theta_k\right),$$

and

$$(\text{XV}) = 2\Bigg(\sum_{i=1}^{I_k}\int_{\psi_i^k}^{\psi_{i+1}^k}((u_{i,N}^k)_{\theta_k}(u_{i,N}^k)_{\tau_k\theta_k} - (u_{i,N+1}^k)_{\theta_k}(u_{i,N+1}^k)_{\tau_k\theta_k})(\ln\rho,\theta_k)\,\mathrm{d}\,\theta_k\Bigg).$$

Once more using Theorem 4.1 we can show that

$$\begin{aligned}&|(\text{XIV})| + |(\text{XV})| \\ &\le C\,(\ln N)^2\left(\sum_{\widetilde{\gamma}_l\subseteq\widetilde{B}_\rho^k}(\|[(u^k)]\|_{0,\widetilde{\gamma}_l}^2 + \|[(u^k)_{\tau_k}^a]\|_{1/2,\widetilde{\gamma}_l}^2 + \|[(u^k)_{\theta_k}^a]\|_{1/2,\widetilde{\gamma}_l}^2)\right) \\ &\quad + \frac{1}{32}\sum_{i=1}^{I_k}\|u_{i,N}^k(\tau_k,\theta_k)\|_{2,\widetilde{\Omega}_{i,N}^k}^2 + \frac{\alpha}{32}\sum_{i=1}^{I_k}\|u_{i,N+1}^k(x,y)\|_{2,\Omega_{i,N+1}^k}^2.\end{aligned} \tag{3.68}$$

We now consider the term $|[(u^{p+1})_x](D_i^m)|^2$. There exist two domains Ω_t^{p+1} and Ω_s^{p+1} such that $\partial\Omega_s^{p+1}\bigcap\partial\Omega_t^{p+1} = \gamma_l$ and D_i^m is an end point of the curve γ_l. Let us assume that $\gamma_l\bigcap\partial\Omega_t^{p+1}$ is the image of the mapping M_t^{p+1} of the boundary $\eta = 1$ of S and $\gamma_l\bigcap\partial\Omega_s^{p+1}$ is the image of the mapping M_s^{p+1} of the boundary $\eta = 0$ of S. Further let D_i^m correspond to the image of the point $\xi = 0$ for both these cases.

Now

$$\begin{aligned}&|[(u^{p+1})_x](D_i^m)|^2 \\ &\le 3\{(((u_t^{p+1})_x^a(\xi,1) - (u_s^{p+1})_x^a(\xi,0))|_{\xi=0})^2 \\ &\quad + (((u_t^{p+1})_x - (u_t^{p+1})_x^a)(\xi,1)|_{\xi=0})^2 \\ &\quad + (((u_s^{p+1})_x - (u_s^{p+1})_x^a)(\xi,0)|_{\xi=0})^2\}.\end{aligned} \tag{3.69}$$

Moreover $((u_t^{p+1})_x^a(\xi,1) - (u_s^{p+1})_x^a(\xi,0))$ is a polynomial in ξ of degree at most $2N$.

Now by Theorem 4.79 of [12] we have that if $p\,(s)$ is polynomial of degree M defined on [0, 1] then

$$\|p\|^2_{L^\infty[0,1]} \leq C\,(1+\ln M)\,\|p\|^2_{1/2,[0,1]}\,.$$

Hence we obtain

$$3(((u_t^{p+1})_x^a(\xi,1)-(u_s^{p+1})_x^a(\xi,0))|_{\xi=0})^2$$
$$\leq K\ln N \|[(u^{p+1})_x^a]\|^2_{1/2,\gamma_l}.$$

Now

$$|((u_t^{p+1})_x-(u_t^{p+1})_x^a)(0,1)|^2$$
$$\leq 2(|((u_t^{p+1})_\xi(\xi_x-\widehat{\xi}_x))(0,1)|^2+|((u_t^{p+1})_\eta(\eta_x-\widehat{\eta}_x))(0,1)|^2).$$

