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Algebraic stacks

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Abstract. This is an expository article on the theory of algebraic stacks. After introducing the general theory, we concentrate in the example of the moduli stack of vector bundles, giving a detailed comparison with the moduli scheme obtained via geometric invariant theory.

Keywords. 2 categories; algebraic stacks; moduli spaces; vector bundles.

1. Introduction

The concept of algebraic stack is a generalization of the concept of scheme, in the same sense that the concept of scheme is a generalization of the concept of projective variety. In many moduli problems, the functor that we want to study is not representable by a scheme. In other words, there is no fine moduli space. Usually this is because the objects that we want to parametrize have automorphisms. But if we enlarge the category of schemes (following ideas that go back to Grothendieck and Giraud [Gi], and were developed by Deligne, Mumford and Artin [DM, Ar2]) and consider algebraic stacks, then we can construct the 'moduli stack', that captures all the information that we would like in a fine moduli space. For other sources on stacks, see [E, La, LaM, Vi].

The idea of enlarging the category of algebraic varieties to study moduli problems is not new. In fact Weil invented the concept of abstract variety to give an algebraic construction of the Jacobian of a curve.

These notes are an introduction to the theory of algebraic stacks. I have tried to emphasize ideas and concepts through examples instead of detailed proofs (I give references where these can be found). In particular, §3 is a detailed comparison between the moduli *scheme* and the moduli *stack* of vector bundles.

First I will give a quick introduction in subsection 1.1, just to give some motivations and get a flavor of the theory of algebraic stacks.

Section 2 has a more detailed exposition. There are mainly two ways of introducing stacks. We can think of them as 2-functors (I learnt this approach from Nitsure and Sorger, cf. subsection 2.1), or as categories fibered on groupoids. (This is the approach used in the references, cf. subsection 2.2.) From the first point of view it is easier to see in which sense stacks are generalizations of schemes, and the definition looks more natural, so conceptually it seems more satisfactory. But since the references use categories fibered on groupoids, after we present both points of view, we will mainly use the second.

The concept of stack is merely a categorical concept. To do geometry we have to add some conditions, and then we get the concept of algebraic stack. This is done in subsection 2.3.

In subsection 2.4 we introduce a third point of view to understand stacks: as groupoid spaces.

In subsection 2.5 we define for algebraic stacks many of the geometric properties that are defined for schemes (smoothness, irreducibility, separatedness, properness, etc...). In subsection 2.6 we introduce the concept of point and dimension of an algebraic stack, and in subsection 2.7 we define sheaves on algebraic stacks.

In §3 we study in detail the example of the moduli of vector bundles on a scheme X, comparing the moduli stack with the moduli scheme.

Prerequisites. In the examples, I assume that the reader has some familiarity with the theory of moduli spaces of vector bundles. A good source for this material is [HL]. The necessary background on Grothendieck topologies, sheaves and algebraic spaces is in Appendix A, and the notions related to the theory of 2-categories are explained in Appendix B.

1.1 *Quick introduction to algebraic stacks*

We will start with an example: vector bundles (with fixed prescribed Chern classes and rank) on a projective scheme X over an algebraically closed field k. What is the moduli stack $\mathcal{M}_X$ of vector bundles on X? I do not know a short answer to this, but instead it is easy to define what is a morphism from a scheme B to the moduli stack $\mathcal{M}_X$. It is just a family of vector bundles parametrized by B. More precisely, it is a vector bundle V on $B \times X$ (hence flat over B) such that the restriction to the slices $b \times X$ have prescribed Chern classes and rank. In other words, $\mathcal{M}_X$ has the property that we expect from a fine moduli space: the set of morphisms $\mathrm{Hom}(B, \mathcal{M}_X)$ is equal to the set of families parametrized by B.

We will say that a diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "g"'] & B' \arrow[d, "g'"] \\ & \mathcal{M}_X \end{tikzcd} \tag{1}$$

is commutative if the vector bundle V on $B \times X$ corresponding to g is isomorphic to the vector bundle $(f \times \mathrm{id}_X)^* V'$, where V' is the vector bundle corresponding to g'. Note that in general, if L is a line bundle on B, then V and $V \otimes p_B^* L$ won't be isomorphic, and then the corresponding morphisms from B to $\mathcal{M}_X$ will be different, as opposed to what happens with moduli schemes.

A k-point in the stack $\mathcal{M}_X$ is a morphism $u : \mathrm{Spec}\ k \to \mathcal{M}_X$, in other words, it is a vector bundle V on X, and we say that two points are isomorphic if they correspond to isomorphic vector bundles. But we should not think of $\mathcal{M}_X$ just as a set of points, it should be thought of as a category. The objects of $\mathcal{M}_X$ are points[1], i.e. vector bundles on X, and a morphism in $\mathcal{M}_X$ is an isomorphism of vector bundles. This is the main difference between a scheme and an algebraic stack: the points of a scheme form a *set*, whereas the points of a stack form a *category*, in fact a *groupoid* (i.e. a category in which all morphisms are isomorphisms). Each point comes with a group of automorphisms. Roughly speaking, a scheme (or more generally, an algebraic space [Ar1, K]) can be

[1]To be precise, we should consider also B-valued points, for any scheme B, but we will only consider k-valued points for the moment.

thought of as an algebraic stack in which these groups of automorphisms are all trivial. If p is the k-point in $\mathcal{M}_X$ corresponding to a vector bundle V on X, then the group of automorphisms associated to p is the group of vector bundle automorphisms of V. This is why algebraic stacks are well suited to serve as moduli of objects that have automorphisms.

An algebraic stack has an atlas. This is a scheme U and a (representable) surjective morphism $u : U \to \mathcal{M}_X$ (with some other properties). As we have seen, such a morphism u is equivalent to a family of vector bundles parametrized by U. The precise definition of representable surjective morphism of stacks will be given in §2. In this situation it implies that for every vector bundle V over X there is at least one point in U whose corresponding vector bundle is isomorphic to V. The existence of an atlas for an algebraic stack is the analog of the fact that for a scheme B there is always an *affine* scheme U and a surjective morphism $U \to B$ (if $\{U_i \to B\}$ is a covering of B by affine subschemes, take U to be the disjoint union $\coprod U_i$). Many local properties (smooth, normal, reduced...) can be studied by looking at the atlas U. It is true that in some sense an algebraic stack looks, locally, like a scheme, but we shouldn't take this too far. For instance the atlas of the classifying stack BG (parametrizing principal G-bundles, cf. Example 2.18) is just a single point. The dimension of an algebraic stack $\mathcal{M}_X$ will be defined as the dimension of U minus the relative dimension of the morphism u. The dimension of an algebraic stack can be negative (for instance, $\dim(BG) = -\dim(G)$).

We will see that many geometric concepts that appear in the theory of schemes have an analog in the theory of algebraic stacks. For instance, one can define coherent sheaves on them. We will give a precise definition in §2, but the idea is that a coherent sheaf L on an algebraic stack $\mathcal{M}_X$ is a functor that, for each morphism $g : B \to \mathcal{M}_X$, gives a coherent sheaf L_B on B, and for each commutative diagram like (1), gives an isomorphism between $f^*L_{B'}$ and L_B. The coherent sheaf L_B should be thought of as the pullback 'g^*L' of L under g (the compatibility condition for commutative diagrams is just the condition that $(g' \circ f)^*L$ should be isomorphic to $f^*g'^*L$).

Let's look at another example: the moduli quotient (Example 2.18). Let G be an affine algebraic group acting on X. For simplicity, assume that there is a normal subgroup H of G that acts trivially on X, and that $\overline{G} = G/H$ is an affine group acting freely on X and furthermore there is a quotient by this action $X \to B$ and this quotient is a principal $\overline{G}$-bundle. We call $B = X/G$ the *quotient scheme*. Each point corresponds to a G-orbit of the action. But note that B is also equal to the quotient $X/\overline{G}$, because H acts trivially and then G-orbits are the same thing as $\overline{G}$-orbits. We can say that the quotient scheme 'forgets' H.

One can also define the *quotient stack* $[X/G]$. Roughly speaking, a point p of $[X/G]$ again corresponds to a G-orbit of the action, but now each point comes with an automorphism group: given a point p in $[X/G]$, choose a point $x \in X$ in the orbit corresponding to p. The automorphism group attached to p is the stabilizer G_x of x. With the assumptions that we have made on the action of G, the automorphism group of any point is always H. Then the quotient stack $[X/G]$ is not a scheme, since the automorphism groups are not trivial. The action of H is trivial, but the moduli stack still 'remembers' that there was an action by H. Observe that the stack $[X/\overline{G}]$ is not isomorphic to the stack $[X/G]$ (as opposed to what happens with the quotient schemes). Since the action of $\overline{G}$ is free on X, the automorphism group corresponding to each point of $[X/\overline{G}]$ is trivial, and it can be shown that, with the assumptions that we made, $[X/\overline{G}]$ is represented by the scheme B (this terminology will be made precise in §2).

2. Stacks

2.1 *Stacks as 2-functors: Sheaves of sets*

Given a scheme M over a base scheme S, we define its (contravariant) functor of points $\mathrm{Hom}_S(-,M)$

$$\begin{array}{cccc} \mathrm{Hom}_S(-,M): & (\mathrm{Sch}/S) & \longrightarrow & (\mathrm{Sets}) \\ & B & \longmapsto & \mathrm{Hom}_S(B,M) \end{array}$$

where (Sch/S) is the category of S-schemes, B is an S-scheme, and $\mathrm{Hom}_S(B,M)$ is the set of S-scheme morphisms. If we give (Sch/S) the Zariski (or étale, or fppf) topology, $\tilde{M} = \mathrm{Hom}_S(-,M)$ is a sheaf (see Appendix A for the definition of topologies and sheaves on categories). Furthermore, given schemes M and N there is a bijection (given by Yoneda Lemma) between the set of morphisms of schemes $\mathrm{Hom}_S(M,N)$ and the set of natural transformations between the associated functors $\tilde{M}$ and $\tilde{N}$, hence the category of schemes is a full subcategory of the category of sheaves on (Sch/S).

A sheaf of sets on (Sch/S) with a given topology is called a space[2] with respect to that topology (this is the definition given in ([La], 0)).

Then schemes can be thought of as sheaves of sets. Moduli problems can usually be described by functors. We say that a sheaf of sets F is representable by a scheme M if F is isomorphic to the functor of points $\mathrm{Hom}_S(-,M)$. The scheme M is then called the fine moduli scheme. Roughly speaking, this means that there is a one to one correspondence between families of objects parametrized by a scheme B and morphisms from B to M.

Example 2.1 (*Vector bundles*). Let X be a projective scheme over an algebraically closed field k. We define the moduli functor $\underline{M}'_X$ of vector bundles of fixed rank r and Chern classes c_i by sending the scheme B to the set $\underline{M}'_X(B)$ of isomorphism classes of vector bundles on $B \times X$ (hence flat over B) with rank r and whose restriction to the slices $\{b\} \times X$ have Chern classes c_i. These vector bundles should be thought of as families of vector bundles parametrized by B. A morphism $f: B' \to B$ is sent to $\underline{M}'_X(f) = f^*: \underline{M}'_X(B) \to \underline{M}'_X(B')$, the map of sets induced by the pullback. Usually we will also fix a polarization H in X and restrict our attention to stable or semistable vector bundles with respect to this polarization (see [HL] for definitions), and then we consider the corresponding functors $\underline{M}'^{s}_X$ and $\underline{M}'^{ss}_X$.

Example 2.2 (*Curves*). The moduli functor M_g of smooth curves of genus g over a Noetherian base S is the functor that sends each scheme B to the set $M_g(B)$ of isomorphism classes of smooth and proper morphisms $C \to B$ (where C is an S-scheme) whose fibers are geometrically connected curves of genus g. Each morphism $f: B' \to B$ is sent to the map of sets induced by the pullback f^*.

None of these examples are sheaves (then none of these are representable), because of the presence of automorphisms. They are just presheaves (= functors). For instance, given a curve C over S with nontrivial automorphisms, it is possible to construct a family $f: \mathcal{C} \to B$ such that every fiber of f is isomorphic to C, but $\mathcal{C}$ is not isomorphic to $B \times C$ (see [E]). This implies that M_g does not satisfy the monopresheaf axiom.

