Charged Harmonic Oscillator in the Presence of Electric and Magnetic Fields.

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Summary. — A new treatment is presented for the time evolution of a fixed charged oscillator under the combined action of arbitrarily specified electric and magnetic fields. The Heisenberg equations of motion have been solved and quasi-coherent states are utilized to calculate the Green's function. We also examine the connection between the Schrödinger wave functions of pseudostationary and quasi-coherent states. The Bloch density matrix is calculated in a special case.

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1. – Introduction.

In a previous communication (1) we have considered the problem of a two-dimensional harmonic oscillator with arbitrary time-dependent coupling. In the present paper we turn our attention to the more fundamental problem of a three-dimensional charged oscillator (a simple model of an atom) under the combined action of arbitrary electric and magnetic fields. This is obviously of interest in quantum optics (2-4) and also in electromagnetic theory (5,6). In recent years much

(1) M. S. ABDALLA: *Phys. Rev. A*, 35, 4160 (1987).
(2) M. S. ABDALLA, R. K. COLEGRAVE and A. A. SILEM: *Physica A*, to appear (1988).
(3) M. S. ABDALLA, S. S. HASSAN and A. S. F. OBADA: *Phys. Rev. A*, 34, 4869 (1986).
(4) R. K. COLEGRAVE and A. VAHABPOUR-ROUNDSARI: *Opt. Acta*, 33, 645 (1986).
(5) J. D. JACKSON: *Classical Electrodynamics* (John Wiley & Sons Inc., New York, N.Y., 1970).
(6) H. GOLDSTEIN: *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1980).

effort has been devoted to finding an exact solution of the Green's function for a system with constant magnetic field [7,8] either by using a semi-classical method or by formally using a Feynman path integral approach [9] and this of course leads to the Bloch density matrix [10]. In the present paper we shall treat the problem of finding the propagator for a charged particle from a quantum-mechanical viewpoint in a more general case when the magnetic field is time dependent and in addition an external electric field is present. Also we shall consider the Boson operators for this system and the connection with the Green's function. Our starting point is the Lagrangian for a charged particle of charge e and unit mass in a variable magnetic field $\mathscr{H}(t)$ in the z-direction which has the form

$$L=\frac{1}{2}\left[\sum_{i=1}^{3}(\dot{q}_i^2-\omega_i^2 q_i^2)+\lambda(t)(q_1\dot{q}_2-q_2\dot{q}_1)\right], \tag{1.1}$$

where ω_i $(i=1,2,3)$ are, respectively, the frequencies along the x, y and z directions and $\lambda(t)=e\mathscr{H}(t)/C$ is the time-dependent Larmor frequency. The Hamiltonian for this system is obtained from

$$H(p,q,t)=\sum_i\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i-L(q,\dot{q},t). \tag{1.2}$$

From eqs. (1.1) and (1.2) we find that

$$H(p,q,t)=\frac{1}{2}\sum_{i=1}^{3}(p_i^2+\Omega_i^2(t)q_i^2)-(\lambda(t)/2)(q_1p_2-q_2p_1), \tag{1.3}$$

where

$$\Omega_i(t)=\sqrt{\omega_i^2+\frac{\lambda^2(t)}{4}}.$$

In the presence of the external electric field the Hamiltonian becomes

$$H(p,q,t)=\frac{1}{2}\left[\sum_{i=1}^{3}(p_i^2+\Omega_i^2(t)q_i^2-2E_i(t)q_i)-\lambda(t)(q_1p_2-q_2p_1)\right], \tag{1.4}$$

where $E_i(t)$ is an arbitrary function of the time.

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(7) B. K. Cheng: *J. Phys. A*, **17**, 819 (1984).
(8) I. M. Davies: *J. Phys. A*, **8**, 2737 (1985).
(9) R. P. Feynman and A. R. Hibbs: *Quantum Mechanics and Path Integral* (McGraw Hill, New York, N.Y., 1965).
(10) J. M. Manayan: *J. Phys. A*, **19**, 3013 (1986).

Since we can separate the motion in z-direction, therefore without loss of generality we can drop the terms in this direction, and by restricting ourselves in the present paper to the case $\omega_1 = \omega_2 = \omega$, then eq. (1.4) can be written in the form

$$H(t) = \frac{1}{2}[(p_1^2 + p_2^2) + \Omega^2(t)(q_1^2 + q_2^2)] - \left[\frac{\lambda(t)}{2}(q_1 p_2 - p_1 q_2) + E_1(t)\, q_1 + E_2(t)\, q_2\right]. \tag{1.5}$$

In the absence of the external electric field eq. (1.5) is exactly eq. (92) of ref. ([11]) and eq. (1) of ref. ([12]). Our intention in the following sections of this paper is to give an exact solution for the Heisenberg equations of motion in sect. **2**, followed by the Green's function in sect. **3**. In sect. **4** we use the Boson operators to find the quasi-coherent states and to give an alternative derivation of the Green's function. In sect. **5** we give an exact solution for the Schrödinger equation and the connection to the quasi-coherent states. Finally we give a discussion.

2. – The equations of motion and their solution.