Using (3.51) and the Sobolev's embedding theorem we can conclude that

$$|((u_t^{p+1})_x-(u_t^{p+1})_x^a)(0,1)|^2 \leq \frac{C}{N^4}\|u_t^{p+1}(\xi,\eta)\|^2_{5/2,S}.$$

And as before we can show

$$|((u_t^{p+1})_x-(u_t^{p+1})_x^a)(0,1)|^2 \leq \frac{C}{N^2}\|u_t^{p+1}(\xi,\eta)\|^2_{2,S}.$$

Choosing N large enough we obtain

$$|((u_t^{p+1})_x-(u_t^{p+1})_x^a)(0,1)|^2 \leq \frac{\alpha}{32}\|u_t^{p+1}(\xi,\eta)\|^2_{2,S}.$$

And so we can conclude that

$$C\ln N\left(\sum_{m=k}^{k+1}(|[(u^{p+1})_x](D_i^m)|^2+|[(u^{p+1})_y](D_i^m)|^2)\right)$$
$$\leq K\,(\ln N)^2\left(\sum_{m=k}^{k+1}\sum_{\gamma_s\subseteq\partial T^k\bigcap\Gamma_m}(\|[(u^{p+1})_x^a]\|^2_{1/2,\gamma_s}\right.$$
$$\left.+\|[(u^{p+1})_y^a]\|^2_{1/2,\gamma_s})\right)+\frac{12\alpha}{32}\sum_{l\in L^k}\|u_l^{p+1}\|^2_{2,S}. \qquad (3.70)$$

In the same way, we can conclude that

$$C\ln N\sum_{i=2}^{I_k}(|[(u^k)_{\tau_k}](G_i^k)|^2+|[(u^k)_{\theta_k}](G_i^k)|^2)$$
$$\leq K\,(\ln N)^2\left(\sum_{\widetilde{\gamma}_l\subseteq\widetilde{B}_\rho^k}(\|[(u^k)_{\tau_k}]\|^2_{1/2,\widetilde{\gamma}_l}+\|[(u^k)_{\theta_k}]\|^2_{1/2,\widetilde{\gamma}_l})\right). \qquad (3.71)$$

Substituting (3.68), (3.70) and (3.71) into (3.67) we get the required result. □

We have now obtained an energy inequality for any of the p subdomains S^k into which we had divided our original domain. Combining these estimates we can now prove the main theorem of this paper which can be interpreted as a stability estimate for the whole domain.

Theorem 3.3. *Consider the whole domain Ω. Then for N large enough there exists a constant C such that*

$$\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\|u_{i,j}^k(\tau_k,\theta_k)\|_{2,\widetilde{\Omega}_{i,j}^k}^2+\sum_{l=1}^{L}\|u_l^{p+1}(\xi,\eta)\|_{2,S}^2$$
$$\leq C(\ln N)^2\left\{\left(\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\|\Delta u_{i,j}^k(\tau_k,\theta_k)\|_{0,\widetilde{\Omega}_{i,j}^k}^2\right.\right.$$
$$+\sum_{k=1}^{p}\sum_{\widetilde{\gamma}_l\subseteq\widetilde{\Omega}^k}(\|[(u^k)]\|_{0,\widetilde{\gamma}_l}^2+\|[(u^k)_{\tau_k}]\|_{1/2,\widetilde{\gamma}_l}^2+\|[(u^k)_{\theta_k}]\|_{1/2,\widetilde{\gamma}_l}^2)$$
$$\left.+\sum_{k=1}^{p}\sum_{m=k}^{k+1}\sum_{\widetilde{\gamma}_l\subseteq\widetilde{\Gamma}_m\bigcap\partial\widetilde{\Omega}^k}(\|(u^k)\|_{0,\widetilde{\gamma}_l}^2+\|(u^k)_{\tau_k}\|_{1/2,\widetilde{\gamma}_l}^2)\right)$$
$$+\left(\sum_{k=1}^{p}\sum_{\widetilde{\gamma}_l\subseteq\widetilde{B}_\rho^k}(\|[(u^k)]\|_{0,\widetilde{\gamma}_l}^2+\|[(u^k)_{\tau_k}^a]\|_{1/2,\widetilde{\gamma}_l}^2+\|[(u^k)_{\theta_k}^a]\|_{1/2,\widetilde{\gamma}_l}^2)\right)$$
$$+\left(\sum_{l=1}^{L}\|((L_l^{p+1})^a u_l^{p+1})(\xi,\eta)\|_{0,S}^2\right.$$
$$+\sum_{\gamma_s\subseteq\Omega^{p+1}}(\|[(u^{p+1})]\|_{0,\gamma_s}^2+\|[(u^{p+1})_x^a]\|_{1/2,\gamma_s}^2+\|[(u^{p+1})_y^a]\|_{1/2,\gamma_s}^2)$$
$$\left.\left.+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_k}(\|(u^{p+1})\|_{0,\gamma_s}^2+\|(u^{p+1})_{\sigma_k}^a\|_{1/2,\gamma_s}^2)\right)\right\}. \tag{3.72}$$