[2]Note that the concept of space is just a categorical concept. To do geometry we need to add some algebraic and technical conditions (existence of an atlas, quasi-separatedness,...). After we add these conditions (see Definitions 4.3 or 4.4), we have an algebraic space.

This can be solved by taking the sheaf associated to the presheaf (sheafification). In the examples, this amounts to change isomorphism classes of families to equivalence classes of families, declaring two families to be equivalent if they are locally (using the étale topology over the parametrizing scheme B) isomorphic. In the case of vector bundles, this is the reason why one usually considers two vector bundles V and V' on $X \times B$ equivalent if $V \cong V' \otimes p_B^* L$ for some line bundle L on B. The functor obtained with this equivalence relation is denoted $\underline{M}_X$ (and analogously for $\underline{M}_X^s$ and $\underline{M}_X^{ss}$).

Note that if two families V and V' are equivalent in this sense, then they are locally isomorphic. The converse is only true if the vector bundles are simple (only automorphisms are scalar multiplications). This will happen, for instance, if we are considering the functor $\underline{M}_X'^s$ of stable vector bundles, since stable vector bundles are simple. In general, if we want the functor to be a sheaf, we have to use a weaker notion of equivalence, but this is not done because for other reasons there is only hope of obtaining a fine moduli space if we restrict our attention to stable vector bundles.

Once this modification is made, there are some situations in which these examples are representable (for instance, stable vector bundles on curves with coprime rank and degree), but in general they will still not be representable, because in general we do not have a universal family:

DEFINITION 2.3 (Universal family)

Let F be a representable functor, and let $\phi : F \to \mathrm{Hom}_S(-, X)$ be the isomorphism. The object of $F(X)$ corresponding to the element id_X of $\mathrm{Hom}_S(X, X)$ is called the universal family.

Example 2.4 (*Vector bundles*). If V is a universal vector bundle (over $M \times X$, where M is the fine moduli space), it has the property that for any family W of vector bundles (i.e. W is a vector bundle over $B \times X$ for some parameter scheme B) there exists a morphism $f : B \to M$ such that $(f \times \mathrm{id}_X)^* V$ is equivalent to W.

In other words, the functor $\underline{M}_X$ is represented by the scheme M iff there exists a universal vector bundle on $M \times X$.

When a moduli functor F is not representable and then there is no scheme X whose functor of points is isomorphic to F, one can still try to find a scheme X whose functor of points is an approximation to F in some sense. There are two different notions:

DEFINITION 2.5 (Corepresents) ([S], p. 60), ([HL], Definition 2.2.1)

We say that a scheme M corepresents the functor F if there is a natural transformation of functors $\phi : F \to \mathrm{Hom}_S(-, M)$ such that

- Given another scheme N and a natural transformation $\psi : F \to \mathrm{Hom}_S(-, N)$, there is a unique natural transformation $\eta : \mathrm{Hom}_S(-, M) \to \mathrm{Hom}_S(-, N)$ with $\psi = \eta \circ \phi$.

$$\begin{tikzcd} F \arrow[d, "\phi"] \arrow[dr, "\psi"] & \\ \mathrm{Hom}_S(-, M) \arrow[r, "\eta"] & \mathrm{Hom}_S(-, N) \end{tikzcd}$$

This characterizes M up to unique isomorphism. Let $(\mathrm{Sch}/S)'$ be the functor category, whose objects are contravariant functors from (Sch/S) to (Sets) and whose morphisms are natural transformation of functors. Then M represents F iff $\mathrm{Hom}_S(Y,M) = \mathrm{Hom}_{(\mathrm{Sch}/S)'}(\mathcal{Y},F)$ for all schemes Y, where $\mathcal{Y}$ is the functor represented by Y. On the other hand, one can check that M corepresents F iff $\mathrm{Hom}_S(M,Y) = \mathrm{Hom}_{(\mathrm{Sch}/S)'}(F,\mathcal{Y})$ for all schemes Y. If M represents F, then it corepresents it, but the converse is not true. From now on we will denote a scheme and the functor that it represents by the same letter.

DEFINITION 2.6 (Coarse moduli)

A scheme M is called a coarse moduli scheme if it corepresents F and furthermore

- for any algebraically closed field k, the map $\phi(k) : F(\mathrm{Spec}k) \to \mathrm{Hom}_S(\mathrm{Spec}k, M)$ is bijective.

If M corepresents F (in particular, if M is a coarse moduli space), given a family of objects parametrized by B we get a morphism from B to M, but we don't require the converse to be true, i.e. not all morphisms are induced by families.

Example 2.7 (*Vector bundles*). There is a scheme M_X^{ss} that corepresents $\underline{M}_X^{ss}$ (see [HL]). It fails to be a coarse moduli scheme because its closed points are in one to one correspondence with S-equivalence classes of vector bundles, and not with isomorphism classes of vector bundles. Of course, this can be solved 'by hand' by modifying the functor and considering two vector bundles equivalent if they are S-equivalent. Once this modification is done, M_X^{ss} is a coarse moduli space.

But in general M_X^{ss} doesn't represent the moduli functor $\underline{M}_X^{ss}$. The reason for this is that vector bundles have always nontrivial automorphisms (multiplication by scalar), but the moduli functor does not record information about automorphisms: recall that to a scheme B it associates just the set of equivalence classes of vector bundles. To record the automorphisms of these vector bundles, we define

$$\begin{array}{rccc} \mathcal{M}_X : & (\mathrm{Sch}/S) & \longrightarrow & (\text{groupoids}) \\ & B & \longmapsto & \mathcal{M}_X(B), \end{array}$$

where $\mathcal{M}_X(B)$ is the category whose objects are vector bundles V on $B \times X$ of rank r and with fixed Chern classes (note that the objects are vector bundles, not isomorphism classes of vector bundles), and whose morphisms are vector bundle isomorphisms (note that we use isomorphisms of vector bundles, not S-equivalence nor equivalence classes as before). This defines a 2-functor between the 2-category associated to (Sch/S) and the 2-category (groupoids) (for the definition of 2-categories and 2-functors, see Appendix B).

DEFINITION 2.8

Let (groupoids) be the 2-category whose objects are groupoids, 1-morphisms are functors between groupoids, and 2-morphisms are natural transformation between these functors. A presheaf in groupoids (also called quasi-functor) is a contravariant 2-functor $\mathcal{F}$ from (Sch/S) to (groupoids). For each scheme B we have a groupoid $\mathcal{F}(B)$ and for each morphism $f : B' \to B$ we have a functor $\mathcal{F}(f) : \mathcal{F}(B) \to \mathcal{F}(B')$ that is denoted by f^* (usually it is actually defined by a pull-back).

Example 2.9 (*Vector bundles*) ([La], 1.3.4). $\mathcal{M}_X$ is a presheaf. For each object B of (Sch/S) it gives the groupoid $\mathcal{M}_X(B)$ defined in Example 2.7. For each 1-morphism

$f : B' \to B$ it gives the functor $\mathcal{F}(f) = f^* : \mathcal{M}_X(B) \to \mathcal{M}_X(B')$ given by pull-back, and for every diagram

$$B'' \xrightarrow{g} B' \xrightarrow{f} B \tag{2}$$

it gives a natural transformation of functors (a 2-isomorphism) $\epsilon_{g,f} : g^* \circ f^* \to (f \circ g)^*$. This is the only subtle point. First recall that the pullback f^*V of a vector bundle (or more generally, any fiber product) is not uniquely defined: it is only defined up to unique isomorphism. First choose once and for all a pullback f^*V for each f and V. Then, given a diagram like 2, in principle $g^*(f^*V)$ and $(f \circ g)^*V$ are not the same, but (because both solve the same universal problem) there is a canonical isomorphism (the unique isomorphism of the universal problem) $g^*(f^*V) \to (f \circ g)^*V$ between them, and this defines the natural transformation of functors $\epsilon_{g,f} : g^* \circ f^* \to (f \circ g)^*$. By a slight abuse of language, usually we will not write explicitly these isomorphisms $\epsilon_{g,f}$, and we will write $g^* \circ f^* = (f \circ g)^*$. Since they are uniquely defined this will cause no ambiguity.

Example 2.10 (*Stable curves*) ([DM], Definition 1.1). Let B be an S-scheme. Let $g \geq 2$. A stable curve of genus g over B is a proper and flat morphism $\pi : C \to B$ whose geometric fibers are reduced, connected and one-dimensional schemes C_b such that

1. The only singularities of C_b are ordinary double points.
2. If E is a non-singular rational component of C_b, then E meets the other components of C_b in at least 3 points.
3. $\dim H^1(\mathcal{O}_{C_b}) = g$.

Condition 2 is imposed so that the automorphism group of C_b is finite. A stable curve over B should be thought of as a family of stable curves (over S) parametrized by B.

For each object B of (Sch/S), let $\overline{\mathcal{M}}_g(B)$ be the groupoid whose objects are stable curves over B and whose (iso)morphisms are Cartesian diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd} .$$

For each morphism $f : B' \to B$ of (Sch/S), we define the pullback functor $f^* : \overline{\mathcal{M}}_g(B) \to \overline{\mathcal{M}}_g(B')$, sending an object $X \to B$ to $f^*X \to B'$ (and a morphism $\varphi : X_1 \to X_2$ of curves over B to $f^*\varphi : f^*X_1 \to f^*X_2$). And finally, for each diagram

$$B'' \xrightarrow{g} B' \xrightarrow{f} B$$

we have to give a natural transformation of functors (i.e. a 2-isomorphism in (groupoids)) $\epsilon_{g,f} : g^* \circ f^* \to (f \circ g)^*$. As in the case of vector bundles, this is defined by first choosing once an for all a pullback f^*X for each curve X and morphism f, and then $\epsilon_{g,f}$ is given by the canonical isomorphism between $g^*(f^*X)$ and $(f \circ g)^*X$. Since this isomorphism is canonical, by a slight abuse of language we usually write $g^* \circ f^* = (f \circ g)^*$.

Now we will define the concept of stack. First we have to choose a Grothendieck topology on (Sch/S), either the étale or the fppf topology. Later on, when we define algebraic stack, the étale topology will lead to the definition of a Deligne-Mumford stack ([DM, Vi, E]), and the fppf to an Artin stack ([La]). For the moment we will give a unified description.

In the following definition, to simplify notation we denote by $X|_i$ the pullback $f_i^* X$ where $f_i : U_i \to U$ and X is an object of $\mathcal{F}(U)$, and by $X_i|_{ij}$ the pullback $f_{ij,i}^* X_i$ where $f_{ij,i} : U_i \times_U U_j \to U_i$ and X_i is an object of $\mathcal{F}(U_i)$. We will also use the obvious variations of this convention, and will simplify the notation using Remark 5.3.

DEFINITION 2.11 (Stack)

A stack is a sheaf of groupoids, i.e. a 2-functor (= presheaf) that satisfies the following sheaf axioms. Let $\{U_i \to U\}_{i \in I}$ be a covering of U in the site (Sch/S). Then

1. *Glueing of morphisms*. If X and Y are two objects of $\mathcal{F}(U)$, and $\varphi_i : X|_i \to Y|_i$ are morphisms such that $\varphi_i|_{ij} = \varphi_j|_{ij}$, then there exists a morphism $\eta : X \to Y$ such that $\eta|_i = \varphi_i$.
2. *Monopresheaf*. If X and Y are two objects of $\mathcal{F}(U)$, and $\varphi : X \to Y$, $\psi : X \to Y$ are morphisms such that $\varphi|_i = \psi|_i$, then $\varphi = \psi$.
3. *Glueing of objects*. If X_i are objects of $\mathcal{F}(U_i)$ and $\varphi_{ij} : X_j|_{ij} \to X_i|_{ij}$ are morphisms satisfying the cocycle condition $\varphi_{ij}|_{ijk} \circ \varphi_{jk}|_{ijk} = \varphi_{ik}|_{ijk}$, then there exists an object X of $\mathcal{F}(U)$ and $\varphi_i : X|_i \xrightarrow{\cong} X_i$ such that $\varphi_{ji} \circ \varphi_i|_{ij} = \varphi_j|_{ij}$.

