Ann. Univ. Ferrara - Sez. VII - Sc. Mat.
Vol. XLI, 73-84 (1995)

Mean Fractional-Order-Derivatives Differential Equations and Filters.

MICHELE CAPUTO(*)

SUNTO - Si generalizza la soluzione di equazioni differenziali di ordine frazionario al caso in cui le derivate frazionarie sono integrate rispetto all'ordine di differenziazione. La soluzione formale è trovata a mezzo della Trasformata di Laplace. Le soluzioni delle equazioni integrodifferenziali, definite a mezzo delle derivate di ordine frazionario e dei loro integrali rispetto all'ordine di differenziazione, sono discusse a mezzo della teoria dei filtri.

ABSTRACT - The solution of differential equations of fractional order is generalized to the case when the fractional order derivatives are integrated with respect to the order of differentiation. The formal solution is found by means of the Laplace Transform. The solutions of the integro-differential equations, defined by means of derivatives of fractional order and of their integrals with respect to the order of differentiation, are also discussed in terms of filtering.

Introduction.

Since long time physics and mathematics have given great amphasis to the modelling of the energy dissipation and dispersion in the propagation of elastic waves (Bagley and Torvik 1983) and perturbations in solid anelastic media and of electromagnetic waves and perturbations in plasmas, liquid and solid dielectric (Heaviside 1989, Cisotti 1991, Cole and Cole 1941).

Laboratory experiments (e.g., Bagley and Torvik 1983, 1986, Cole and

(*) Indirizzo dell'autore: Dipartimento di Fisica, Università «La Sapienza», Piazzale A. Moro 2, 00185 Roma.

Cole 1941, Hasted 1973, Jacquelin 1991, Körnig and Müller 1989) have confirmed that the introduction of memory mechanisms in the constitutive equations of the propagation of the above mentioned waves and perturbations represents adequately their phenomena of dispersion and energy dissipation.

The most succesfull memory mechanism used to represent dispersion and energy dissipation is that of fractional order derivative (Caputo 1969, Caputo and Mainardi 1971, Bagley and Torvik 1983, 1986, Jacquelin 1991) which actually transforms the constitutive equations from differential into integrodifferential and leads to the discussion of interesting mathematical problems (e.g. Bagley 1989, Mainardi 1994, Caputo 1994a, b).

In a recent work Podlubny (1994a, 1994b) solved the differential equations of fractional order of the following type where $f(t)$ is the unknown function of the real variable t

$$\sum_{k=0}^{m-1} A_k(t)\,{}_aD_t^{z(k)} f(t) = g(t)\, f(t) \tag{1}$$

where

$${}_cD_t^z f(t) = (1/\Gamma(n-z))(d/dt)^n \int_c^t f(u)\, du/(t-u)^{z-n+1} \tag{2}$$

with $(n-1) < z < n$, $n > 0$ integer, $c > 0$ and $z > 0$ real, is the Riemann Liouville fractional derivative of order z. We note that in (1) appear derivatives of fractional order with m different orders.

In this note we specialize the derivative defined in (2) with

$$d^{n+z} f(t)/dt^{n+z} = (1/\Gamma(1-z)) \int_0^t f^{(n+1)}(u)\, du/(t-u)^z \tag{3}$$

and, on the other hand, generalize equation (1) substituting the summation with an integral in the following way

$$\int_a^b A(z)(d^{m+z} f(t)/dt^{m+z})dz = \tag{4}$$

$$= \int_a^b [A(z)\, dz/\Gamma(1-z)] \int_0^t [\, f^{(m+1)}(u)/(t-u)^z du] = g(t)$$

with $A(z)$ limited in the interval a, b limited by $0 < a < b < 1$, m integer and positive.

The left member of (4) was used by Caputo (1969) to generalize the stress strain relation of anelastic media; Caputo (1969) wrote the equilibrium condition equations of the anelastic medium, solved them to determine formally the eigenfunctions of the torsional modes of anelastic or dielectric spherical shells and infinite plates (Caputo 1989) and gave the formulae to estimate the split periods of their free modes.