Here $(L_l^{p+1})^a u_l^{p+1}(\xi,\eta)$ is the approximate representation of the elliptic differential operator L_l^{p+1} acting on $u_l^{p+1}(\xi,\eta)$ as defined immediately after (3.62). Moreover $(\partial u_l^{p+1}/\partial x)^a$ and $(\partial u_l^{p+1}/\partial y)^a$ are the approximate representations of the derivatives $\partial u_l^{p+1}/\partial x$ and $\partial u_l^{p+1}/\partial y$ in (ξ,η) variables as defined in (3.48a)–(3.49b). Similarly $\|(u^{p+1})_{\sigma_k}^a\|_{1/2,\gamma_s}^2$ is the approximate representation of $\|(u^{p+1})_{\sigma_k}\|_{1/2,\gamma_s}^2$ as defined in (3.59b) where $(u^{p+1})_{\sigma_k}$ denotes the tangential derivative of u^{p+1} along Γ_k. This representation is obtained by replacing the coefficients of these differential operators, which are analytic functions of ξ and η, by polynomials approximations of degree at most $(N-1)$ in each of the degrees of freedom of the function elements in their respective domains.

Summing the estimate (3.65) in Theorem 3.2 over k and estimating terms as before the result follows. □

4. Technical results

In this section we prove the results which we frequently refer to in §3.

Lemma 4.1. *Let $w(\theta)$ be a piecewise smooth function defined for $\theta\in[\theta_1,\ldots\theta_{M+1}]$ which has discontinuities only at the points $\theta_2,\theta_3,\ldots\theta_M$. Then*

$$\int_{\theta_1}^{\theta_{M+1}} w^2(\theta)\,\mathrm{d}\theta \le 2\left(\frac{(\theta_{M+1}-\theta_1)^2}{2}\sum_{k=1}^{M}\int_{\theta_k}^{\theta_{k+1}}\left(\frac{\mathrm{d}w}{\mathrm{d}\theta}\right)^2\mathrm{d}\theta \right.$$
$$\left. +M(\theta_{M+1}-\theta_1)\left(w^2(\theta_1)+\sum_{j=2}^{M}(w(\theta_j^+)-w(\theta_j^-))^2\right)\right). \quad (4.1)$$

Here

$$w(\theta_j^+) = \lim_{\theta>\theta_j,\theta\to\theta_j} w(\theta),$$

and

$$w(\theta_j^-) = \lim_{\theta<\theta_j,\theta\to\theta_j} w(\theta).$$

Define a function $s(\theta)$ as follows:

$$s(\theta) = \begin{cases} w(\theta_1) & \text{for } \theta_1 \le \theta < \theta_2, \\ w(\theta_1) + \sum_{j=2}^{k}(w(\theta_j^+) - w(\theta_j^-)) & \text{for } \theta_k \le \theta < \theta_{k+1}, 2 \le k \le M \end{cases}.$$

Then $w(\theta)$ may be written as

$$w(\theta) = h(\theta) + s(\theta)$$

where $h(\theta)$ is a continuous function which is differentiable a.e.