At first sight this might seem very complicated, but if we check in a particular example we will see that it is a very natural definition:

Example 2.12 (*Stable curves*). It is easy to check that the presheaf $\overline{\mathcal{M}}_g$ defined in 2.10 is a stack (all properties hold because of descent theory). We take the étale topology on (Sch/S) (we will see that the reason for this is that the automorphism group of a stable curve is finite). Let $\{U_i \to U\}_{i \in I}$ be a cover of U. Item 1 says that if we have two curves X and Y over U, and we have isomorphisms $\varphi_i : X|_i \to Y|_i$ on the restriction for each U_i, then these isomorphisms glue to give an isomorphism $\eta : X \to Y$ over U if the restrictions to the intersections $\varphi_i|_{ij}$ and $\varphi_j|_{ij}$ coincide.

Item 2 says that two morphisms of curves over U coincide if the restrictions to all U_i coincide.

Finally, item 3 says that if we have curves X_i over U_i and we are given isomorphisms φ_{ij} over the intersections U_{ij}, then we can glue the curves to get a curve over U if the isomorphisms satisfy the cocycle condition.

Example 2.13 (*Vector bundles*). It is also easy to check that the presheaf of vector bundles $\mathcal{M}_X$ is a sheaf. In this case we take the fppf topology on (Sch/S) (we will see that the reason for this choice is that the automorphism group of a vector bundle is not finite, because it includes multiplication by scalars).

Let us stop for a moment and look at how we have enlarged the category of schemes by defining the category of stacks. We can draw the following diagram

$$\begin{tikzcd} & \text{Algebraic Stacks} \arrow[r] & \text{Stacks} \arrow[r] & \text{Presheaves of groupoids} \\ \mathrm{Sch}/S \arrow[r] \arrow[ur] & \text{Algebraic Spaces} \arrow[r] \arrow[u] & \text{Spaces} \arrow[r] \arrow[u] & \text{Presheaves of sets} \arrow[u] \end{tikzcd}$$

where $A \to B$ means that the category A is a subcategory B. Recall that a presheaf of sets is just a functor from (Sch/S) to the category (Sets), a presheaf of groupoids is just a 2-functor to the 2-category (groupoids). A sheaf (for example a space or a stack) is a presheaf that satisfies the sheaf axioms (these axioms are slightly different in the context of categories or 2-categories), and if this sheaf satisfies some geometric conditions (that we have not yet specified), we will have an algebraic stack or algebraic space.

2.2 *Stacks as categories: Groupoids*

There is an alternative way of defining a stack. From this point of view a stack will be a category, instead of a functor.

DEFINITION 2.14

A category over (Sch/S) is a category $\mathcal{F}$ and a covariant functor $p_{\mathcal{F}} : \mathcal{F} \to (\mathrm{Sch}/S)$ (called the structure functor). If X is an object (resp. ϕ is a morphism) of $\mathcal{F}$, and $p_{\mathcal{F}}(X) = B$ (resp. $p_{\mathcal{F}}(\phi) = f$), then we say that X lies over B (resp. ϕ lies over f).

DEFINITION 2.15 (Groupoid)

A category $\mathcal{F}$ over (Sch/S) is called a category fibered on groupoids (or just groupoid) if

1. For every $f : B' \to B$ in (Sch/S) and every object X with $p_{\mathcal{F}}(X) = B$, there exists at least one object X' and a morphism $\phi : X' \to X$ such that $p_{\mathcal{F}}(X') = B'$ and $p_{\mathcal{F}}(\phi) = f$.

$$\begin{tikzcd} X' \arrow[r, dashed, "\phi"] \arrow[d, dashed, mapsto] & X \arrow[d, mapsto] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

2. For every diagram

$$\begin{tikzcd} X_3 \arrow[rr, "\psi"] \arrow[dd, mapsto] & & X_1 \arrow[dd, mapsto] \\ & X_2 \arrow[ur, "\phi"] \arrow[dd, mapsto] & \\ B_3 \arrow[rr, "f \circ f'"] \arrow[dr, "f'"'] & & B_1 \\ & B_2 \arrow[ur, "f"'] & \end{tikzcd}$$

(where $p_{\mathcal{F}}(X_i) = B_i$, $p_{\mathcal{F}}(\phi) = f$, $p_{\mathcal{F}}(\psi) = f \circ f'$), there exists a unique $\varphi : X_3 \to X_2$ with $\psi = \phi \circ \varphi$ and $p_{\mathcal{F}}(\varphi) = f'$.

Condition 2 implies that the object X' whose existence is asserted in condition 1 is unique up to canonical isomorphism. For each X and f we choose once and for all such an X' and call it f^*X. Another consequence of condition 2 is that ϕ is an isomorphism if and only if $p_{\mathcal{F}}(\phi) = f$ is an isomorphism.

Let B be an object of (Sch/S). We define $\mathcal{F}(B)$, the fiber of $\mathcal{F}$ over B, to be the subcategory of $\mathcal{F}$ whose objects lie over B and whose morphisms lie over id_B. It is a groupoid.

The association $B \to \mathcal{F}(B)$ in fact defines a presheaf of groupoids (note that the 2-isomorphisms $\epsilon_{f,g}$ required in the definition of presheaf of groupoids are well defined thanks to condition 2). Conversely, given a presheaf of groupoids $\mathcal{G}$ on (Sch/S), we can define the category $\mathcal{F}$ whose objects are pairs (B, X) where B is an object of (Sch/S) and

X is an object of $\mathcal{G}(B)$, and whose morphisms $(B', X') \to (B, X)$ are pairs (f, α) where $f : B' \to B$ is a morphism in (Sch/S) and $\alpha : f^*X \to X'$ is an isomorphism, where $f^* = \mathcal{G}(f)$. This gives the relationship between both points of view. Since we have a canonical one-to-one relationship between presheaves of groupoids and groupoids over S, by a slight abuse of language, we denote both by the same letter.

Example 2.16 (*Vector bundles*). The groupoid of vector bundles $\mathcal{M}_X$ on a scheme X is the category whose objects are vector bundles over $B \times X$ (for B a scheme), and whose morphisms are isomorphisms

$$\varphi : V' \xrightarrow{\cong} (f \times \mathrm{id})^* V,$$

where V (resp. V') is a vector bundle over $B \times X$ (resp. $B' \times X$) and $f : B' \to B$ is a morphism of schemes. The structure functor sends a vector bundle over $B \times X$ to the scheme B, and a morphism φ to the corresponding morphism of schemes f.

Example 2.17 (*Stable curves*) ([DM], Definition 1.1). We define $\overline{\mathcal{M}}_g$, the groupoid over S whose objects are stable curves over B of genus g (see Definition 2.10), and whose morphisms are Cartesian diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd} \tag{3}$$

The structure functor sends a curve over B to the scheme B, and a morphism as in (3) to f.

Example 2.18 (*Quotient by group action*) ([La], 1.3.2), ([DM], Example 4.8), ([E], Example 2.2). Let X be an S-scheme (assume all schemes are Noetherian), and G an affine flat group S-scheme acting on the right on X. We define the groupoid $[X/G]$ whose objects are principal G-bundles $\pi : E \to B$ together with a G-equivariant morphism $f : E \to X$. A morphism is Cartesian diagram

$$\begin{tikzcd} E' \arrow[r, "p"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "g"] & B \end{tikzcd} \tag{4}$$

such that $f \circ p = f'$.

The structure functor sends an object $(\pi : E \to B, f : E \to X)$ to the scheme B, and a morphism as in (4) to g.

DEFINITION 2.19 (Stack)

A stack is a groupoid that satisfies

1. (*Prestack*). For all scheme B and pair of objects X, Y of $\mathcal{F}$ over B, the contravariant functor

$$\begin{array}{cccc} \mathrm{Iso}_B(X,Y): & (\mathrm{Sch}/B) & \longrightarrow & (\mathrm{Sets}) \\ & (f : B' \to B) & \longmapsto & \mathrm{Hom}(f^*X, f^*Y) \end{array}$$

   is a sheaf on the site (Sch/B).
2. Descent data is effective (this is just condition 3 in the Definition 2.11 of sheaf).

Example 2.20. If G is smooth and affine, the groupoid $[X/G]$ is a stack ([La], 2.4.2), ([Vi], Example 7.17), ([E], Proposition 2.2). Then also $\overline{\mathcal{M}}_g$ (cf. Example 2.17) is a stack, because it is isomorphic to a quotient stack of a subscheme of a Hilbert scheme by $PGL(N)$ ([E], Theorem 3.2), [DM]. The groupoid $\mathcal{M}_X$ defined in Example 2.16 is also a stack ([La], 2.4.4).

From now on we will mainly use this approach. Now we will give some definitions for stacks.

Morphisms of stacks. A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is a functor between the categories, such that $p_{\mathcal{G}} \circ f = p_{\mathcal{F}}$. A commutative diagram of stacks is a diagram

$$\begin{tikzcd} & \mathcal{G} \arrow[dr, "g"] \arrow[d, Rightarrow, "\alpha"] & \\ \mathcal{F} \arrow[ur, "f"] \arrow[rr, "h"'] & {} & \mathcal{H} \end{tikzcd}$$

such that $\alpha : g \circ f \to h$ is an isomorphism of functors. If f is an equivalence of categories, then we say that the stacks $\mathcal{F}$ and $\mathcal{G}$ are isomorphic. We denote by $\mathrm{Hom}_S(\mathcal{F}, \mathcal{G})$ the category whose objects are morphisms of stacks and whose morphisms are natural transformations.

Stack associated to a scheme. Given a scheme U over S, consider the category (Sch/U). Define the functor $p_U : (\mathrm{Sch}/U) \to (\mathrm{Sch}/S)$ which sends the U-scheme $f : B \to U$ to the composition $B \xrightarrow{f} U \to S$, and sends the U-morphism $(B' \to U) \to (B \to U)$ to the S-morphism $(B' \to S) \to (B \to S)$. Then (Sch/U) becomes a stack. Usually we denote this stack also by U. From the point of view of 2-functors, the stack associated to U is the 2-functor that for each scheme B gives the category whose objects are the elements of the set $\mathrm{Hom}_S(B, U)$, and whose only morphisms are identities.

We say that a stack is represented by a scheme U when it is isomorphic to the stack associated to U. We have the following very useful lemmas:

Lemma 2.21. *If a stack has an object with an automorphism other that the identity, then the stack cannot be represented by a scheme.*

Proof. In the definition of stack associated with a scheme we see that the only automorphisms are identities. $\square$

Lemma 2.22 ([Vi], 7.10). *Let $\mathcal{F}$ be a stack and U a scheme. The functor*

$$u : \mathrm{Hom}_S(U, \mathcal{F}) \to \mathcal{F}(U)$$

that sends a morphism of stacks $f : U \to \mathcal{F}$ to $f(id_U)$ is an equivalence of categories.

Proof. Follows from Yoneda lemma. $\square$

This useful observation that we will use very often means that an object of $\mathcal{F}$ that lies over U is equivalent to a morphism (of stacks) from U to $\mathcal{F}$.

Fiber product. Given two morphisms $f_1 : \mathcal{F}_1 \to \mathcal{G}$, $f_2 : \mathcal{F}_2 \to \mathcal{G}$, we define a new stack $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ (with projections to $\mathcal{F}_1$ and $\mathcal{F}_2$) as follows. The objects are triples (X_1, X_2, α) where X_1 and X_2 are objects of $\mathcal{F}_1$ and $\mathcal{F}_2$ that lie over the same scheme U, and $\alpha : f_1(X_1) \to f_2(X_2)$ is an isomorphism in $\mathcal{G}$ (equivalently, $p_{\mathcal{G}}(\alpha) = \mathrm{id}_U$). A morphism from (X_1, X_2, α) to (Y_1, Y_2, β) is a pair (ϕ_1, ϕ_2) of morphisms $\phi_i : X_i \to Y_i$ that lie over the same morphism of schemes $f : U \to V$, and such that $\beta \circ f_1(\phi_1) = f_2(\phi_2) \circ \alpha$. The fiber product satisfies the usual universal property.

Representability. A stack $\mathcal{X}$ is said to be representable by an algebraic space (resp. scheme) if there is an algebraic space (resp. scheme) X such that the stack associated to X is isomorphic to $\mathcal{X}$. If 'P' is a property of algebraic spaces (resp. schemes) and $\mathcal{X}$ is a representable stack, we will say that $\mathcal{X}$ has 'P' iff X has 'P'.