However the equation (4) was not studied by Caputo (1969) or by other authors as will be done in this preliminary work.

The left hand member of (4), with the introduction of one more parameter (a and b instead of z), renders the operators (3) more flexible because it includes a variety of memory mechanisms and is more apt to represent the dispersion acting with several slightly different relaxations. In other words the left hand member of (4) offers a more flexible operator to be used instead of (3).

For the Laplace Transform (LT) of the derivatives of order z we shall use the following property (Caputo 1969)

$$(5)\qquad LT[d^{m+z}f(t)/dt^{m+z}] = p^{z+m}TL[f(t)] - \sum_{n=0}^{m} p^{n+z-1}f^{(m-n)}(0)$$

where p is the LT variable.

The solutions.

Equation (4) may be formally solved by means of the Laplace Transform (LT) obtaining first

$$(6)\qquad \int_0^\infty \exp(-pt)\,dt\left[\int_a^b [A(z)/\Gamma(1-z)]\,dz\left[\int_0^t [f^{(m+1)}(u)/(t-u)^z]\,du\right]\right] = G(p),$$

$$G(p) = \mathrm{LT}\,(g(t))\,.$$

It is shown in Appendix A that it is possible to change the order of integration of dz and dt in (6) and write

$$(7)\qquad \int_a^b [A(z)/\Gamma(1-z)]\left[\int_0^\infty \exp(-pt)\left[\int_0^t [f^{(m+1)}(u)/(t-u)^z]\,du\right]dt\right]dz = G(p)$$

or (Caputo 1969)

$$(8)\qquad \int_a^b A(z)\,dz\left[p^{m+z}F(p) - p^z\sum_{n=0}^{m} p^{n-1}f^{(m-n)}(0)\right] = G(p)\,,$$

$$F(p) = \mathrm{LT}\,[f(t)]\,.$$

Integrating the left hand member of (8) and expliciting $F(p)$ we find the solution

$$(9) \quad \begin{cases} F(p) = G(p)/p^m \int_a^b A(z) p^z dz + \sum_{n=0}^{m} p^{-n-1} f^{(n)}(0), \\ f(t) = g(t)^* \, TL^{-1}\left(1/p^m \int_a^b A(z) p^z dz\right) + \sum_{n=0}^{m} t^n f^{(n)}(0)/n! \end{cases}$$

When $A(z)$ is an analytic function of z then we may write

$$(10) \quad A(z) = \sum_0^\infty A_j z^j / j!$$

and we find

$$(11) \quad \begin{cases} F(p) = G(P)/p^m \sum_{j=0}^{\infty} \int_a^b (A_j z^j p^z / j!) dz + \sum_{n=0}^{m} p^{-n-1} f^{(n)}(0), \\ f(t) = g(t)^* \, TL^{-1}\left(1/p^m \sum_{j=0}^{\infty} \int_a^b A_j z^j p^z dz / j!\right) + \sum_{n=0}^{m} t^n f^{(n)}(0)/n!. \end{cases}$$

The filtering effect.

The derivatives of order z_1 and z_2 with $z_2 > z_1$, when $f(0) = 0$, imply filtering the function $f(t)$ with a high pass filters whose response functions are

$$(12) \quad p^{z_2},$$

$$(13) \quad p^{z_1}.$$

Since

$$(14) \quad \{|p^{z_2}| - |p^{z_1}|\}/|p^{z_1}| = |p^{z_2 - z_1}| - 1$$

is an increasing function of $|p|$, the response function (13) is increasingly more severe than the (12) in cutting the high frequencies (see figure 1). Setting $p = i\Omega$, it is seen that the phase shift is $\pi z/2$ when $\Omega > 0$ and $-\pi z/2$ when $\Omega < 0$.

When $A(z) = \delta(z - z_0)$, $(a \leq z_0 \leq b)$, equation (9) is reduced to the classic case considered above.