In this section, we shall consider the solution in the Heisenberg picture of quantum mechanics. For the Hamiltonian given by eq. (1.5) we have

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = p_1 + \frac{\lambda(t)}{2} q_2 , \tag{2.1a}$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -\Omega^2(t)\, q_1 + \frac{\lambda(t)}{2} p_2 + E_1(t) , \tag{2.1b}$$

$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = p_2 - \frac{\lambda(t)}{2} q_1 , \tag{2.1c}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = -\Omega^2(t)\, q_2 - \frac{\lambda(t)}{2} p_1 + E_2(t) . \tag{2.1d}$$

In order to solve eqs. (2.1*a*-*d*) we introduce the transformation

$$\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \cos f(t) & \sin f(t) & 0 & 0 \\ -\sin f(t) & \cos f(t) & 0 & 0 \\ 0 & 0 & \cos f(t) & \sin f(t) \\ 0 & 0 & -\sin f(t) & \cos f(t) \end{bmatrix} \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix},$$

([11]) H. R. Lewis and W. B. Riesenfeld: *J. Math. Phys.* (*N.Y.*), **10**, 1458 (1969).
([12]) M. S. Abdalla: *Phys. Rev. A*, **37**, 4026 (1988).

where

$$f(t) = \frac{1}{2}\int_0^t \lambda(\dot{t})\,\mathrm{d}\dot{t}. \tag{2.2}$$

From eqs. (2.1) and (2.2) we find

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p_x, \tag{2.3a}$$

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -\Omega^2(t)\,x + E_x(t), \tag{2.3b}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = p_y, \tag{2.3c}$$

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = -\Omega^2(t)\,y + E_y(t). \tag{2.3d}$$

$E_x(t)$ and $E_y(t)$ are

$$E_x(t) = E_1(t)\cos f(t) - E_2(t)\sin f(t), \tag{2.4a}$$

$$E_y(t) = E_2(t)\cos f(t) + E_1(t)\sin f(t). \tag{2.4b}$$

Following the technique used in ref. ([13]) and after some minor algebra the result in physical coordinates and momenta is

$$q_1(t) = r(t)[q_1(0)\cos f(t) + q_2(0)\sin f(t)] + \\ + s(t)[p_1(0)\cos f(t) + p_2(0)\sin f(t)] + [\bar{\xi}_x(t)\cos f(t) + \bar{\xi}_y(t)\sin f(t)], \tag{2.5a}$$

$$q_2(t) = r(t)[q_2(0)\cos f(t) - q_1(0)\sin f(t)] + \\ + s(t)[p_2(0)\cos f(t) - p_1(0)\sin f(t)] + [\bar{\xi}_y(t)\cos f(t) - \bar{\xi}_x(t)\sin f(t)], \tag{2.5b}$$

$$p_1(t) = v(t)[p_1(0)\cos f(t) + p_2(0)\sin f(t)] + \\ + w(t)[q_1(0)\cos f(t) + q_2(0)\sin f(t)] + [\dot{\bar{\xi}}_x(t)\cos f(t) + \dot{\bar{\xi}}_y(t)\sin f(t)], \tag{2.5c}$$

$$p_2(t) = v(t)[p_2(0)\cos f(t) - p_1(0)\sin f(t)] + \\ + w(t)[q_2(0)\cos f(t) - q_1(0)\sin f(t)] + [\dot{\bar{\xi}}_y(t)\cos f(t) - \dot{\bar{\xi}}_x(t)\sin f(t)], \tag{2.5d}$$

([13]) M. S. ABDALLA: *Phys. Rev. A*, 34, 4598 (1986).

where

$$r(t)=\sqrt{\frac{\mu(0)}{\mu(t)}}\cos\eta(t)+\frac{\dot\mu(0)/2\mu(0)}{\sqrt{\mu(t)\mu(0)}}\sin\eta(t)\,, \tag{2.6a}$$

$$s(t)=\frac{\sin\eta(t)}{\sqrt{\mu(t)\mu(0)}}\,, \tag{2.6b}$$

$$v(t)=\sqrt{\frac{\mu(t)}{\mu(0)}}\cos\eta(t)-\frac{\dot\mu(t)/2\mu(t)}{\sqrt{\mu(t)\mu(0)}}\sin\eta(t) \tag{2.6c}$$

and

$$w(t)=\left[\sqrt{\frac{\mu(t)}{\mu(0)}}\left(\frac{\dot\mu(0)}{2\mu(0)}\right)-\sqrt{\frac{\mu(0)}{\mu(t)}}\left(\frac{\dot\mu(t)}{2\mu(t)}\right)\right]\cos\eta(t)-$$
$$-\sqrt{\mu(t)\mu(0)}\left[1+\frac{(\dot\mu(t)/2\mu(t))(\dot\mu(0)/2\mu(0))}{\mu(t)\mu(0)}\right]\sin\eta(t)\,, \tag{2.6d}$$

while

$$\eta(t)=\int_0^t\mu(\dot t)\,\mathrm{d}\dot t\,, \tag{2.7a}$$

$$\sqrt{\mu(t)}\,\bar\xi_x(t)=\xi_x(t)=[(I_x^c(t)-I_x^c(0))\sin\eta(t)-(I_x^s(t)-I_x^s(0))\cos\eta(t)]\,, \tag{2.7b}$$

$$I_x^c(t)=\int\frac{E_x(t)}{\sqrt{\mu(t)}}\cos\eta(t)\,\mathrm{d}t\,, \tag{2.7c}$$

$$I_x^s(t)=\int\frac{E_x(t)}{\sqrt{\mu(t)}}\sin\eta(t)\,\mathrm{d}t \tag{2.7d}$$

and $\mu(t)$ is given by

$$\mu(t)=\frac{1}{\rho^2(t)},\qquad \ddot\rho(t)+\Omega^2(t)\rho(t)=\frac{1}{\rho^3(t)}. \tag{2.7e}$$

Equation (2.7*e*) is a nonlinear differential equation and has a solution presented in ref. ([14]), see also ref. ([12]) for the physical interpretation. From eqs. (2.5) it is easy to check that the commutation relations $[q_i, p_j]=i\hbar\delta_{ij}$ hold.

([14]) C. J. ELIEZER and A. GRAY: *Siam J. Appl. Math.*, **30**, 463 (1976).