A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is said to be representable if for all objects U in (Sch/S) and morphisms $U \to \mathcal{G}$, the fiber product stack $U \times_{\mathcal{G}} \mathcal{F}$ is representable by an algebraic space. Let P be a property of morphisms of schemes that is local in nature on the target for the topology chosen on (Sch/S) (étale or fppf), and it is stable under arbitrary base change. For instance: separated, quasi-compact, unramified, flat, smooth, étale, surjective, finite type, locally of finite type,.... Then, for a representable morphism f, we say that f has P if for every $U \to \mathcal{G}$, the pullback $U \times_{\mathcal{G}} \mathcal{F} \to U$ has P ([La], p.17, [DM], p. 98).

Diagonal. Let $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ be the obvious diagonal morphism. A morphism from a scheme U to $\mathcal{F} \times_S \mathcal{F}$ is equivalent to two objects X_1, X_2 of $\mathcal{F}(U)$. Taking the fiber product of these we have

$$\begin{tikzcd} \mathrm{Iso}_U(X_1, X_2) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\ U \arrow[r, "{(X_1,X_2)}"] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd}$$

hence the group of automorphisms of an object is encoded in the diagonal morphism.

PROPOSITION 2.23 ([La], Corollary 2.12), ([Vi], Proposition 7.13)

The following are equivalent

1. *The morphism* $\Delta_{\mathcal{F}}$ *is representable.*
2. *The stack* $\mathrm{Iso}_U(X_1, X_2)$ *is representable for all* U, X_1 *and* X_2.
3. *For all scheme* U, *every morphism* $U \to \mathcal{F}$ *is representable.*
4. *For all schemes* U, V *and morphisms* $U \to \mathcal{F}$ *and* $V \to \mathcal{F}$, *the fiber product* $U \times_{\mathcal{F}} V$ *is representable.*

Proof. The implications $1 \Leftrightarrow 2$ and $3 \Leftrightarrow 4$ follow easily from the definitions.
$(1 \Rightarrow 4)$ Assume that $\Delta_{\mathcal{F}}$ is representable. We have to show that $U \times_{\mathcal{F}} V$ is representable for any $f : U \to \mathcal{F}$ and $g : V \to \mathcal{F}$. Check that the following diagram is Cartesian

$$\begin{tikzcd} U \times_{\mathcal{F}} V \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\ U \times_S V \arrow[r, "f \times g"] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd} .$$

Then $U \times_{\mathcal{F}} V$ is representable.
$(1 \Leftarrow 4)$ First note that the Cartesian diagram defined by $h : U \to \mathcal{F} \times_S \mathcal{F}$ and $\Delta_{\mathcal{F}}$ factors as follows:

$$\begin{tikzcd} U \times_{\mathcal{F} \times_S \mathcal{F}} \mathcal{F} \arrow[r] \arrow[d] & U \times_{\mathcal{F}} U \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ U \arrow[r, "\Delta_U"] & U \times_S U \arrow[r] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd} .$$

The outer (big) rectangle and the right square are Cartesian, so the left square is also Cartesian. By hypothesis $U \times_{\mathcal{F}} U$ is representable, then $U \times_{\mathcal{F}\times_S\mathcal{F}} \mathcal{F}$ is also representable. □

2.3 *Algebraic stacks*

Now we will define the notion of algebraic stack. As we have said, first we have to choose a topology on (Sch/S). Depending of whether we choose the étale or fppf topology, we get different notions.

DEFINITION 2.24 (Deligne-Mumford stack)

Let (Sch/S) be the category of S-schemes with the étale topology. Let $\mathcal{F}$ be a stack. Assume

1. *Quasi-separatedness*. The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact and separated.
2. There exists a scheme U (called atlas) and an étale surjective morphism $u : U \to \mathcal{F}$.

Then we say that $\mathcal{F}$ is a Deligne-Mumford stack.

The morphism of stacks u is representable because of Proposition 2.23 and the fact that the diagonal $\Delta_{\mathcal{F}}$ is representable. Then the notion of étale is well defined for u. In [DM] this was called an algebraic stack. In the literature, algebraic stack usually refers to Artin stack (that we will define later). To avoid confusion, we will use 'algebraic stack' only when we refer in general to both notions, and we will use 'Deligne-Mumford' or 'Artin' stack when we want to be specific.

Note that the definition of Deligne-Mumford stack is the same as the definition of algebraic space, but in the context of stacks instead of spaces. Following the terminology used in scheme theory, a stack such that the diagonal $\Delta_{\mathcal{F}}$ is quasi-compact and separated is called quasi-separated. We always assume this technical condition, as it is usually done both with schemes and algebraic spaces.

Sometimes it is difficult to find explicitly an étale atlas, and the following proposition is useful.

PROPOSITION 2.25 ([DM], Theorem 4.21), [E]

Let $\mathcal{F}$ be a stack over the étale site (Sch/S). Assume

1. *The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact, separated and unramified.*
2. *There exists a scheme U of finite type over S and a smooth surjective morphism $u : U \to \mathcal{F}$.*

Then $\mathcal{F}$ is a Deligne-Mumford stack.

Now we define the analog for the fppf topology [Ar2].

DEFINITION 2.26 (Artin stack)

Let (Sch/S) be the category of S-schemes with the fppf topology. Let $\mathcal{F}$ be a stack. Assume

1. *Quasi-separatedness*. The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact and separated.
2. There exists a scheme U (called atlas) and a smooth (hence locally of finite type) and surjective morphism $u : U \to \mathcal{F}$.

Then we say that $\mathcal{F}$ is an Artin stack.

For propositions analogous to proposition 2.25, see [La, 4].

PROPOSITION 2.27 ([Vi], Proposition 7.15), ([La], Lemma 3.3)

If $\mathcal{F}$ is a Deligne-Mumford (*resp. Artin*) *stack, then the diagonal $\Delta_{\mathcal{F}}$ is unramified* (*resp. finite type*).

Recall that $\Delta_{\mathcal{F}}$ is unramified (resp. finite type) if for every scheme B and objects X, Y of $\mathcal{F}(B)$, the morphism $\mathrm{Iso}_B(X, Y) \to B$ is unramified (resp. finite type). If $B = \operatorname{Spec} S$ and $X = Y$, then this means that the automorphism group of X is discrete and reduced for a Deligne-Mumford stack, and it is of finite type for an Artin stack.

Example 2.28 (*Vector bundles*). The stack $\mathcal{M}_X$ is an Artin stack, locally of finite type ([La], 4.14.2.1). The atlas is constructed as follows: Let P^H_{r,c_i} be the Hilbert polynomial corresponding to locally free sheaves on X with rank r and Chern classes c_i. Let Quot $(\mathcal{O}(-m)^{\oplus N}, P^H_{r,c_i})$ be the Quot scheme parametrizing quotients of sheaves on X,

$$\mathcal{O}(-m)^{\oplus N} \to V, \tag{5}$$

where V is a coherent sheaf on X with Hilbert polynomial P^H_{r,c_i}. Let $R_{N,m}$ be the subscheme corresponding to quotients (5) such that V is a vector bundle with $H^p(V(m)) = 0$ for $p > 0$ and the morphism (5) induces an isomorphism on global sections

$$H^0(\mathcal{O})^{\oplus N} \xrightarrow{\cong} H^0(V(m)).$$

The scheme $R_{N,m}$ has a universal vector bundle, induced from the universal bundle of the Quot scheme, and then there is a morphism $u_{N,m} : R_{N,m} \to \mathcal{M}_X$. Since H is ample, for every vector bundle V, there exist integers N and m such that $R_{N,m}$ has a point whose corresponding quotient is V, and then if we take the infinite disjoint union of these morphisms we get a surjective morphism

$$u : \left(\coprod_{N,m>0} R_{N,m}\right) \to \mathcal{M}_X.$$

It can be shown that this morphism is smooth, and then it gives an atlas. Each scheme $R_{N,m}$ is of finite type, so the union is locally of finite type, which in turn implies that the stack $\mathcal{M}_X$ is locally of finite type.

Example 2.29 (*Quotient by group action*). The stack $[X/G]$ is an Artin stack ([La], 4.14.1.1). If G is smooth, an atlas is defined as follows (for more general G, see ([La], 4.14.1.1)): Take the trivial principal G-bundle $X \times G$ over X, and let the map $f : X \times G \to X$ be the action of the group. This defines an object of $[X/G](X)$, and by Lemma 2.22, it defines a morphism $u : X \to [X/G]$. It is representable, because if B is a scheme and $g : B \to [X/G]$ is the morphism corresponding to a principal G-bundle E over B with an equivariant morphism $f : E \to X$, then $B \times_{[X/G]} X$ is isomorphic to the scheme E, and in fact we have a Cartesian diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "u"] \\ B \arrow[r, "g"] & {[X/G]} \end{tikzcd}.$$

The morphism u is surjective and smooth because π is surjective and smooth for every g (if G is not smooth, but only separated, flat and of finite presentation, then u is not an atlas, but if we apply Artin's theorem ([Ar2], Theorem 6.1), ([La], Theorem 4.1), we conclude that there is a smooth atlas).

If either G is étale over S ([DM], Example 4.8) or the stabilizers of the geometric points of X are finite and reduced ([Vi], Example 7.17), then $[X/G]$ is a Deligne-Mumford stack.

Note that if the action is not free, then $[X/G]$ is not representable by Lemma 2.21. On the other hand, if there is a scheme Y such that $X \to Y$ is a principal G-bundle, then $[X/G]$ is represented by Y.

Let G be a reductive group acting on X. Let H be an ample line bundle on X, and assume that the action is polarized. Let X^s and X^{ss} be the subschemes of stable and semistable points. Let $Y = X//G$ be the GIT quotient. Recall that there is a good quotient $X^{ss} \to Y$, and that the restriction to the stable part $X^s \to Y$ is a principal bundle. There is a natural morphism $[X^{ss}/G] \to X^{ss}//G$. By the previous remark, the restriction $[X^s/G] \to Y^s$ is an isomorphism of stacks.

If $X = S$ (with trivial action of G on S), then $[S/G]$ is denoted BG, the classifying groupoid of principal G-bundles.

Example 2.30 (*Stable curves*). The stack $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack ([DM], Proposition 5.1), [E]. The idea of the proof is to show that $\overline{\mathcal{M}}_g$ is the quotient stack $[\overline{H}_g/PGL(N)]$ of a scheme $\overline{H}_g$ by a smooth group $PGL(N)$. This gives a smooth atlas. Then one shows that the diagonal is unramified, and finally we apply Proposition 2.25.

2.4 *Algebraic stacks as groupoid spaces*

We will introduce a third equivalent definition of stack. First consider a category C. Let U be the set of objects and R the set of morphisms. The axioms of a category give us four maps of sets

$$R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{e} R \quad R \times_{s,U,t} R \xrightarrow{m} R,$$

where s and t give the source and target for each morphism, e gives the identity morphism, and m is composition of morphisms. If the category is a groupoid then we have a fifth morphism

$$R \xrightarrow{i} R$$

that gives the inverse. These maps satisfy

1. $s \circ e = t \circ e = \mathrm{id}_U$, $s \circ i = t$, $t \circ i = s$, $s \circ m = s \circ p_2$, $t \circ m = t \circ p_1$.
2. *Associativity.* $m \circ (m \times \mathrm{id}_R) = m \circ (\mathrm{id}_R \times m)$.
3. *Identity.* Both compositions

$$R = R \times_{s,U} U = U \times_{U,t} R \underset{e \times \mathrm{id}_R}{\overset{\mathrm{id}_R \times e}{\rightrightarrows}} R \times_{s,U,t} R \xrightarrow{m} R$$

   are equal to the identity map on R.
4. *Inverse.* $m \circ (i \times \mathrm{id}_R) = e \circ s$, $m \circ (\mathrm{id}_R \times i) = e \circ t$.

DEFINITION 2.31 (Groupoid space) ([La], 1.3.3), ([DM], pp. 668–669)

A groupoid space is a pair of spaces (sheaves of sets) U, R, with five morphisms s, t, e, m, i with the same properties as above.

DEFINITION 2.32 ([La], 1.3.3).

Given a groupoid space, define the groupoid over (Sch/S) as the category $[R,U]'$ over (Sch/S) whose objects over the scheme B are elements of the set $U(B)$ and whose morphisms over B are elements of the set $R(B)$. Given $f: B' \to B$ we define a functor $f^*: [R,U]'(B) \to [R,U]'(B')$ using the maps $U(B) \to U(B')$ and $R(B) \to R(B')$.