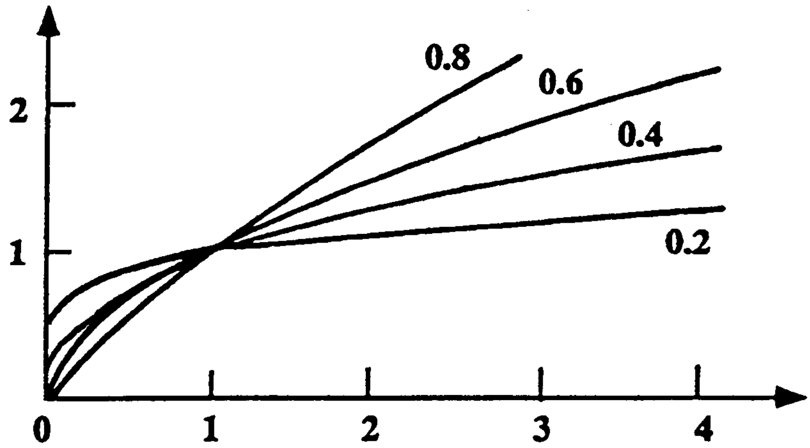

Figure 1. – Transfer functions of the filters associated to the derivatives of fractional order indicated near each curve which represents the response of the filter applied to $f(t)$ when $A(z) = \delta(z - z_0)$ for the values of z_0 indicated near each curve. The abscissa is frequency.

When

$$A(z) = kz + h \tag{15}$$

then

$$\int_a^b p^z(kz+h)\,dz = (((kb+h)p^b - (ka+h)p^a) - k(p^b - p^a)/(\ln p))/\ln p\,. \tag{16}$$

The case when $k = 0$ is simple and of interest because it gives the same weight to all the derivatives of fractional order in the range $[a, b]$. The response function $\Phi(p)$ and its modulus as function of Ω, taking the principal values of p^a, p^b and $\ln p$, are

$$\begin{cases} \Phi(p) = (p^b - p^a)p^m/\ln p\,, \\ |\Phi(\Omega)| = \\ = |\Omega^{2b} + \Omega^{2a} - 2\Omega^{a+b}\cos\pi(b-a)/2|^{1/2}\,|\Omega^m|/|(\ln\Omega)^2 + (\pi/2)^2|^{1/2}\,. \end{cases} \tag{17}$$

When $m = 0$ the properties of the response function $\Phi(p)$ at $p = \infty$ are governed by p^b, while at $p = 0$ they are governed by p^a. This property represents a relevant difference between $\Phi(p)$ and the response to the simple derivative of order z since it allows to model a filter with independent properties at zero and at infinity frequency.

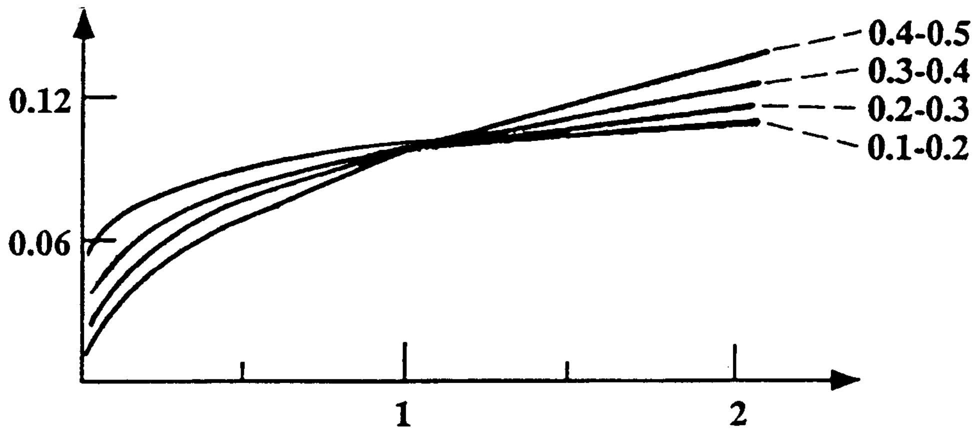

Figure 2. – Transfer functions operating on the $f(t)$ function when $A(z) = h =$ constant and with the values of the lower and upper limits of integration on z given near each curve. The abscissa is frequency.