Now let us define the following pairs of operators A and B such that

$$A(t) = (2\hbar\mu(t))^{-1/2}\left[\left(\mu(t) + i\frac{\dot{\mu}(t)}{2\mu(t)}\right)(q_1\cos f(t) - q_2\sin f(t)) + \right. \qquad (2.8a)$$

$$\left. + i(p_1\cos f(t) - p_2\sin f(t)) - K_x(t)\right],$$

$$K_x(t) = \overline{\xi}_x(t)\left(\mu(t) + i\frac{\dot{\xi}_x(t)}{\xi_x(t)}\right), \qquad (2.8b)$$

$$B(t) = (2\hbar\mu(t))^{-1/2}\left[\left(\mu(t) + i\frac{\dot{\mu}(t)}{2\mu(t)}\right)(q_2\cos f(t) + q_1\sin f(t)) + \right. \qquad (2.9a)$$

$$\left. + i(p_2\cos f(t) + p_1\sin f(t)) - K_y(t)\right],$$

$$K_y(t) = \overline{\xi}_y(t)\left(\mu(t) + i\frac{\dot{\xi}_y(t)}{\xi_y(t)}\right), \qquad (2.9b)$$

since these operators and their adjoints satisfy the canonical relations

$$[A, A^\dagger] = 1 = [B, B^\dagger]. \qquad (2.10)$$

Then

$$\begin{cases} A^\dagger A|n(t),\, m(t)\rangle = n\,|n(t),\, m(t)\rangle\,, & n = 0, 1, 2, \ldots, \\ B^\dagger B|n(t),\, m(t)\rangle = m\,|n(t),\, m(t)\rangle\,, & m = 0, 1, 2, \ldots. \end{cases} \qquad (2.11)$$

By using eqs. (2.5a-d), we may write the expressions for the expectation values of the potential energy $V(t)$, the kinetic energy $T(t)$ and the Hamiltonian $H(t)$ with respect to the states $|n, m\rangle$. Thus

$$\langle V(t)\rangle = \frac{\hbar}{2}(n + m + 1)(\Omega^2(t)/\mu(t)) + \frac{1}{2}(\overline{\xi}_x^2(t) + \overline{\xi}_y^2(t))\Omega^2(t)\,, \qquad (2.12a)$$

$$\langle T(t)\rangle = \frac{\hbar}{2}(n + m + 1)\mu^{-1}(t)\left[\mu^2(t) + \left(\frac{\dot{\mu}(t)}{2\mu(t)}\right)^2\right] + \frac{1}{2}(\dot{\overline{\xi}}_x^2(t) + \dot{\overline{\xi}}_y^2(t))\,, \qquad (2.12b)$$

$$\langle H(t)\rangle = \frac{\hbar}{2}(n + m + 1)\mu^{-1}(t)\left[\mu^2 + \frac{d}{dt}\left(\frac{\dot{\mu}(t)}{2\mu(t)}\right)\right] + F(t)\,, \qquad (2.12c)$$

where

$$F(t)=\frac{1}{2}[\dot{\bar{\xi}}_x^2(t)+\dot{\bar{\xi}}_y^2(t)+(\Omega^2(t)\,\bar{\xi}_y(t)+\lambda(t)\,\dot{\bar{\xi}}_x(t))\,\bar{\xi}_y(t)+(\Omega^2(t)\,\bar{\xi}_x(t)-\lambda(t)\,\dot{\bar{\xi}}_y(t))\,\bar{\xi}_x(t)]\,.$$

3. – Green's function.

Having solved the problem in the Heisenberg picture, we are in position to construct the Green's function and the corresponding Bloch density matrix for the system given by Hamiltonian (1.5).

From eqs. (2.5*a*-*d*) and by using the transformation in eq. (2.2) we obtain

$$x(t)=r(t)\,x(0)+s(t)\,p_x(0)+\bar{\xi}_x(t)\,, \tag{3.1a}$$

$$p_x(t)=v(t)\,p_x(0)+w(t)\,x(0)+\dot{\bar{\xi}}_x(t)\,. \tag{3.1b}$$

Following the technique presented in ref. ([13,15,16]) we find that

$$G(x,x_0,t)=(2\pi\hbar|s(t)|)^{-1/2}\cdot \tag{3.2a}$$

$$\cdot\exp\left[\frac{i}{2\hbar s(t)}[r(t)\,x_0^2+v(t)\,x^2-2xx_0+2\bar{\xi}_x(t)\,x_0+2(s(t)\,\dot{\bar{\xi}}_x(t)-\bar{\xi}_x(t)\,v(t))\,x]\right],$$

where x_0 is given by the equation

$$x(0)\,\delta(x-x_0)=x_0\,\delta(x-x_0)\,. \tag{3.2b}$$

Similarly we can show that

$$G(y,y_0,t)=(2\pi\hbar|s(t)|)^{-1/2}\cdot \tag{3.2c}$$

$$\cdot\exp\left[\frac{i}{2\hbar s(t)}[r(t)\,y_0^2+v(t)\,y^2-2yy_0+2\bar{\xi}_y(t)\,y_0+2(s(t)\,\dot{\bar{\xi}}_y(t)-\bar{\xi}_y(t)\,v(t))\,y]\right].$$

([15]) L. F. LANDOVITZ, A. M. LEVINE, E. OZIZMIR and W. M. SCHREIBER: *J. Chem. Phys*, 78, 291 (1983).

([16]) M. SARGENT, M. O. SCULLY and W. E. LAMB: *Laser Physics* (Addison-Wisley, Reading, Mass., 1974).