The groupoid $[R,U]'$ is in general only a prestack. We denote by $[R,U]$ the associated stack. The stack $[R,U]$ can be thought of as the sheaf associated to the presheaf of groupoids $B \mapsto [R,U]'(B)$ ([La], 2.4.3).

Example 2.33 (*Quotient by group action*). Let X be a scheme and G an affine group scheme. We denote by the same letters the associated spaces (functors of points). We take $U = X$ and $R = X \times G$. Using the group action we can define the five morphisms (t is the action of the group, $s = p_1$, m is the product in the group, e is defined with the identity of G, and i with the inverse).

The objects of $[X \times G, X]'(B)$ are morphisms $f: B \to X$. Equivalently, they are trivial principal G-bundles $B \times G$ over B and a map $B \times G \to X$ defined as the composition of the action of G and f. The stack $[X \times G, X]$ is isomorphic to $[X/G]$.

Example 2.34 (*Algebraic stacks*). Let R, U be a groupoid space such that R and U are algebraic spaces, locally of finite presentation (equivalently locally of finite type if S is noetherian). Assume that the morphisms s, t are flat, and that $\delta = (s,t): R \to U \times_S U$ is separated and quasi-compact. Then $[R,U]$ is an Artin stack, locally of finite type ([La], Corollary 4.7).

In fact, any Artin stack $\mathcal{F}$ can be defined in this fashion. The algebraic space U will be the atlas of $\mathcal{F}$, and we set $R = U \times_{\mathcal{F}} U$. The morphisms s and t are the two projections, i exchanges the factors, e is the diagonal, and m is defined by projection to the first and third factor.

Let $\delta: R \to U \times_S U$ be an equivalence relation in the category of spaces. One can define a groupoid space, and $[R,U]$ is to be thought of as the stack-theoretic quotient of this equivalence relation, as opposed to the quotient space, used for instance to define algebraic spaces (for more details and the definition of equivalence relation see appendix A).

2.5 *Properties of algebraic stacks*

So far we have only defined scheme-theoretic properties for representable stacks and morphisms. We can define some properties for arbitrary algebraic stacks (and morphisms among them) using the atlas.

Let P be a property of schemes, local in nature for the smooth (resp. étale) topology. For example: regular, normal, reduced, of characteristic $p, \ldots$ Then we say that an Artin (resp. Deligne-Mumford) stack has P iff the atlas has P ([La], p. 25), ([DM], p. 100).

Let P be a property of morphisms of schemes, local on source and target for the smooth (resp. étale) topology, i.e. for any commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "p"] \arrow[dr, "f''"'] & Y' \times_Y X \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
 & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

with p and g smooth (resp. étale) and surjective, f has P iff f'' has P. For example: flat, smooth, locally of finite type, ... For the étale topology we also have: étale, unramified,.... Then if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of Artin (resp. Deligne-Mumford) stacks, we say that f has P iff for one (and then for all) commutative diagram of stacks

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[dr, "f''"'] & Y' \times_Y \mathcal{X} \arrow[r, "g'"] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\ & Y' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd},$$

where X', Y' are schemes and p, g are smooth (resp. étale) and surjective, f'' has P ([La], pp. 27–29).

For Deligne-Mumford stacks it is enough to find a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f''"'] & \mathcal{X} \arrow[d, "f"] \\ Y' \arrow[r, "g"] & \mathcal{Y} \end{tikzcd},$$

where p and g are étale and surjective and f'' has P. Then it follows that f has P ([DM], p. 100).

Other notions are defined as follows.

DEFINITION 2.35 (Substack) ([La], Definition 2.5), ([DM], p. 102).

A stack $\mathcal{E}$ is a substack of $\mathcal{F}$ if it is a full subcategory of $\mathcal{F}$ and

1. If an object X of $\mathcal{F}$ is in $\mathcal{E}$, then all isomorphic objects are also in $\mathcal{E}$.
2. For all morphisms of schemes $f : U \to V$, if X is in $\mathcal{E}(V)$, then f^*X is in $\mathcal{E}(U)$.
3. Let $\{U_i \to U\}$ be a cover of U in the site (Sch/S). Then X is in $\mathcal{E}$ iff $X|_i$ is in $\mathcal{E}$ for all i.

DEFINITION 2.36 ([La], Definition 2.13)

A substack $\mathcal{E}$ of $\mathcal{F}$ is called open (resp. closed, resp. locally closed) if the inclusion morphism $\mathcal{E} \to \mathcal{F}$ is *representable* and it is an open immersion (resp. closed immersion, resp. locally closed immersion).

DEFINITION 2.37 (Irreducibility) ([La], Definition 3.10), ([DM], p. 102)

An algebraic stack $\mathcal{F}$ is irreducible if it is not the union of two distinct and nonempty proper closed substacks.

DEFINITION 2.38 (Separatedness) ([La], Definition 3.17), ([DM], Definition 4.7)

An algebraic stack $\mathcal{F}$ is separated, if the (representable) diagonal morphism $\Delta_{\mathcal{F}}$ is universally closed (and hence proper, because it is automatically separated and of finite type).

A morphism $f : \mathcal{F} \to \mathcal{G}$ of algebraic stacks is separated if for all $U \to \mathcal{G}$ with U affine, $U \times_{\mathcal{G}} \mathcal{F}$ is a separated (algebraic) stack.

For Deligne-Mumford stacks, $\Delta_{\mathcal{F}}$ is universally closed iff it is finite. There is a valuative criterion of separatedness, similar to the criterion for schemes. Recall that by Yoneda lemma (Lemma 2.22), a morphism $f : U \to \mathcal{F}$ between a scheme and a stack is equivalent to an object in $\mathcal{F}(U)$. Then we will say that α is an isomorphism between two morphisms $f_1, f_2 : U \to \mathcal{F}$ when α is an isomorphism between the corresponding objects of $\mathcal{F}(U)$.

PROPOSITION 2.39 (Valuative criterion of separatedness (stacks)) ([La], Proposition 3.19), ([DM], Theorem 4.18)

An algebraic stack $\mathcal{F}$ is separated (over S) if and only if the following holds. Let A be a valuation ring with fraction field K. Let $g_1 : \operatorname{Spec} A \to \mathcal{F}$ and $g_2 : \operatorname{Spec} A \to \mathcal{F}$ be two morphisms such that:

1. $f_{p_{\mathcal{F}}} \circ g_1 = f_{p_{\mathcal{F}}} \circ g_2$.
2. *There exists an isomorphism $\alpha : g_1|_{\operatorname{Spec} K} \to g_2|_{\operatorname{Spec} K}$.*

$$\begin{tikzcd}
 & & \mathcal{F} \arrow[d, "p_{\mathcal{F}}"] \\
\operatorname{Spec} K \arrow[r, "i"] \arrow[urr, bend left] & \operatorname{Spec} A \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r] & S
\end{tikzcd}$$

then there exists an isomorphism (in fact unique) $\tilde{\alpha} : g_1 \to g_2$ that extends α, i.e. $\tilde{\alpha}|_{\operatorname{Spec} K} = \alpha$.

Remark 2.40. It is enough to consider complete valuation rings A with algebraically closed residue field ([La], 3.20.1). If furthermore S is locally Noetherian and $\mathcal{F}$ is locally of finite type, it is enough to consider discrete valuation rings A ([La], 3.20.2).

Example 2.41. The stack BG will not be separated if G is not proper over S ([La], 3.20.3), and since we assumed G to be affine, this will not happen if it is not finite.

In general the moduli stack of vector bundles $\mathcal{M}_X$ is not separated. It is easy to find families of vector bundles that contradict the criterion.

The stack of stable curves $\overline{\mathcal{M}}_g$ is separated ([DM], Proposition 5.1).

The criterion for morphisms is more involved because we are working with stacks and we have to keep track of the isomorphisms.

PROPOSITION 2.42 (Valuative criterion of separatedness (morphisms)) ([La], Proposition 3.19)

A morphism of algebraic stacks $f : \mathcal{F} \to \mathcal{G}$ is separated if and only if the following holds. Let A be a valuation ring with fraction field K. Let $g_1 : \operatorname{Spec} A \to \mathcal{F}$ and $g_2 : \operatorname{Spec} A \to \mathcal{F}$ be two morphisms such that:

1. *There exists an isomorphism $\beta : f \circ g_1 \to f \circ g_2$.*
2. *There exists an isomorphism $\alpha : g_1|_{\operatorname{Spec} K} \to g_2|_{\operatorname{Spec} K}$.*
3. $f(\alpha) = \beta|_{\operatorname{Spec} K}$.

Then there exists an isomorphism (in fact unique) $\tilde{\alpha} : g_1 \to g_2$ that extends α, i.e. $\tilde{\alpha}|_{\operatorname{Spec} K} = \alpha$ and $f(\tilde{\alpha}) = \beta$.

Remark 2.40 is also true in this case.

DEFINITION 2.43 ([La], Definition 3.21), ([DM], Definition 4.11)

An algebraic stack $\mathcal{F}$ is proper (over S) if it is separated and of finite type, and if there is a scheme X proper over S and a (representable) surjective morphism $X \to \mathcal{F}$.

A morphism $\mathcal{F} \to \mathcal{G}$ is proper if for any affine scheme U and morphism $U \to \mathcal{G}$, the fiber product $U \times_{\mathcal{G}} \mathcal{F}$ is proper over U.

For properness we only have a satisfactory criterion for stacks (see ([La], Proposition 3.23 and Conjecture 3.25) for a generalization for morphisms).

PROPOSITION 2.44 (Valuative criterion of properness) ([La], Proposition 3.23), ([DM], Theorem 4.19)

Let $\mathcal{F}$ be a separated algebraic stack (over S). It is proper (over S) if and only if the following condition holds. Let A be a valuation ring with fraction field K. For any commutative diagram

$$\begin{tikzcd} & & \mathcal{F} \arrow[d, "p_{\mathcal{F}}"] \\ \operatorname{Spec} K \arrow[urr, "g"] \arrow[r, "i"] & \operatorname{Spec} A \arrow[r] & S \end{tikzcd}$$

there exists a finite field extension K' of K such that g extends to $\operatorname{Spec}(A')$, *where A' is the integral closure of A in K'.*

$$\begin{tikzcd} & & \mathcal{F} \arrow[dd, "p_{\mathcal{F}}"] \\ \operatorname{Spec} K' \arrow[urr, "g \circ u"] \arrow[r] \arrow[d, "u"] & \operatorname{Spec} A' \arrow[ur, dashed] \arrow[d] & \\ \operatorname{Spec} K \arrow[r, "i"] & \operatorname{Spec} A \arrow[r] & S \end{tikzcd}.$$

Example 2.45 (*Stable curves*). The Deligne-Mumford stack of stable curves $\overline{\mathcal{M}}_g$ is proper ([DM], Theorem 5.2).

2.6 *Points and dimension*

We will introduce the concept of point of an algebraic stack and dimension of a stack at a point. The reference for this is ([La], Chapter 5).

DEFINITION 2.46

Let $\mathcal{F}$ be an algebraic stack over S. The set of points of $\mathcal{F}$ is the set of equivalence classes of pairs (K,x), with K a field over S (i.e. a field with a morphism of schemes $\operatorname{Spec}K \to S$) and $x : \operatorname{Spec}K \to \mathcal{F}$ a morphism of stacks over S. Two pairs (K',x') and (K'',x'') are equivalent if there is a field K extension of K' and K'' and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} K' \arrow[d, "x'"] \\ \operatorname{Spec} K'' \arrow[r, "x''"] & \mathcal{F} \end{tikzcd}.$$

Given a morphism $\mathcal{F} \to \mathcal{G}$ of algebraic stacks and a point of $\mathcal{F}$, we define the image of that point in $\mathcal{G}$ by composition.

Every point of an algebraic stack is the image of a point of an atlas. To see this, given a point represented by $\operatorname{Spec}K \to \mathcal{F}$ and an atlas $X \to \mathcal{F}$, take any point $\operatorname{Spec}K' \to X \times_{\mathcal{F}} \operatorname{Spec}K$. The image of this point in X maps to the given point.