The transfer function of the filter (17) is shown in figure 2 for some values of a and b and $m = 0$.

Applications.

The filter transfer functions considered in the previous paragraph, when solving equation (8), appear in the denominator of the right hand member. The effect of the filter response applied to $G(p)$ is therefore that of the reciprocal of the filter response applied to $F(p)$, that is the filter from high pass becomes a low pass as one verifies directly in the case when $A(z) = \delta(z - z_0)$ and equation (9) is reduced to the classic case

$$(18) \qquad \begin{cases} F(p) = G(p)/p^{m+z_0} + \sum_{n=0}^{m} p^{-n-1} f^{(n)}(0), \\ f(t) = g(t)^* \, t^{m+z_0-1}/(m+z-1)! + \sum_{n=0}^{\infty} t^n f^{(n)}(0)/n!\,. \end{cases}$$

When $A(z)$ is as in (15) with $k = 0$, the transfer function is $1/\Phi(p)$, with $\Phi(p)$ given in (17), which now appears in the denominator of $G(p)$.

The filter curves applied on $g(t)$ are shown in figure 3 for some values of z_0 and $m = 0$.

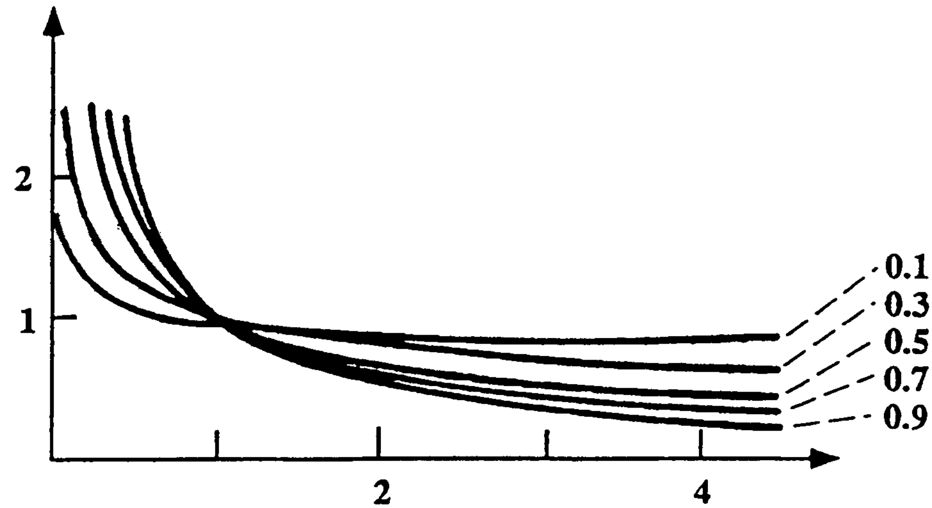

Figure 3. – Transfer functions operating on the $g(t)$ function in the equation (3) in the case when $A(z) = \delta(z - z_0)$. The value of z_0 is indicated near each curve. The result of the filtering gives the solution of the equation (3) when the initial values of $f(t)$ and its derivatives of order up to m are zero. The abscissa is the dimensionless parameter $\tau\Omega$.

Assuming $k < 0$, $h > 0$ in (16) implies that the transfer functions of the derivatives with lower order are weighted relatively less.