Then from eqs. (3.2a) and (3.2c), the Green's function has the form

$$(3.3a)\qquad G(q_1, q_2, \overline{q}_1, \overline{q}_2, t) = (2\hbar\pi|s(t)|)^{-1}\cdot$$

$$\cdot \exp\Bigg[\frac{i}{2\hbar s(t)}\{r(t)(\overline{q}_1^2+\overline{q}_2^2)+v(t)(q_1^2+q_2^2)-2[(\overline{q}_1 q_1+\overline{q}_2 q_2)\cos f(t)+$$

$$+(q_1\overline{q}_2-\overline{q}_1 q_2)\sin f(t)]+2[\overline{q}_1(\overline{\xi}_x(t)\cos f(t)+\overline{\xi}_y(t)\sin f(t))+$$

$$+\overline{q}_2(\overline{\xi}_y(t)\cos f(t)-\overline{\xi}_x(t)\sin f(t))]+$$

$$+2[q_1((s(t)\dot{\overline{\xi}}_x(t)-v(t)\overline{\xi}_x(t))\cos f(t)+(s(t)\dot{\overline{\xi}}_y(t)-v(t)\overline{\xi}_y(t))\sin f(t))+$$

$$+q_2((s(t)\dot{\overline{\xi}}_y(t)-v(t)\overline{\xi}_y(t))\cos f(t)-(s(t)\dot{\overline{\xi}}_x(t)-v(t)\overline{\xi}_x(t))\sin f(t))]\}\Bigg],$$

where

$$(3.3b)\qquad \overline{q}_1 = x_0\cos f(t) + y_0\sin f(t),$$

$$(3.3c)\qquad \overline{q}_2 = y_0\cos f(t) - x_0\sin f(t).$$

For the case when the magnetic and the electric fields are constant so that λ, E_1 and E_2 are constant eq. (3.3a) assumes the form

$$(3.4a)\qquad G(q_1, q_2, \overline{q}_1, \overline{q}_2, t) = \left(\frac{\Omega}{2\hbar\pi|\sin(\Omega t)|}\right)\cdot$$

$$\cdot\exp\left[\frac{i\Omega}{2\hbar\sin\Omega t}\left[(\overline{q}_1^2+\overline{q}_2^2+q_1^2+q_2^2)\cos(\Omega t)-\right.\right.$$

$$\left.\left.-2\left[(\overline{q}_1 q_1+\overline{q}_2 q_2)\cos\left(\frac{\lambda}{2}t\right)+(q_1\overline{q}_2-\overline{q}_1 q_2)\sin\left(\frac{\lambda}{2}t\right)\right]\right]\right]\cdot$$

$$\cdot\exp\left[\frac{i\Omega}{\hbar\sin\Omega t}(l(t)\overline{q}_1+h(t)\overline{q}_2+J(t)q_1+k(t)q_2)\right],$$

$$(3.4b)\qquad l(t) = \frac{E}{\omega^2}\left[\cos\theta-\frac{\lambda/2}{\Omega}\sin(\Omega t)\sin(\lambda/2\,t-\theta)-\cos(\Omega t)\cos(\lambda/2\,t-\theta)\right],$$

$$(3.4c)\qquad h(t) = \frac{E}{\omega^2}\left[\sin\theta-\frac{\lambda/2}{\Omega}\sin(\Omega t)\cos(\lambda/2\,t-\theta)+\cos(\Omega t)\sin(\lambda/2\,t-\theta)\right],$$

(3.4*d*) $$J(t) = -\frac{E}{\omega^2}\left[\frac{\lambda/2}{\Omega}\sin\theta\sin(\Omega t) + \cos(\Omega t)\cos\theta - \cos(\lambda/2\,t - \theta)\right],$$

(3.4*e*) $$k(t) = \frac{E}{\omega^2}\left[\frac{\lambda/2}{\Omega}\cos\theta\sin(\Omega t) - \cos(\Omega t)\sin\theta - \sin(\lambda/2\,t - \theta)\right]$$

and

(3.4*f*) $$E = (E_1^2 + E_2^2)^{1/2} \quad , \quad \theta = \mathrm{tg}^{-1}(E_2/E_1)\,.$$

To find the Bloch density matrix we just replace t by $(-i\hbar\bar{\beta})$ in eq. (3.4*a*). Thus

(3.5) $$C(q_1, q_2, \bar{q}_1, \bar{q}_2, \bar{\beta}) = [\Omega/2\hbar\pi \sinh(\bar{\beta}\hbar\Omega)]\cdot$$
$$\cdot\exp\left[\left[-\frac{\Omega}{2\hbar\sinh(\bar{\beta}\hbar\Omega)}\right]\left[(\bar{q}_1^2 + \bar{q}_2^2 + q_1^2 + q_2^2)\cosh(\bar{\beta}\hbar\Omega) - \right.\right.$$
$$\left.\left. -2\left((\bar{q}_1 q_1 + \bar{q}_2 q_2)\cosh\left(\bar{\beta}\frac{\hbar}{2}\lambda\right) - i(q_1\bar{q}_2 - \bar{q}_1 q_2)\sinh\left(\bar{\beta}\frac{\hbar}{2}\lambda\right)\right)\right]\right]\cdot$$
$$\cdot\exp\left[-\frac{\Omega}{\hbar\sinh(\bar{\beta}\hbar\Omega)}(l(\bar{\beta})\bar{q}_1 + h(\bar{\beta})\bar{q}_2 + J(\bar{\beta})q_1 + k(\bar{\beta})q_2)\right].$$

In order to find the average energy for this system, we must calculate first the partition function $z(\bar{\beta})$ such that

(3.6) $$z(\bar{\beta}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(q_1, q_2, \bar{\beta})\,\mathrm{d}q_1\,\mathrm{d}q_2\,.$$