To define the concept of dimension, recall that if X and Y are locally Noetherian schemes and $f : X \to Y$ is flat, then for any point $x \in X$ we have

$$\dim_x(X) = \dim_x(f) + \dim_{f(x)}(Y),$$

with $\dim_x(f) = \dim_x(X_{f(x)})$, where X_y is the fiber of f over y.

DEFINITION 2.47

Let $f : \mathcal{F} \to \mathcal{G}$ be a representable morphism, locally of finite type, between two algebraic spaces. Let ξ be a point of $\mathcal{F}$. Let Y be an atlas of $\mathcal{G}$. Take a point x in the algebraic space $Y \times_{\mathcal{G}} \mathcal{F}$ that maps to ξ,

$$\begin{tikzcd} Y \times_{\mathcal{G}} \mathcal{F} \arrow[r] \arrow[d, "\tilde{f}"'] & \mathcal{F} \arrow[d, "f"] \\ Y \arrow[r] & \mathcal{G} \end{tikzcd}$$

and define the dimension of the morphism f at the point ξ as

$$\dim_\xi(f) = \dim_x(\tilde{f}).$$

It can be shown that this definition is independent of the choices made.

DEFINITION 2.48

Let $\mathcal{F}$ be a locally Noetherian algebraic stack and ξ a point of $\mathcal{F}$. Let $u : X \to \mathcal{F}$ be an atlas, and x a point of X mapping to ξ. We define the dimension of $\mathcal{F}$ at the point ξ as

$$\dim_\xi(\mathcal{F}) = \dim_x(X) - \dim_x(u).$$

The dimension of $\mathcal{F}$ is defined as

$$\dim(\mathcal{F}) = \mathrm{Sup}_\xi(\dim_\xi(\mathcal{F})).$$

Again, this is independent of the choices made.

Example 2.49 (*Quotient by group action*). Let X be a smooth scheme of dimension $\dim(X)$ and G a smooth group of dimension $\dim(G)$ acting on X. Let $[X/G]$ be the quotient stack defined in Example 2.18. Using the atlas defined in Example 2.29, we see that

$$\dim[X/G] = \dim(X) - \dim(G).$$

Note that we have not made any assumption on the action. In particular, the action could be trivial. The dimension of an algebraic stack can then be negative. For instance, the dimension of the classifying stack BG defined in Example 2.18 has dimension $\dim(BG) = -\dim(G)$.

2.7 *Quasi-coherent sheaves on stacks*

DEFINITION 2.50 ([Vi], Definition 7.18), ([La], Definition 6.11, Proposition 6.16). A quasi-coherent sheaf $\mathcal{S}$ on an algebraic stack $\mathcal{F}$ is the following set of data:

1. For each morphism $X \to \mathcal{F}$ where X is a scheme, a quasi-coherent sheaf $\mathcal{S}_X$ on X.
2. For each commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & \mathcal{F} \end{tikzcd}$$

an isomorphism $\varphi_f : \mathcal{S}_X \xrightarrow{\cong} f^*\mathcal{S}_Y$, satisfying the cocycle condition, i.e. for any commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[dl] \\ & \mathcal{F} & \end{tikzcd} \tag{6}$$

we have $\varphi_{g\circ f} = \varphi_f \circ f^*\varphi_g$.

We say that $\mathcal{S}$ is coherent (resp. finite type, finite presentation, locally free) if $\mathcal{S}_X$ is coherent (resp. finite type, finite presentation, locally free) for all X.

A morphism of quasi-coherent sheaves $h : \mathcal{S} \to \mathcal{S}'$ is a collection of morphisms of sheaves $h_X : \mathcal{S}_X \to \mathcal{S}'_X$ compatible with the isomorphisms φ

Remark 2.51. Since a sheaf on a scheme can be obtained by glueing the restriction to an affine cover, it is enough to consider affine schemes.

Example 2.52 (*Structure sheaf*). Let $\mathcal{F}$ be an algebraic stack. The structure sheaf $\mathcal{O}_{\mathcal{F}}$ is defined by taking $(\mathcal{O}_{\mathcal{F}})_X = \mathcal{O}_X$.

Example 2.53 (*Sheaf of differentials*). Let $\mathcal{F}$ be a Deligne-Mumford stack. To define the sheaf of differentials $\Omega_{\mathcal{F}}$, if $U \to \mathcal{F}$ is an étale morphism we set $(\Omega_{\mathcal{F}})_U = \Omega_U$, the sheaf of differentials of the scheme U. If $V \to \mathcal{F}$ is another étale morphism and we have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr] & V \arrow[d] \\ & \mathcal{F} \end{tikzcd}$$

then f has to be étale, there is a canonical isomorphism $\varphi_f : \Omega_{U/S} \to f^*\Omega_{V/S}$, and these canonical isomorphisms satisfy the cocycle condition.

Once we have defined $(\Omega_{\mathcal{F}})_U$ for étale morphisms $U \to \mathcal{F}$, we can extend the definition for any morphism $X \to \mathcal{F}$ with X an arbitrary scheme as follows: take an (étale) atlas $U = \coprod U_i \to \mathcal{F}$. Consider the composition morphism

$$X \times_{\mathcal{F}} U \xrightarrow{p_2} U \to \mathcal{F},$$

and define $(\Omega_{\mathcal{F}})_{X\times_{\mathcal{F}}U} = p_2^*\Omega_U$. The cocycle condition for Ω_{U_i} and étale descent implies that $(\Omega_{\mathcal{F}})_{X\times_{\mathcal{F}}U}$ descends to give a sheaf $(\Omega_{\mathcal{F}})_X$ on X. It is easy to check that this doesn't depend on the atlas U used, and that given a commutative diagram like (6), there are canonical isomorphisms φ satisfying the cocycle condition.

Example 2.54 (*Universal vector bundle*). Let $\mathcal{M}_X$ be the moduli stack of vector bundles on a scheme X defined in 2.9. The universal vector bundle V on $\mathcal{M}_X \times X$ is defined as follows:

Let U be a scheme and $f = (f_1, f_2) : U \to \mathcal{M}_X \times X$ a morphism. By Lemma 2.22, the morphism $f_1 : U \to \mathcal{M}_X$ is equivalent to a vector bundle W on $U \times X$. We define V_U as $\tilde{f}^*W$, where $\tilde{f} = (\mathrm{id}_U, f_2) : U \to U \times X$. Let

$$\begin{tikzcd} U' \arrow[r, "g"] \arrow[dr, "f'"'] & U \arrow[d, "f"] \\ & \mathcal{M}_X \times X \end{tikzcd}$$

be a commutative diagram. Recall that this means that there is an isomorphism $\alpha : f \circ g \to f'$, and looking at the projection to $\mathcal{M}_X$ we have an isomorphism $\alpha_1 : f_1 \circ g \to f_1'$. Using Lemma 2.22, $f_1 \circ g$ and f_1' correspond respectively to the vector bundles $(g \times \mathrm{id}_X)^* W$ and W' on $U' \times X$, and (again by Lemma 2.22) α_1 gives an isomorphism between them. It is easy to check that these isomorphisms satisfy the cocycle condition for diagrams of the form (6).

3. Vector bundles: Moduli stack vs. moduli scheme

In this section we will compare, in the context of vector bundles, the new approach of stacks versus the standard approach of moduli schemes via geometric invariant theory (GIT) (for background on moduli schemes of vector bundles, see [HL]).

Fix a scheme X over , a positive integer r and classes $c_i \in H^{2i}(X)$. All vector bundles over X in this section will have rank r and Chern classes c_i. We will also consider vector bundles on products $B \times X$ where B is a scheme. We will always assume that these vector bundles are flat over B, and that the restriction to the slices $\{p\} \times X$ are vector bundles with rank r and Chern classes c_i. Fix also a polarization on X. All references to stability or semistability of vector bundles will mean Gieseker stability with respect to this fixed polarization.

Recall that the functor $\underline{M}^s_X$ (resp. $\underline{M}^{ss}_X$) is the functor from (Sch/S) to (Sets) that for each scheme B gives the set of *equivalence* classes of vector bundles over $B \times X$, flat over B and such that the restrictions $V|_b$ to the slices $p \times X$ are stable (resp. semistable) vector bundles with fixed rank and Chern classes, where two vector bundles V and V' on $B \times X$ are considered *equivalent* if there is a line bundle L on B such that V is isomorphic to $V' \otimes p_B^* L$.

Theorem 3.1. *There are schemes M^s_X and M^{ss}_X, called moduli schemes, corepresenting the functors $\underline{M}^s_X$ and $\underline{M}^{ss}_X$.*

The moduli scheme M^{ss}_X is constructed using the Quot schemes introduced in Example 2.28 (for a detailed exposition of the construction, see [HL]). Since the set of *semistable* vector bundles is bounded, we can choose once and for all N and m (depending only on the Chern classes and rank) with the property that for any semistable vector bundle V there is a point in $R = R_{N,m}$ whose corresponding quotient is isomorphic to V.

The scheme R parametrizes vector bundles V on X together with a basis of $H^0(V(m))$ (up to multiplication by scalar). Recall that $N = h^0(V(m))$. There is an action of $GL(N)$ on R, corresponding to change of basis but since two basis that only differ by a scalar give the same point on R, this $GL(N)$ action factors through $PGL(N)$. Then the moduli scheme M^{ss}_X is defined as the GIT quotient $R//PGL(N)$.

The closed points of M^{ss}_X correspond to S-equivalence classes of vector bundles, so if there is a strictly semistable vector bundle, the functor $\underline{M}^{ss}_X$ is not representable.

Now we will compare this scheme with the moduli stack $\mathcal{M}_X$ defined on Example 2.9. We will also consider the moduli stack $\mathcal{M}^s_X$ defined in the same way, but with the extra requirement that the vector bundles should be stable. The moduli stack $\mathcal{M}^s_X$ is a substack (Definition 2.35) of $\mathcal{M}_X$. The following are some of the differences between the moduli scheme and the moduli stack:

1. The stack $\mathcal{M}_X$ parametrizes all vector bundles, but the scheme M^{ss}_X only parametrizes semistable vector bundles.

2. From the point of view of the scheme M_X^{ss}, we identify two vector bundles on X (i.e. they give the same closed point on M_X^{ss}) if they are S-equivalent. On the other hand, from the point of view of the moduli stack, two vector bundles are identified (i.e. give isomorphic objects on $\mathcal{M}_X(\mathrm{Spec}\ k)$) only if they are isomorphic as vector bundles.
3. Let V and V' be two families of vector bundles parametrized by a scheme B, i.e. two vector bundles (flat over B) on $B \times X$. If there is a line bundle L on B such that V is isomorphic to $V' \otimes p_B^* L$, then from the point of view of the moduli scheme, V and V' are identified as being the same family. On the other hand, from the point of view of the moduli stack, V and V' are identified only if they are isomorphic as vector bundles on $B \times X$.
4. The subscheme M_X^s corresponding to stable vector bundles is sometimes representable by a scheme, but the moduli stack $\mathcal{M}_X^s$ is never representable by a scheme. To see this, note that any vector bundle has automorphisms different from the identity (multiplication by scalars) and apply Lemma 2.21.

Now we will restrict our attention to stable bundles, i.e. to the scheme M_X^s and the stack $\mathcal{M}_X^s$. For stable bundles the notions of S-equivalence and isomorphism coincide, so the points of M_X^s correspond to isomorphism classes of vector bundles. Consider $R^s \subset R$, the subscheme corresponding to stable bundles. There is a map $\pi : R^s \to M_X^s = R^s/PGL(N)$, and π is in fact a principal $PGL(N)$-bundle (this is a consequence of Luna's étale slice theorem).

Remark 3.2 (*Universal bundle on moduli scheme*). The scheme M_X^s represents the functor $\underline{M}_x^s$ if there is a universal family. Recall that a universal family for this functor is a vector bundle E on $M_X^s \times X$ such that the isomorphism class of $E|_{p\times X}$ is the isomorphism class corresponding to the point $p \in M_X^s$, and for any family of vector bundles V on $B \times X$ there is a morphism $f : B \to M_X^s$ and a line bundle L on B such that $V \otimes p_B^* L$ is isomorphic to $(f \times \mathrm{id}_X)^* E$. Note that if E is a universal family, then $E \otimes p_{M_X^s}^* L$ will also be a universal family for any line bundle L on M_X^s.

The universal bundle for the Quot scheme gives a universal family $\tilde{V}$ on $R^s \times X$, but this family does not always descend to give a universal family on the quotient M_X^s.