When $k = 0$, all the derivatives and the corresponding transfer functions are weighted equally; substituting (15) in (9) gives

$$(19)\qquad \begin{cases} F(p) = (G(p)\ln p)/h(p^b - p^a)p^m + \sum\limits_{n=0}^{m} p^{-n-1} f^{(n)}(0)\,, \\ f(t) = g(t) * \mathrm{LT}^{-1}(\ln p/h(p^b - p^a)p^m) + \sum\limits_{n=0}^{m} t^n f^{(n)}(0)/n!\,. \end{cases}$$

It is verified that

$$(20)\qquad \begin{cases} \lim\limits_{p\to\infty} p \ \ln p/(p^{b+m} - p^{a+m}) = \begin{cases} \infty, & O(1-b) \text{ if } m = 0\,, \\ 0, & O(m+b-1) \text{ if } m > 0\,, \end{cases} \\ \lim\limits_{p\to\infty} p \ \ln p/(p^{b+m} - p^{a+m}) = \begin{cases} 0, & O(1-a) \text{ if } m = 0\,, \\ \infty, & O(m+a-1) \text{ if } m > 0\,, \end{cases} \end{cases}$$

and also that

$$(21)\qquad \lim_{p\to\infty} \ln p/(p^{b+m)} - p^{a+m}) = 0, \qquad O(m+b) \text{ if } m \geqslant 0$$

where the order of the infinitesimal or of the infinity are indicated in brackets.

The limit (21) verifies that (19), with $k = 0$, may be a LT; (20) imply that

$$(22)\qquad \begin{cases} \lim\limits_{t\to\infty} \mathrm{LT}^{-1}(\ln p/(p^{b+m} - p^{a+m})) = \begin{cases} \infty, & O(1-b) \text{ if } m = 0, \\ 0, & O(m+b-1) \text{ if } m > 0, \end{cases} \\ \lim\limits_{t\to\infty} \mathrm{LT}^{-1}(\ln p/(p^{b+m} - p^{a+m})) = \begin{cases} 0, & O(1-a) \text{ if } m = 0, \\ \infty, & O(m+a-1) \text{ if } m > 0, \end{cases} \end{cases}$$

which give the values of the filter weight function at $t = 0$ and $t = \infty$.

The first term of the right member of (19) may be written

$$(23)\qquad g(t) * [\{[\mathrm{LT}^{-1}\ln p/p^{a})] * [\mathrm{TL}^{-1}(-1/(p^{b-a}-1))]\} * t^{m-1}/\Gamma(m)]$$

where the $\mathrm{LT}^{-1}(-1/(p^{b-a}-1))$ exists and is discussed in Caputo (1984), while the $LT^{-1}(\ln p/p^{a})$ must be computed numerically for each value of a.

In order to see the filtering effect of $1/\Phi(\Omega)$, with $k = 0$, on $g(t)$, let us assume $p = i\Omega, f^{(j)}(0) = 0$, $(j = 0, 1, 2, \ldots, m)$ in (29) and consider its modulus, we obtain, assuming the principal values of p^a, p^b and $\ln i\Omega$,

$$(24)\qquad |F(i\Omega) = |G(i\Omega)|\,|\ln(i\Omega)|/|(i\Omega)^b - (i\Omega)^a|\,h\Omega^m =$$

$$= |G(i\Omega)|\,|(\ln\Omega)^2 + (\pi/2)^2|^{1/2}/h|\Omega^b + \Omega^a - 2\Omega^{a+b}\cos\pi(a-b)/2|^{1/2}\Omega^m$$

which implies a low pass filtering of $g(t)$ whose transfer function is shown in figure 4 for several values of a and b and $m = 0$. In general the filter response function in (24) causes phase changes in $f(t)$.