By using eq. (3.5) we obtain

(3.7*a*) $$z(\bar{\beta}) = \frac{1}{4}\Gamma^{-1}(\beta)\exp\left[\frac{4E^2}{\hbar^3\omega^4\Omega}\mathrm{tgh}\left(\frac{\bar{\beta}\hbar\Omega}{2}\right)(\chi^2(\bar{\beta})\Gamma^{-1}(\bar{\beta}))\right],$$

where

(3.7*b*) $$\Gamma(\bar{\beta}) = \sinh\left[\frac{\bar{\beta}\hbar}{2}(\Omega + \lambda/2)\right]\sinh\left[\frac{\bar{\beta}\hbar}{2}(\Omega - \lambda/2)\right],$$

(3.7*c*) $$\chi(\bar{\beta}) = \frac{\partial}{\partial\bar{\beta}}\left[\cosh\left(\frac{\bar{\beta}\hbar\Omega}{2}\right)\cosh\left(\frac{\bar{\beta}\hbar\lambda}{2}\right)\right].$$

In the absence of an external electric field, we get agreement with eq. (5) of ref. (10). Since the average energy can be given from the equation (17)

$$\langle E\rangle = -\frac{\partial}{\partial\bar{\beta}} \ln[z(\bar{\beta})], \tag{3.8}$$

we find that

$$\langle E\rangle = \dot{\Gamma}(\bar{\beta})\Gamma(\bar{\beta})^{-1}(1+\sigma(\bar{\beta})) - 2\sigma(\bar{\beta})\left[\frac{\partial}{\partial\bar{\beta}}\ln\chi(\bar{\beta}) + \frac{\hbar\Omega}{4}\operatorname{cosech}(\bar{\beta}\hbar\Omega)\right], \tag{3.9a}$$

$$\sigma(\bar{\beta}) = \ln[4z(\bar{\beta})\Gamma(\bar{\beta})]. \tag{3.9b}$$

4. – The quasi-coherent states and their relation to the Green's function.

In this section we shall introduce an alternative method for finding the Green's function by first deriving the quasi-coherent states. From the Boson operators, given by eqs. (2.8a) and (2.9a), and since these operators and their adjoints satisfy the canonical relations given by eq. (2.10) on using the Heisenberg equations (2.5) it easily follows that

$$A(t) = A(0)\exp[-i\eta(t)] \quad \text{and} \quad B(t) = B(0)\exp[-i\eta(t)]. \tag{4.1}$$

To derive the quasi-coherent states, let α and β be the eigenvalues for the operators $A(t)$ and $B(t)$, respectively, such that

$$A(t)|\alpha(t), \beta(t)\rangle = \alpha(t)|\alpha(t), \beta(t)\rangle, \tag{4.2a}$$

$$B(t)|\alpha(t), \beta(t)\rangle = \beta(t)|\alpha(t), \beta(t)\rangle. \tag{4.2b}$$

Therefore, from eqs. (2.8a) and (2.9a) after some calculations we find

$$\psi_\alpha(x,t) = \left(\frac{\mu(t)}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{1}{2}(\alpha(t)^2+|\alpha|^2)\right]\cdot \\ \cdot\exp\left[-\frac{1}{2\hbar}\left(\mu(t)+i\frac{\dot{\mu}(t)}{2\mu(t)}\right)(x-\bar{\xi}_x(t))^2+\alpha(t)\sqrt{\frac{2\mu(t)}{\hbar}}(x-\bar{\xi}_x(t))\right]\cdot \\ \cdot\exp\left[\frac{i}{\hbar}[(x-\bar{\xi}_x(t))\dot{\bar{\xi}}_x(t)+\delta_x(t)]\right], \tag{4.3a}$$

(17) W. H. LOUISELL: *Quantum Statistical Properties of Radiation* (John Wiley & Sons Inc., New York, N.Y., 1973).

where x is given from eq. (2.2) and $\delta_x(t)$ is an arbitrary time-dependent phase factor adjusted to have the form

$$\delta_x(t) = \int_0^t \left[E_x(\grave{t})\,\bar{\xi}_x(\grave{t}) - \frac{1}{2}(\Omega^2(\grave{t})\,\bar{\xi}_x^2(\grave{t}) - \dot{\bar{\xi}}_x^2(\grave{t})) \right] \mathrm{d}\grave{t}\,, \tag{4.3b}$$

similarly for the operator B we just replace χ and α by y and β, respectively, in eqs. (4.3*a*, *b*). Then the wave function in a coherent state is

$$\psi_{\alpha\beta}(q_1, q_2, t) = \left(\frac{\mu(t)}{\pi\hbar}\right)^{1/2} \exp\left[-\frac{1}{2}(\alpha(t)^2 + \beta(t)^2 + |\alpha|^2 + |\beta|^2)\right] \cdot \tag{4.4a}$$

$$\cdot \exp\left[-\frac{1}{2\hbar}\left(\mu(t) + i\frac{\dot{\mu}(t)}{2\mu(t)}\right)[(q_1^2 + q_2^2) + (\bar{\xi}_x^2(t) + \bar{\xi}_y^2(t)) - \right.$$

$$\left. - 2q_1(\bar{\xi}_x(t)\,\cos f(t) + \bar{\xi}_y(t)\,\sin f(t)) - 2q_2(\bar{\xi}_y(t)\,\cos f(t) - \bar{\xi}_x(t)\,\sin f(t))]\right] \cdot$$

$$\cdot \exp\left[\frac{i}{\hbar}[q_1(\dot{\bar{\xi}}_x(t)\,\cos f(t) + \dot{\bar{\xi}}_y(t)\,\sin f(t)) + q_2(\dot{\bar{\xi}}_y(t)\,\cos f(t) - \dot{\bar{\xi}}_x(t)\,\sin f(t))]\right] \cdot$$

$$\cdot \exp\left[\sqrt{\frac{2\mu(t)}{\hbar}}[q_1(\alpha(t)\,\cos f(t) + \beta(t)\,\sin f(t)) + q_2(\beta(t)\,\cos f(t) - \alpha(t)\,\sin f(t))]\right] \cdot$$