Let $X \xrightarrow{G} Y$ be a principal G-bundle. A vector bundle V on X descends to Y if the action of G on X can be lifted to V. In our case, if certain numerical criterion involving r and c_i is satisfied (if X is a smooth curve this criterion is $\gcd(r, c_1) = 1$), then we can find a line bundle L on R^s such that the $PGL(N)$ action on R^s can be lifted to $\tilde{V} \otimes p_{R^s}^* L$, and then this vector bundle descends to give a universal family on $M_X^s \times X$. But in general the best that we can get is a universal family on an étale cover of M_X^s.

Recall from Example 2.29 that there is a morphism $[R^{ss}/PGL(N)] \to M_X^{ss}$, and that the morphism $[R^s/PGL(N)] \to M_X^s$ is an isomorphism of stacks.

PROPOSITION 3.3

There is a commutative diagram of stacks

$$\begin{tikzcd} {[R^s/GL(N)]} \arrow[r, "q"] \arrow[d, "g"', "\simeq"] & {[R^s/PGL(N)]} \arrow[d, "h", "\simeq"'] \\ \mathcal{M}_X^s \arrow[r, "\varphi"'] & M_X^s, \end{tikzcd}$$

where g and h are isomorphisms of stacks, but q and φ are not. If we change 'stable' by 'semistable' we still have a commutative diagram, but the corresponding morphism h^{ss} is not an isomorphism of stacks.

Proof. The morphism φ is the composition of the natural morphism $\mathcal{M}^s_X \to \underline{M}^s_X$ (sending each category to the set of isomorphism classes of objects) and the morphism $\underline{M}^s_X \to M^s_X$ given by the fact that the scheme $M^s_X = R^s//PGL(N)$ corepresents the functor $\underline{M}^s_X$.

The morphism h was constructed in Example 2.18.

The key ingredient needed to define g is the fact that the $GL(N)$ action on the Quot scheme lifts to the universal bundle, i.e. the universal bundle on the Quot scheme has a $GL(N)$-linearization. Let

$$\begin{tikzcd} \widetilde{B} \arrow[r, "f"] \arrow[d] & R^{ss} \\ B & \end{tikzcd}$$

be an object of $[R^{ss}/GL(N)]$. Since R^{ss} is a subscheme of a Quot scheme, by restriction we have a universal bundle on $R^{ss} \times X$, and this universal bundle has a $GL(N)$-linearization. Let $\tilde{E}$ be the vector bundle on $\tilde{B} \times X$ defined by the pullback of this universal bundle. Since f is $GL(N)$-equivariant, $\tilde{E}$ is also $GL(N)$-linearized. Since $\tilde{B} \times X \to B \times X$ is a principal bundle, the vector bundle $\tilde{E}$ descends to give a vector bundle E on $B \times X$, i.e. an object of $\mathcal{M}^{ss}_X$. Let

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\phi"'] \arrow[d] \arrow[rr, bend left, "f"] & \widetilde{B}' \arrow[r, "f'"'] \arrow[d] & R^{ss} \\ B \arrow[r, equal] & B & \end{tikzcd}$$

be a morphism in $[R^{ss}/GL(N)]$. Consider the vector bundles $\tilde{E}$ and $\tilde{E}'$ defined as before. Since $f' \circ \phi = f$, we get an isomorphism of $\tilde{E}$ with $(\phi \times \mathrm{id})^* \tilde{E}'$. Furthermore this isomorphism is $GL(N)$-equivariant, and then it descends to give an isomorphism of the vector bundles E and E' on $B \times X$, and we get a morphism in $\mathcal{M}^{ss}_X$.

To prove that this gives an equivalence of categories, we construct a functor $\overline{g}$ from $\mathcal{M}^{ss}_X$ to $[R^{ss}/GL(N)]$. Given a vector bundle E on $B \times X$, let $q : \tilde{B} \to B$ be the $GL(N)$-principal bundle associated with the vector bundle $p_{B*}E$ on B. Let $\tilde{E} = (q \times \mathrm{id})^* E$ be the pullback of E to $\tilde{B} \times X$. It has a canonical $GL(N)$-linearization because it is defined as a pullback by a principal $GL(N)$-bundle. The vector bundle $p_{\tilde{B}*}\tilde{E}$ is canonically isomorphic to the trivial bundle $\mathcal{O}^N_{\tilde{B}}$, and this isomorphism is $GL(N)$-equivariant, so we get an *equivariant* morphism $\tilde{B} \to R^{ss}$, and hence an object of $[R^{ss}/GL(N)]$.

If we have an isomorphism between two vector bundles E and E' on $B \times X$, it is easy to check that it induces an isomorphism between the associated objects of $[R^{ss}/GL(N)]$.

It is easy to check that there are natural isomorphisms of functors $g \circ \tilde{g} \cong \mathrm{id}$ and $\tilde{g} \circ g \cong \mathrm{id}$, and then g is an equivalence of categories.

The morphism q is defined using the following lemma, with $G = GL(N)$, H the subgroup consisting of scalar multiples of the identity, $\overline{G} = PGL(N)$ and Y=R^{ss}. □

Lemma 3.4. *Let Y be an S-scheme and G an affine flat group S-scheme, acting on Y on the right. Let H be a normal closed subgroup of G. Assume that $\overline{G} = G/H$ is affine. If H*

acts trivially on Y, then there is a morphism of stacks

$$[Y/G] \to [Y/\overline{G}].$$

If H is nontrivial, then this morphism is not faithful, so it is not an isomorphism.

Proof. Let

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ B & \end{tikzcd}$$

be an object of $[Y/G]$. There is a scheme E/H such that π factors

$$E \xrightarrow{q} E/H \xrightarrow{\pi'} B.$$

To construct E/H, note that there is a local étale cover U_i of B and isomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times G$, with transition functions $\psi_{ij} = \phi_i \circ \phi_j^{-1}$. Since these isomorphisms are G-equivariant, they descend to give isomorphisms $\overline{\psi}_{ij} : U_j \times G/H \to U_i \times G/H$, and using these transition functions we get E/H. This construction shows that π' is a principal $\overline{G}$-bundle. Furthermore, q is also a principal H-bundle ([HL], Example 4.2.4), and in particular it is a categorical quotient.

Since f is H-invariant, there is a morphism $\overline{f} : E/H \to Y$, and this gives an object of $[Y/\overline{G}]$.

If we have a morphism in $[Y/G]$, given by a morphism $g : E \to E'$ of principal G-bundles over B, it is easy to see that it descends (since g is equivariant) to a morphism $\overline{g} : E/H \to E'/H$, giving a morphism in $[Y/\overline{G}]$.

This morphism is not faithful, since the automorphism $E \xrightarrow{\cdot z} E$ given by multiplication on the right by a nontrivial element $z \in H$ is sent to the identity automorphism $E/H \to E/H$, and then $\mathrm{Hom}(E, E) \to \mathrm{Hom}(E/H, E/H)$ is not injective. □

If X is a smooth curve, then it can be shown that $\mathcal{M}_X$ is a smooth stack of dimension $r^2(g-1)$, where r is the rank and g is the genus of X. In particular, the open substack $\mathcal{M}_X^{ss}$ is also smooth of dimension $r^2(g-1)$, but the moduli scheme M_X^{ss} is of dimension $r^2(g-1)+1$ and might not be smooth. Proposition 3.3 explains the difference in the dimensions (at least on the smooth part): we obtain the moduli stack by taking the quotient by the group $GL(N)$, of dimension N^2, but the moduli scheme is obtained by a quotient by the group $PGL(N)$, of dimension N^2-1. The moduli scheme M_X^{ss} is not smooth in general because in the strictly semistable part of R^{ss} the action of $PGL(N)$ is not free. On the other hand, the smoothness of a stack quotient doesn't depend on the freeness of the action of the group.

Appendix A: Grothendieck topologies, sheaves and algebraic spaces

The standard reference for Grothendieck topologies is SGA (*Séminaire de Géométrie Algébrique*). For an introduction see [T] or [MM]. For algebraic spaces, see [K] or [Ar1].

An open cover in a topological space U can be seen as family of morphisms in the category of topological spaces $f_i : U_i \to U$, with the property that f_i is an open inclusion and the union of their images is U, i.e we are choosing a class of morphisms (open inclusions) in the category of topological spaces. A Grothendieck topology on an arbitrary category is basically a choice of a class of morphisms, that play the role of 'open sets'. A morphism $f: V \to U$ in this class is to be thought of as an 'open set' in the object U. The concept of intersection of open sets is replaced by the fiber product: the 'intersection' of $f_1: U_1 \to U$ and $f_2: U_2 \to U$ is $f_{12}: U_1 \times_U U_2 \to U$.

A category with a Grothendieck topology is called a site. We will consider two topologies on (Sch/S).

fppf topology. Let U be a scheme. Then a cover of U is a finite collection of morphisms $\{f_i: U_i \to U\}_{i\in I}$ such that each f_i is a finitely presented flat morphism (for Noetherian schemes, this is equivalent to flat and finite type), and U is the (set theoretic) union of the images of f_i. In other words, $\coprod U_i \to U$ is *'fidèlement plat de présentation finie'*.

Étale topology. Same definition, but substituting flat by étale.

DEFINITION 4.1 (Presheaf of sets)

A presheaf of sets on (Sch/S) is a contravariant functor F from (Sch/S) to (Sets).

As usual, we will use the following notation: if $X \in F(U)$ and $f_i: U_i \to U$ is a morphism, then $X|_i$ is the element of $F(U_i)$ given by $F(f_i)(X)$, and we will call $X|_i$ the 'restriction of X to U_i', even if f_i is not an inclusion. If $X_i \in F(U_i)$, then $X_i|_{ij}$ is the element of $F(U_{ij})$ given by $F(f_{ij,i})(X_i)$ where $f_{ij,i}: U_i \times_U U_j \to U_i$ is the pullback of f_j.

DEFINITION 4.2 (Sheaf of sets)

Choose a topology on (Sch/S). We say that F is a sheaf (or an S-space) with respect to that topology if for every cover $\{f_i: U_i \to U\}_{i\in I}$ in the topology the following two axioms are satisfied:

1. *Mono*. Let X and Y be two elements of $F(U)$. If $X|_i = Y|_i$ for all i, then $X = Y$.
2. *Glueing*. Let X_i be an object of $F(U_i)$ for each i such that $X_i|_{ij} = X_j|_{ij}$, then there exists $X \in F(U)$ such that $X|_i = X_i$ for each i.

We define morphisms of S-spaces as morphisms of sheaves (i.e. natural transformations of functors). Note that a scheme M can be viewed as an S-space via its functor of points $\mathrm{Hom}_S(-, M)$, and a morphism between two such S-spaces is equivalent to a scheme morphism between the schemes (by the Yoneda embedding lemma), then the category of S-schemes is a full subcategory of the category of S-spaces.

Equivalence relation and quotient space. An equivalence relation in the category of S-spaces consists of two S-spaces R and U and a monomorphism of S-spaces

$$\delta: R \to U \times_S U$$

such that for all S-scheme B, the map $\delta(B): R(B) \to U(B) \times U(B)$ is the graph of an equivalence relation between sets. A quotient S-space for such an equivalence relation is by definition the sheaf cokernel of the diagram

$$R \underset{p_1\circ\delta}{\overset{p_2\circ\delta}{\rightrightarrows}} U.$$

DEFINITION 4.3 (Algebraic space) ([La], 0).

An S-space F is called an algebraic space if it is the quotient S-space for an equivalence relation such that R and U are S-schemes, $p_1 \circ \delta$, $p_2 \circ \delta$ are étale (morphisms of S-schemes), and δ is a quasi-compact morphism (of S-schemes).

Roughly speaking, an algebraic space is a quotient of a scheme by an étale equivalence relation. The following is an equivalent definition.

DEFINITION 4.4 ([K], Definition 1.1)

An S-space F is called an algebraic space if there exists a scheme U (atlas) and a morphism of S-spaces $u : U \to F$ such that

1. *The morphism u is étale*. For any S-scheme V and morphism $V \to F$, the (sheaf) fiber product $U \times_F V$ is representable by a scheme, and the map $U \times_F V \to V$ is an étale morphism of schemes.
2. *Quasi-separatedness*. The morphism $U \times_F U \to U \times_S U$ is quasi-compact.