In particular when

$$(25)\qquad A(z) = Az^n$$

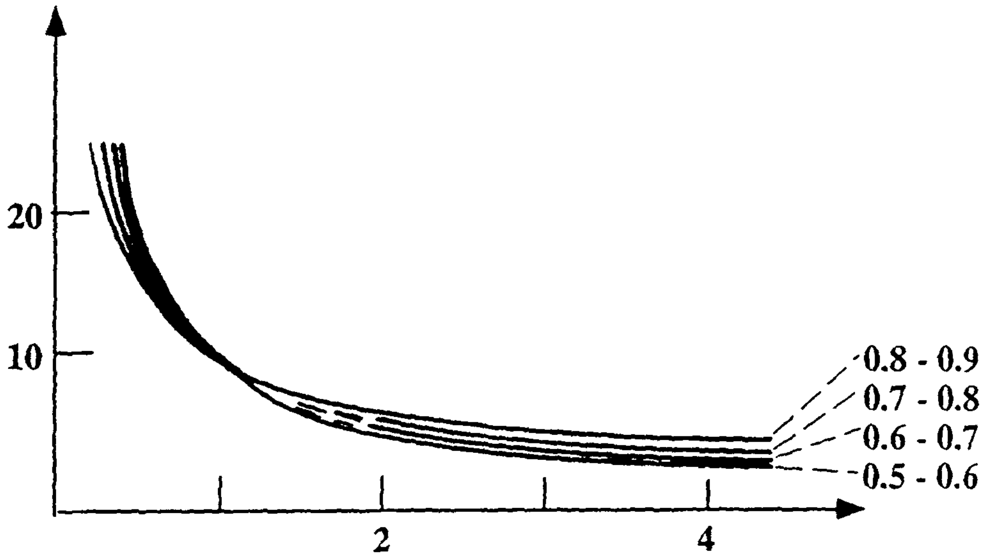

Figure 4. – Transfer functions operating on the $g(t)$ function in the case when $A(z) = h =$ constant with the values of the lower and upper limits of integration on z given near each curve. The result of the filtering gives the solution of the equation (3) when the initial values of $f(t)$ and of its derivatives of order up to m are zero. The abscissa is the dimensionless parameter $\tau\Omega$.

we find

$$(26)\quad f(t) = g(t) * \mathrm{LT}^{-1}\Big[1/Ap^m\Big[\sum_{n=0}^{m}(-1)^n m(m-1)\ldots$$
$$\ldots(m-n+1)(b^{m-n}p^b - p^a a^{m-n})/(\ln p)^{n+1}\Big]\Big].$$

A generalization.

Equation (4) may be generalized as follows

$$(27)\qquad \sum_{j=1}^{n}\int_{a_j}^{b_j} A_j(z)[d^{m_j+z}f(t)/dt^{m_j+z}]\,dz = g(t)$$

where m and n are positive integers and $0 < a_j < b_j < 1$.

By taking the LT of both members of (27) and interchanging the integrations in dt and dz, the formal solution is readily obtained

$$(28)\quad F(p) = G(p) \Bigg/ \left[\sum_{j=1}^{n} p^{m_j} \int_{a_j}^{b_j} p^z A_j(z)\, dz \right] + \\ + \left[\int_{j=1}^{n} [p^{-1} f(0) + p^{-2} f^{(1)}(0) + \dots + p^{-m_j} f^{(m_j - 1)}(0) + \right. \\ \left. + p^{-m_j - 1} f^{(m_j)}(0)] p^{m_j + z} \int_{a_j}^{b_j} p^z A_j(z)\, dz \right] \Bigg/ \left[\sum_{j=1}^{n} p^{m_j} \int_{a_j}^{b_j} p^z A_j(z)\, dz \right].$$

Constraints for physical applications.

It is be noted that, in the formulae previously obtained, after the Laplace transformation and the integration with respect to z, may appear sums of powers of p (or of ω) which have different dimensions, which would be physically unacceptable.

This limiting aspect of the result may be avoided by assuming

$$(29)\qquad A(z) = \tau^z B(z)$$

where τ has the dimension of time.