Moreover $h(\theta_1) = 0$. Now

$$\int_{\theta_1}^{\theta_{M+1}} w^2(\theta)\,\mathrm{d}\theta \le 2\left(\int_{\theta_1}^{\theta_{M+1}} h^2(\theta)\,\mathrm{d}\theta + \int_{\theta_1}^{\theta_{M+1}} s^2(\theta)\,\mathrm{d}\theta\right).$$

Clearly

$$h(\theta) = \int_{\theta_1}^{\theta}\frac{\mathrm{d}h}{\mathrm{d}\phi}\mathrm{d}\phi.$$

Hence

$$h^2(\theta) \le (\theta-\theta_1)\int_{\theta_1}^{\theta_{M+1}}\left(\frac{\mathrm{d}h}{\mathrm{d}\theta}\right)^2\mathrm{d}\theta.$$

From which we can conclude that

$$\int_{\theta_1}^{\theta_{M+1}} h^2(\theta)\,\mathrm{d}\theta \le \frac{(\theta_{M+1}-\theta_1)^2}{2}\int_{\theta_1}^{\theta_{M+1}}\left(\frac{\mathrm{d}h}{\mathrm{d}\theta}\right)^2\mathrm{d}\theta.$$

Now

$$\int_{\theta_1}^{\theta_{M+1}} s^2(\theta)\,\mathrm{d}\theta \le (w(\theta_1))^2\,\Delta\theta_1$$
$$+\sum_{k=2}^{M} k\Delta\theta_k\left((w(\theta_1))^2+\sum_{j=2}^{k}(w(\theta_j^+)-w(\theta_j^-))^2\right)$$
$$\le M(\theta_{M+1}-\theta_1)\left((w(\theta_1))^2+\sum_{j=2}^{M}(w(\theta_j^+)-w(\theta_j^-))^2\right).$$

And so we obtain the estimate. □

Theorem 4.1. *Let $a^p(s)$ and $b^p(s)$ be polynomials of degree P on the finite interval $[\alpha, \beta]$. Then*

$$\left|\int_{\alpha}^{\beta} a^P(s)\frac{\mathrm{d}b^P(s)}{\mathrm{d}s}\mathrm{d}s\right| \leq C \ln P \|a^P\|_{1/2,(\alpha,\beta)}\|b^P\|_{1/2,(\alpha,\beta)}. \tag{4.2}$$

Here $\| \|_{s,\Omega}$ denotes the fractional Sobolev norm on $H^s(\Omega)$ as defined in [9], when s is not an integer. Now for any $0 < \varepsilon < \frac{1}{2}$ we have

$$\left|\int_{\alpha}^{\beta} a^P(s)\frac{\mathrm{d}b^P(s)}{\mathrm{d}s}\mathrm{d}s\right| \leq \|a^P\|_{1/2-\epsilon,(\alpha,\beta)}\|\frac{\mathrm{d}b^P}{\mathrm{d}s}\|_{-1/2+\epsilon,(\alpha,\beta)} \tag{4.3}$$

since the space of infinitely differentiable functions with compact support in (α, β) is dense in $W_q^t(\alpha, \beta)$ for $0 \leq t \leq 1/q$ by Theorem 1.4.2.4 of [9].

Next using Theorem 1.4.4.6 of [9] we have the result that the differentiation operator is a continuous linear operator from $W_q^t(\alpha, \beta)$ to $W_q^{t-1}(\alpha, \beta)$, except when $t = 1/q$, with norm proportional to $1/\left|t - \frac{1}{q}\right|$. Thus we can conclude that

$$\left\|\frac{\mathrm{d}b^P}{\mathrm{d}s}\right\|_{-1/2+\epsilon,(\alpha,\beta)} \leq \frac{K}{\epsilon}\|b^P\|_{(1/2)+\epsilon,(\alpha,\beta)}. \tag{4.4}$$