$$\cdot \exp\left[-\sqrt{\frac{2\mu(t)}{\hbar}}(\alpha(t)\,\bar{\xi}_x(t) + \beta(t)\,\bar{\xi}_y(t)) + \frac{i}{\hbar}\delta(t)\right],$$

where

$$\delta(t) = \int_0^t [(E_x(\grave{t})\,\bar{\xi}_x(\grave{t}) + E_y(\grave{t})\,\bar{\xi}_y(\grave{t})) + \frac{1}{2}(\dot{\bar{\xi}}_x^2(\grave{t}) + \dot{\bar{\xi}}_y^2(\grave{t})) - \frac{1}{2}L(\bar{\xi}_x^2(\grave{t}) + \bar{\xi}_y^2(\grave{t}))]\,\mathrm{d}\grave{t}\,, \tag{4.4b}$$

L is the operator $(\mathrm{d}^2/\mathrm{d}t^2 + \Omega^2(t))$ and

$$\alpha(t) = \alpha(0)\,\exp[-i\eta(t)], \qquad \beta(t) = \beta(0)\,\exp[-i\eta(t)]\,. \tag{4.4c}$$

The coherent states are not orthogonal but form an overcomplete system of states. Therefore, we may calculate the Green's function using the following relation:

$$G(x, x_0, t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_\alpha(x, t)\,\psi_\alpha^*(x_0, 0)\,\mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha)\,. \tag{4.5}$$

Substituting eq. (4.3*a*) into (4.5) we have

$$G(x, x_0, t) = \frac{1}{\pi}\left[\frac{\mu(t)\mu(0)}{(\pi\hbar)^2}\right]^{1/4} \exp\left[-\frac{1}{2\hbar}\left(\mu(t) + i\frac{\dot{\mu}(t)}{2\mu(t)}\right)(x - \bar{\xi}_x(t))^2\right] \cdot \tag{4.6}$$

$$\cdot \exp\left[-\frac{1}{2\hbar}\left(\mu(0) - i\frac{\dot{\mu}(0)}{2\mu(0)}\right)x_0^2\right] \exp\left[\frac{i}{\hbar}[(x - \bar{\xi}_x(t))\dot{\bar{\xi}}_x(t) + \delta_x(t)]\right] \cdot$$

$$\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(\alpha^2(0)\exp[-2i\eta(t)] + \alpha^{*2}(0)) - |\alpha|^2\right] \cdot$$

$$\cdot \exp\left[\alpha(0)\exp[-i\eta(t)]\sqrt{\frac{2\mu(t)}{\hbar}}(x - \bar{\xi}_x(t)) + \alpha^*(0)\sqrt{\frac{2\mu(0)}{\hbar}}x_0\right] \mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha).$$

The integral in eq. (4.6) can be evaluated as follows. Let

$$\alpha(0) = a + ib, \qquad \alpha^*(0) = a - ib.$$

Thus

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}[a^2(3 + \exp[-2i\eta]) - 2a(\gamma + \gamma_0)]\right] \cdot \tag{4.7a}$$

$$\cdot \exp\left[-\frac{1}{2}[b^2(1 - \exp[-2i\eta]) - 2ib(a(\exp[-2i\eta] - 1) + (\gamma - \gamma_0))]\right] \mathrm{d}a\,\mathrm{d}b,$$

where

$$\gamma = \exp[-i\eta]\sqrt{\frac{2\mu(t)}{\hbar}}(x - \bar{\xi}_x(t)), \qquad \gamma_0 = \sqrt{\frac{2\mu(0)}{\hbar}}x_0. \tag{4.7b}$$

By using the identity

$$\int_{-\infty}^{\infty} \exp[-\tau^2]\,\mathrm{d}\tau = \sqrt{\pi}, \tag{4.8}$$

then the result of integration is

$$I = \frac{\pi}{\sqrt{2\sin\eta(t)}} \exp\left[-\frac{1}{2}\left[\frac{\gamma_0^2\exp[-2i\eta] - 2\gamma_0\gamma + \gamma^2}{1 - \exp[-2i\eta]}\right]\right]; \tag{4.9}$$

substituting eq. (4.9) into eq. (4.6) we find

$$(4.10)\qquad G(x, x_0, t) = (2\pi\hbar s(t))^{-1/2}\cdot$$

$$\cdot\exp\left[\frac{i}{2\hbar s(t)}\left[r(t)\,x_0^2 + v(t)\,x^2 - 2xx_0 + 2\bar{\xi}_x(t)\,x_0 + 2(s(t)\,\dot{\bar{\xi}}_x(t) - v(t)\,\bar{\xi}_x(t))\,x\right]\right].$$

It is easy to realize that eq. (4.10) is exactly eq. (3.2*a*). In the previous calculations we have set

$$(4.11)\qquad \delta_x(t) = \bar{\xi}_x(t)\,\dot{\bar{\xi}}_x(t) + \frac{1}{2}\left(\frac{\dot{\mu}(t)}{2\mu(t)} - \mu(t)\,\operatorname{ctg}\eta(t)\right)\bar{\xi}_x^2(t)$$

and to establish this one needs to evaluate the integral in eq. (4.3*b*) which can be done with aid of eq. (2.7*b*). Similarly we can find $G(y,\, y_0, t)$ and this leads to the Green's function (3.3*a*).

5. – The wave function for pseudostationary states.