We recover the first definition by taking $R = U \times_F U$. Then roughly speaking, we can also think of an algebraic space as 'something' that looks locally in the étale topology like an affine scheme, in the same sense that a scheme is something that looks locally in the Zariski topology like an affine scheme.

Algebraic spaces are used, for instance, to give algebraic structure to certain complex manifolds (for instance Moishezon manifolds) that are not schemes, but can be realized as algebraic spaces. All smooth algebraic spaces of dimension 1 and 2 are actually schemes. An example of a smooth algebraic space of dimension 3 that is not a scheme can be found in [H].

But étale topology is useful even if we are only interested in schemes. The idea is that the étale topology is finer than the Zariski topology, and in many situations it is 'fine enough' to do the analog of the manipulations that can be done with the analytic topology of complex manifolds. As an example, consider the affine complex line $\mathrm{Spec}(\mathbb{C}[x])$, and take a (closed) point x_0 different from 0. Assume that we want to define the function $\sqrt{x}$ in a neighborhood of x_0. In the analytic topology we only need to take a neighborhood small enough so that it does not contain a loop that goes around the origin, then we choose one of the branches (a sign) of the square root. In the Zariski topology this cannot be done, because all open sets are too large (have loops going around the origin, so the sign of the square root will change, and $\sqrt{x}$ will be multivaluated). But take the 2:1 étale map $V = \mathrm{Spec}(\mathbb{C}[y, x, x^{-1}]/(y - x^2)) \to \mathrm{Spec}(\mathbb{C}[x])$. The function $\sqrt{x}$ can certainly be defined on V, it is just equal to the function y, so it is in this sense that we say that the étale topology is finer: V is a 'small enough open subset' because the square root can be defined on it.

Appendix B: 2-categories

In this section we recall the notions of 2-category and 2-functor. A 2-category C consists of the following data [Hak]:

(i) A class of objects ob C.
(ii) For each pair X, $Y \in \mathrm{ob}\, C$, a category $\mathrm{Hom}(X, Y)$.

(iii) *Horizontal composition of 1-morphisms and 2-morphisms.* For each triple X, Y, $Z \in \mathrm{ob}C$, a functor

$$\mu_{X,Y,Z} : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$$

with the following conditions

(i') *Identity 1-morphism.* For each object $X \in \mathrm{ob}C$, there exists an object $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ such that

$$\mu_{X,X,Y}(\mathrm{id}_X,) = \mu_{X,Y,Y}(, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}(X,Y)},$$

where $\mathrm{id}_{\mathrm{Hom}(X,Y)}$ is the identity functor on the category $\mathrm{Hom}(X,Y)$.

(ii') *Associativity of horizontal compositions.* For each quadruple X, Y, Z, $T \in \mathrm{ob}C$,

$$\mu_{X,Z,T} \circ (\mu_{X,Y,Z} \times \mathrm{id}_{\mathrm{Hom}(Z,T)}) = \mu_{X,Y,T} \circ (\mathrm{id}_{\mathrm{Hom}(X,Y)} \times \mu_{Y,Z,T}).$$

The example to keep in mind is the 2-category *Cat* of categories. The objects of *Cat* are categories, and for each pair X, Y of categories, $\mathrm{Hom}(X,Y)$ is the category of functors between X and Y.

Note that the main difference between a 1-category (a usual category) and a 2-category is that $\mathrm{Hom}(X,Y)$, instead of being a set, is a category.

Given a 2-category, an object f of the category $\mathrm{Hom}(X,Y)$ is called a 1-morphism of C, and is represented with a diagram

$$\overset{X}{\bullet} \xrightarrow{f} \overset{Y}{\bullet}$$

and a morphism α of the category $\mathrm{Hom}(X,Y)$ is called a 2-morphisms of C, and is represented as

$$\begin{tikzcd} \overset{X}{\bullet} \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \overset{Y}{\bullet} \end{tikzcd}$$

Now we will rewrite the axioms of a 2-category using diagrams.

1. *Composition of 1-morphisms.* Given a diagram

$$\overset{X}{\bullet} \xrightarrow{f} \overset{Y}{\bullet} \xrightarrow{g} \overset{Z}{\bullet} \quad \text{there exist} \quad \overset{X}{\bullet} \xrightarrow{g \circ f} \overset{Z}{\bullet}$$

(this is (iii) applied to objects) and this composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$ (this is (ii') applied to objects).

2. *Identity for* 1-*morphisms.* For each object X there is a 1-morphism id_X such that $f \circ \mathrm{id}_Y = \mathrm{id}_X \circ f = f$ (this is (i')).

3. *Vertical composition of* 2-*morphisms.* Given a diagram

$$\begin{tikzcd} \overset{X}{\bullet} \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \overset{Y}{\bullet} \end{tikzcd} \quad \text{there exists} \quad \begin{tikzcd} \overset{X}{\bullet} \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \overset{Y}{\bullet} \end{tikzcd}$$

and this composition is associative $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$.

4. *Horizontal composition of* 2-*morphisms*. Given a diagram

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=A}] \arrow[r, bend right=50, "f'"'{name=B}] & Y \arrow[r, bend left=50, "g"{name=C}] \arrow[r, bend right=50, "g'"'{name=D}] & Z \arrow[from=A, to=B, Rightarrow, "\alpha"] \arrow[from=C, to=D, Rightarrow, "\beta"] \end{tikzcd} \tag{7}$$

there exists

$$\begin{tikzcd} X \arrow[r, bend left=50, "g\circ f"{name=A}] \arrow[r, bend right=50, "g'\circ f'"'{name=B}] & Z \arrow[from=A, to=B, Rightarrow, "\beta*\alpha"] \end{tikzcd}$$

(this is (iii) applied to morphisms) and it is associative $(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha)$ (this is (ii') applied to morphisms).

5. *Identity for* 2-*morphisms*. For every 1-morphism f there is a 2-morphism id_f such that $\alpha \circ \mathrm{id}_g = \mathrm{id}_f \circ \alpha = \alpha$ (this and item are (ii)). We have $\mathrm{id}_g * \mathrm{id}_f = \mathrm{id}_{g\circ f}$ (this means that $\mu_{X,Y,Z}$ respects the identity).

6. *Compatibility between horizontal and vertical composition of* 2-*morphisms*. Given a diagram

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"{name=A}] \arrow[r, "f'"{name=B, description}] \arrow[r, bend right=60, "f''"'{name=C}] & Y \arrow[r, bend left=60, "g"{name=D}] \arrow[r, "g'"{name=E, description}] \arrow[r, bend right=60, "g''"'{name=F}] & Z \arrow[from=A, to=B, Rightarrow, "\alpha"] \arrow[from=B, to=C, Rightarrow, "\alpha'"] \arrow[from=D, to=E, Rightarrow, "\beta"] \arrow[from=E, to=F, Rightarrow, "\beta'"] \end{tikzcd} ,$$

then $(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha)$ (this is (iii) applied to morphisms).

Two objects X and Y of a 2-category are called equivalent if there exist two 1-morphisms $f : X \to Y$, $g : Y \to X$ and two 2-isomorphisms (invertible 2-morphism) $\alpha : g \circ f \to \mathrm{id}_X$ and $\beta : f \circ g \to \mathrm{id}_Y$.

A commutative diagram of 1-morphisms in a 2-category is a diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd}$$

such that $\alpha : g \circ f \to h$ is a 2-isomorphisms.

Remark 5.1 Note that we do not require $g \circ f = h$ to say that the diagram is commutative, but just require that there is a 2-isomorphisms between them. This is the reason why 2-categories are used to describe stacks.

On the other hand, a diagram of 2-morphisms will be called commutative only if the compositions are actually equal. Now we will define the concept of covariant 2-functor (a contravariant 2-functor is defined in a similar way).

A covariant 2-functor F between two 2-categories C and C' is a law that for each object X in C gives an object $F(X)$ in C'. For each 1-morphism $f : X \to Y$ in C gives a 1-morphism $F(f) : F(X) \to F(Y)$ in C', and for each 2-morphism $\alpha : f \Rightarrow g$ in C gives a

2-morphism $F(\alpha) : F(f) \Rightarrow F(g)$ in C', such that

1. *Respects identity 1-morphism.* $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
2. *Respects identity 2-morphism.* $F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$.
3. *Respects composition of 1-morphism up to a 2-isomorphism.* For every diagram

$$\overset{X}{\bullet} \xrightarrow{f} \overset{Y}{\bullet} \xrightarrow{g} \overset{Z}{\bullet}$$

there exists a 2-isomorphism $\epsilon_{g,f} : F(g) \circ F(f) \to F(g \circ f)$

$$\begin{tikzcd} & \overset{F(Y)}{\bullet} \arrow[dr, "F(g)"] \arrow[d, phantom, "\Downarrow \epsilon_{g,f}"] & \\ F(X)\bullet \arrow[ur, "F(f)"] \arrow[rr, "F(g\circ f)"'] & {} & \bullet F(Z) \end{tikzcd}$$

(a) $\epsilon_{f,\mathrm{id}_X} = \epsilon_{\mathrm{id}_Y,f} = \mathrm{id}_{F(f)}$.
(b) ϵ *is associative*. The following diagram is commutative

$$\begin{tikzcd} F(h) \circ F(g) \circ F(f) \arrow[r, Rightarrow, "\epsilon_{h,g} \times \mathrm{id}"] \arrow[d, Rightarrow, "\mathrm{id} \times \epsilon_{g,f}"'] & F(h \circ g) \circ F(f) \arrow[d, Rightarrow, "\epsilon_{h\circ g,f}"] \\ F(h) \circ F(g \circ f) \arrow[r, Rightarrow, "\epsilon_{h,g\circ f}"] & F(h \circ g \circ f) \end{tikzcd} .$$

4. *Respects vertical composition of* 2*-morphisms.* For every pair of 2-morphisms $\alpha : f \to g$, $\beta : g \to h$, we have $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$.
5. *Respects horizontal composition of* 2*-morphisms.* For every pair of 2-morphisms $\alpha : f \to f'$, $\beta : g \to g'$ as in (7) the following diagram commutes

$$\begin{tikzcd} F(g) \circ F(f) \arrow[r, Rightarrow, "F(\beta)*F(\alpha)"] \arrow[d, Rightarrow, "\epsilon_{g,f}"'] & F(g') \circ F(f') \arrow[d, Rightarrow, "\epsilon_{g',f'}"] \\ F(g \circ f) \arrow[r, Rightarrow, "F(\beta*\alpha)"] & F(g' \circ f') \end{tikzcd} .$$

By a slight abuse of language, condition 5 is usually written as $F(\beta) * F(\alpha) = F(\beta * \alpha)$. Note that strictly speaking this equality doesn't make sense, because the sources (and the targets) do not coincide, but if we chose once and for all the 2-isomorphisms ϵ of condition 3, then there is a unique way of making sense of this equality.

Remark 5.2. Since 2-functors only respect composition of 1-functors up to a 2-isomorphism (condition 3), sometimes they are called pseudofunctors or lax functors.

Remark 5.3. In the applications to stacks, the isomorphism $\epsilon_{g,f}$ of item 3 is canonically defined, and by abuse of language we will say that $F(g) \circ F(f) = F(g \circ f)$, instead of saying that they are isomorphic.

Given a 1-category C (a usual category), we can define a 2-category: we just have to make the set $\mathrm{Hom}(X, Y)$ into a category, and we do this just by defining the unit morphisms for each element.

On the other hand, given a 2-category C there are two ways of defining a 1-category. We have to make each category $\mathrm{Hom}(X, Y)$ into a set. The naive way is just to take the set of objects of $\mathrm{Hom}(X, Y)$, and then we obtain what is called the underlying category of C

(see [Hak]). This has the problem that a 2-functor $F : C \to C'$ is not in general a functor of the underlying categories (because in item 3 we only require the composition of 1-morphisms to be respected up to 2-isomorphism).

The best way of constructing a 1-category from a 2-category is to define the set of morphisms between the objects X and Y as the set of isomorphism classes of objects of $\mathrm{Hom}(X,Y)$: two objects f and g of $\mathrm{Hom}(X,Y)$ are isomorphic if there exists a 2-isomorphism $\alpha : f \Rightarrow g$ between them. We call the category obtained in this way the 1-category associated to C. Note that a 2-functor between 2-categories then becomes a functor between the associated 1-categories.

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