Now by the interpolation inequality from [9]

$$\|b^P\|_{(1/2)+\epsilon,(\alpha,\beta)} \leq C\|b^P\|_{1/2,(\alpha,\beta)}^{1-2\epsilon}\|b^P\|_{1,(\alpha,\beta)}^{2\epsilon}. \tag{4.5}$$

And by the inverse inequality for differentiation in [12]

$$\|b^P\|_{2,(\alpha,\beta)} \leq CP^2\|b^P\|_{1,(\alpha,\beta)}. \tag{4.6}$$

Once more by the interpolation inequatlity

$$\|b^P\|_{1,(\alpha,\beta)} \leq C\|b^P\|_{1/2,(\alpha,\beta)}^{1-1/3}\|b^P\|_{2,(\alpha,\beta)}^{1/3}$$

and from (4.6) we can conclude that

$$\|b^P\|_{1,(\alpha,\beta)} \leq CP^{2/3}\|b^P\|_{1,(\alpha,\beta)}^{1/3}\|b^P\|_{1/2,(\alpha,\beta)}^{1-1/3}.$$

This gives us the inverse inequality for fractional Sobolev norms

$$\|b^P\|_{1,(\alpha,\beta)} \leq CP\|b^P\|_{1/2,(\alpha,\beta)}. \tag{4.7}$$

Using (4.5) and (4.7) we get

$$\|b^P\|_{1/2+\varepsilon,(\alpha,\beta)} \leq CP^{2\varepsilon}\|b^P\|_{1/2,(\alpha,\beta)}. \tag{4.8}$$

Next it is easy to see that

$$\|a^P\|_{(1/2)-\epsilon,(\alpha,\beta)} \leq C\|a^P\|_{1/2,(\alpha,\beta)}. \tag{4.9}$$

Substituting the relations (4.4), (4.8) and (4.9) in (4.3) we get

$$\left|\int_{\alpha}^{\beta} a^P(s)\frac{\mathrm{d}b^P(s)}{\mathrm{d}s}\mathrm{d}s\right| \leq \frac{K}{\epsilon}P^{2\epsilon}\|a^P\|_{1/2,(\alpha,\beta)}\|b^P\|_{1/2,(\alpha,\beta)}. \tag{4.10}$$

Taking the minimum over positive ϵ we get the required result. □

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References

[1] Babuska I and Guo B Q, Regularity of the solution of elliptic problems with piecewise analytic data, Part - I, *SIAM J. Math. Anal.* **19** (1988) 172–203

[2] Babuska I and Guo B Q, The h-p version of the finite element method on domains with curved boundaries, *SIAM J. Num. Anal.* **25** (1988) 837–861

[3] Ciarlet P G, The finite element method for elliptic problems (North Holland, Amsterdam) (1978)

[4] Dutt P, Spectral methods for initial boundary value problems – An alternative approach, *SIAM J. Num. Anal.* **27(4)** (1990) 885–903

[5] Dutt P and Joshi S, Spectral methods for hyperbolic initial boundary value problems on parallel computers, *J. Comp. Appl. Math.* **134** (2001) 165–190

[6] Dutt P, Tomar S and Rathish Kumar B V, h-p Spectral element methods for Dirichlet problems on parallel computers (preprint)

[7] Dutt P and Tomar S, h-p Spectral element methods for mixed problems on parallel problems (preprint)

[8] Dutt P and Tomar S, h-p Spectral element methods for elliptic boundary value problems – The general case (preprint)

[9] Grisvard P, Elliptic problems in non-smooth domains, Pitman Advanced Publishing Program (1985)

[10] Karniadakis G and Spencer Sherwin J, Spectral/hp element methods for CFD (Oxford University Press) (1999)

[11] Kondratiev V A, The smoothness of a solution of Dirichlet's problem for second order elliptic equations in a region with a piecewise smooth boundary, *Differential' nye Uraneniya*, **6(10)** 1831–1843; *Differential Equations* **6** 1392–1401

[12] Schwab Ch, p and h-p Finite element methods (Oxford: Clarendon Press) (1998)