In this section we shall consider the solution in the Schrödinger picture to find the pseudostationary states. The wave function in Schrödinger representation is given by

$$(5.1)\qquad H(t)|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

from eq. (1.5) and (5.1). Thus

$$(5.2)\qquad \frac{\partial^2\psi}{\partial q_1^2} + \frac{\partial^2\psi}{\partial q_2^2} - \frac{\Omega^2(t)}{\hbar^2}(q_1^2 + q_2^2)\,\psi - i\frac{\lambda(t)}{\hbar}\left(q_1\frac{\partial\psi}{\partial q_2} - q_2\frac{\partial\psi}{\partial q_1}\right) +$$

$$+\frac{2}{\hbar^2}(E_1(t)\,q_1 + E_2(t)\,q_2)\,\psi = \frac{-2i}{\hbar}\frac{\partial\psi}{\partial t}.$$

In order to find the solution of eq. (5.2) we shall use the transformation given by eq. (2.2), therefore we need

$$(5.3)\qquad \psi(q_1, q_2, t) \equiv \phi(x, y, t)\,.$$

Thus

$$(5.4a)\qquad \frac{\partial\psi}{\partial q_1} = \frac{\partial\phi}{\partial x}\cos f(t) + \frac{\partial\phi}{\partial y}\sin f(t)\,,$$

(5.4b)
$$\frac{\partial\psi}{\partial q_2}=\frac{\partial\phi}{\partial y}\cos f(t)-\frac{\partial\phi}{\partial x}\sin f(t)\,,$$

(5.4c)
$$\frac{\partial\psi}{\partial t}=\frac{\partial\phi}{\partial t}+\dot f(t)\left(x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right).$$

Substituting eqs. (2.2), (5.3) and (5.4) into eq. (5.2), we have

(5.5)
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\frac{\Omega^2(t)}{\hbar^2}(x^2+y^2)\,\phi+\frac{2}{\hbar^2}(xE_x+yE_y)\,\phi=\frac{-2i}{\hbar}\frac{\partial\phi}{\partial t}\,,$$

where E_x and E_y are given in eqs. (2.4a), (2.4b).

Now let

(5.6)
$$\phi(x,y,t)=\phi_1(x,t)\,\phi_2(y,t)\,.$$

Then eq. (5.5) can be reduced to

(5.7a)
$$\frac{\partial^2\phi_1}{\partial x^2}-\frac{\Omega^2(t)}{\hbar^2}x^2\,\phi_1+2x\frac{E_x}{\hbar^2}\phi_1=\frac{-2i}{\hbar}\;\frac{\partial\phi_1}{\partial t}+\nu(t)\,\phi_1$$

and

(5.7b)
$$\frac{\partial^2\phi_2}{\partial y^2}-\frac{\Omega^2(t)}{\hbar^2}y^2\,\phi_2+2y\frac{E_y}{\hbar^2}\phi_2=\frac{-2i}{\hbar}\,\frac{\partial\phi_2}{\partial t}-\nu(t)\,\phi_2\,,$$

where $\nu(t)$ is a constant of separation which depends on time. To obtain a solution of eq. (5.7a) we shall introduce the following transformation:

(5.8)
$$x=\frac{u}{\sqrt{\mu(t)}}+\bar\xi_x(t)\,,$$

$\bar\xi_x(t)$ and $\mu(t)$ are given by eqs. (2.7b) and (2.7e), respectively. From eqs. (5.8) and (5.7a) we find

(5.9)
$$\frac{\partial^2\bar\phi_1}{\partial u^2}+\frac{2i}{\hbar}\left[\frac{\dot\mu(t)}{2\mu^2(t)}(u+\xi_x(t))-\frac{\dot\xi_x(t)}{\mu(t)}\right]\frac{\partial^2\bar\phi_1}{\partial u}-\frac{\Omega^2(t)}{\hbar^2\mu(t)}[u^2+2\xi_x(t)\,u]\,\bar\phi_1+$$
$$+2\frac{E_x}{\hbar^2\mu^{3/2}(t)}(u+\xi_x(t))\,\bar\phi_1-\frac{\Omega^2(t)}{\hbar^2\mu^2(t)}\xi_x^2(t)\,\bar\phi_1=\frac{-2i}{\hbar\mu(t)}\,\frac{\partial\bar\phi_1}{\partial t}+\frac{\nu(t)}{\mu(t)}\bar\phi_1$$

with $\bar\phi_1(u,t)\equiv\phi_1(x,t)$.

By using eqs. (2.7*b*) and (2.7*e*) we can write the general solution of eq. (5.9) as follows:

$$(5.10)\qquad \overline{\phi}_{1n}(u,t)=\left(\frac{\mu(t)}{\pi\hbar}\right)^{1/4}2^{-n/2}(n!)^{-1/2}H_n\left(\frac{u}{\sqrt{\hbar}}\right)\exp\left[-\frac{1}{2}u^2/\hbar\right]\cdot$$

$$\cdot\exp\left[\frac{-i}{\hbar}\left(\frac{\dot{\mu}(t)}{4\mu^2(t)}u^2-\frac{\dot{\overline{\xi}}_x}{\sqrt{\mu(t)}}u\right)\right]\exp\left[-i\left(n+\frac{1}{2}\right)\eta(t)\right]\cdot$$

$$\cdot\exp\left[\frac{i}{\hbar}\int_0^t\left[E_x(\acute{t})\,\overline{\xi}_x(\acute{t})-\frac{1}{2}(\Omega^2(\acute{t})\,\overline{\xi}_x^2(\acute{t})-\dot{\overline{\xi}}_x^2(\acute{t}))-\frac{\hbar^2\nu(\acute{t})}{2}\right]\mathrm{d}\acute{t}\right].$$

Then the solution of eq. (5.7*a*) is of the form

$$(5.11)\qquad \phi_{1n}(x,t)=\left(\frac{\mu(t)}{\hbar\pi}\right)^{1/4}2^{-n/2}(n!)^{-1/2}H_n\left[\sqrt{\frac{\mu(t)}{\hbar}}(x-\overline{\xi}_x(t))\right]\cdot$$

$$\cdot\exp\left[-\frac{1}{2\hbar}\left(\mu(t)+i\frac{\dot{\mu}(t)}{2\mu(t)}\right)(x-\overline{\xi}_x(t))^2+\frac{i}{\hbar}(x-\overline{\xi}_x(t))\,\dot{\overline{\xi}}_x(t)\right]\cdot$$

$$\cdot\exp\left[-i\left(n+\frac{1}{2}\right)\eta(t)+\frac{i}{\hbar}\left(\delta_x(t)-\frac{\hbar^2}{2}\int_0^t\nu(\acute{t})\,\mathrm{d}\acute{t}\right)\right],$$

$\delta_x(t)$ is given by eq. (4.3*b*).

Solution of eq. (5.7*b*) can be obtained directly from eq. (5.11) by replacing x, n and ν by y, m and $-\nu$, respectively. Then, on reverting to the physical coordinates, the solution of eq. (5.2) becomes

$$(5.12)\qquad \psi_{nm}(q_1,q_2,t)=\left(\frac{\mu(t)}{\pi\hbar}\right)^{1/2}2^{-(n+m)/2}(n!\,m!)^{-1/2}\cdot$$

$$\cdot H_n\left[\sqrt{\frac{\mu(t)}{\hbar}}(q_1\cos f(t)-q_2\sin f(t)-\overline{\xi}_x(t))\right]\cdot$$

$$\cdot H_m\left[\sqrt{\frac{\mu(t)}{\hbar}}(q_2\cos f(t)+q_1\sin f(t)-\overline{\xi}_y(t))\right]\cdot$$

$$\cdot\exp\left[-\frac{1}{2\hbar}\left(\mu(t)+i\frac{\dot{\mu}(t)}{2\mu(t)}\right)[(q_1^2+q_2^2)+(\overline{\xi}_x^2(t)+\overline{\xi}_y^2(t))-\right.$$

$$- 2q_1(\bar{\xi}_x(t)\cos f(t) + \bar{\xi}_y(t)\sin(t)) - 2q_2(\bar{\xi}_y(t)\cos f(t) - \bar{\xi}_x(t)\sin f(t))]\bigg]\cdot$$

$$\cdot\exp\left[\frac{i}{\hbar}[q_1(\dot{\bar{\xi}}_x(t)\cos f(t) + \dot{\bar{\xi}}_y(t)\sin f(t)) + q_2(\dot{\bar{\xi}}_y(t)\cos f(t) - \dot{\bar{\xi}}_x(t)\sin f(t))]\right]\cdot$$

$$\cdot\exp\left[-i(n+m+1)\,\eta(t) + \frac{i}{\hbar}\delta(t)\right].$$

For more details of the technique which we have used see ref. ([1,13]). To find the connection between the wave function in Schrödinger representation and the quasi-coherent states, one can expand eq. (4.4*a*) in a power series of α and β, and then use eq. (5.12). Thus

$$\psi_{\alpha\beta}(q_1, q_2, t) = \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2)\right]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^n(t)\,\beta^m(t)}{\sqrt{n!\,m!}}\,\psi_{nm}(q_1, q_2, t)\,. \tag{5.13}$$

Equation (5.13) can be compared with eq. (4.2) of ref. ([1]) and eq. (12) of ref. ([18]).

6. – Discussion.

In this paper we have considered the problem of a charged particle in a variable magnetic field together with an external electric field. The problem has been reduced to uncoupled form with the same variable frequencies by means of the canonical transformation (2.2). This leads to a situation in which we can use the technique presented in ref. ([1,12]) for finding the solution of the Heisenberg equations of motion. Also we have calculated the Green's function by using two different methods. Our first method was to use the solution in Heisenberg picture and then apply the technique in ref. ([12,15]). Our second method was to construct two pairs of Boson operators leading to coherent states, and then by using the properties of these states we calculated the Green's function. We should like to point out that these operators can be obtained from the wave function in the Schrödinger representation by differentiating the exponential term in eq. (5.12) and adjusting the result of the differentiation to obtain the accurate result (see ref. ([12]) for more details). Alternatively, we can proceed by finding a quadratic invariant for the Hamiltonian (1.5), as described in ref. ([11]).

Also these operators can be obtained from a suitable linear combination of the solutions in the Heisenberg picture equation (2.5). Finally we have presented the exact solution in the Schrödinger representation and connected it up with the

([18]) V. V. Dodonov and V. I. Man'Ko: *Phys. Rev. A*, **20**, 550 (1979).

quasi-coherent states to show the correctness of our calculations. We regard this work as an extension of that presented in ref. ([13]).

* * *

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● RIASSUNTO (*)

Si presenta un nuovo procedimento per l'evoluzione nel tempo di un oscillatore carico fissato sotto l'azione combinata di campi elettrici e magnetici specificati arbitrariamente. Le equazioni di moto di Heisenberg sono state risolte e gli stati quasi coerenti sono utilizzati per calcolare la funzione di Green. Si esamina anche la connessione tra le funzioni d'onda di Schrödinger di stati pseudostazionari e quasi coerenti. La matrice di densità di Bloch è calcolata in un caso speciale.

(*) *Traduzione a cura della Redazione.*

Заряженный гармонический осциллятор в присутствии электрических и магнитных полей.

Резюме (*). — Предлагается новое рассмотрение временной эволюции фиксированного заряженного осциллятора при комбинированном взаимодействии электрических и магнитных полей. Решаются уравнения движения Гайзенберга и используются квазикогерентные состояния для вычисления функции Грина. Мы также исследуем связь между волновыми функциями Шредингера в случае псевдостационарности и квазикогерентных состояний. В частном случае вычисляется матрица плотности Блоха.

(*) *Переведено редакцией.*