Substituting in (9) we find

$$(30)\qquad F(p) = G(p)/p^m \int_a^b B(z)(p\tau)^z\, dz + \sum_{n=0}^{m} p^{-n-1} f^{(n)}(0).$$

As an example we may consider the case when $B(z) = h =$ constant, one finds

$$F(p) = G(p) \ln(p\tau)/hp^m((p\tau)^b - (p\tau)^a) + \sum_{n=0}^{m} p^{-n-1} f^{(n)}(0)$$

When $m = 0$, $f(0) = 0$, and $h = 1$ we obtain a generalization of the case when $A = \delta(z - z_0)$ which is of interest in the study of the propagation of waves in anelastic or dielectric dispersive media (Caputo 1994, Caputo 1995) with a variety of slightly different relaxations; in this case formula (31) is

$$(32)\qquad F(p) = G(p) \ln(p\tau)/((p\tau)^b - (p\tau)^a)$$

where it is seen that the determination of the solution consists in a filtering of the function $g(t)$ with a filter whose transfer function is of the type $1/\Phi(p)$

represented in figures 3 and 4 where however the abscissa, in this case, would be the dimensionless parameter $\Omega\tau$ (where Ω is frequency).

One may note the difference between the curve with $z_0 = 0.5$ of figure 3 and the curve with $a = 0.5$, $b = 0.6$ of figure 4; it is also seen that the lines of figure 4 resulting from (32) ($A(z) = \text{const.}$) have very different geometry than those of figure 3 ($A(z) = \delta(z - z_0)$).

Appendix A.

In order to show that in (6) we may change the order of integration in dz and dt, for the theorem of Fubini Tonelli it is sufficient to show that the following function of z

$$\text{(A.1)}\qquad |[A(z)/\Gamma(1-z)]| \left\{ \int_z^\infty \exp(-pt) \left| \left[\int_0^t f^{(m+1)}(u)(t-u)^{-z}\,du \right] \right| dt \right\}$$

is summable with respect to z in the interval a, b where a and b satisfy the condition $0 < a < b < 1$.

Caputo (1969) has shown that, in the hypothesis that the LT of $f^{(m+1)}(t)$ exists, we may change the order of integration in (A.1) and write

$$\text{(A.2)}\qquad |[A(z)/\Gamma(1-z)]| \left\{ \int_0^t |f^{(m+1)}(u)| \left[\int_z^\infty (\exp(-pt)(t-u)^{-z}\,dt \right] du \right\} =$$

$$= |[A(z)/\Gamma(1-z)]| \left\{ \int_0^t |f^{(m+1)}(u)|\, p^{z-1} \int_z^\infty [(pt - pu)^{-z}(\exp(-pt)\,dpt]\,du \right\} =$$

$$= |[A(z)/\Gamma(1-z)]| \left\{ \int_0^t |f^{(m+1)}(u)|\, p^{z-1} \exp(-pu)\,\Gamma(1-z)\,du \right\} =$$

$$= A(z) \left\{ p^{z-1} \int_0^t | f^{(m+1)}(u) \exp(-pu)\,du | \right\}.$$

The hypothesis is that the integral in du in the last line of (A.2) exists and is finite; since this integral does not depend on z and the factor $A(z)\,p^{z-1}$, as function of z, is summable in the interval $[a, b]$, then (A.1) is summable in $[a, b]$.

BIBLIOGRAFIA

BAGLEY R. L, *The initial value problem for fractional order differential equations with constant coefficients*, Air Force Inst. Technology, AFIT-TR-EN-88-1, September 1989.

BAGLEY R. L. - TORVIK P. J, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheology, **27**, 3 (1983), pp. 201-210.

BAGLEY R. L. - TORVIK P. J, *On the fractional calculus model of viscoelastic behavior*, J. Theology, **30**, 1 (1986), pp. 133-155.

CAPUTO M, *Elasticità e dissipazione*, Zanichelli Printer, Bologna (1969).

CAPUTO M, *Relaxation and free modes of a self gravitating planet*, Geophys. J. R. Astr. Soc., **77** (1984), pp. 789-808.

CAPUTO M, *The rheology of an anelastic medium studied by means of the splitting of its eigenfrequencies*, J. Acoust Soc. Am., **86**, 5 (1989), pp. 1984-1987.

CAPUTO M, *The Riemann sheet solutions of linear anelasticity*, Ann. Mat. Pura Appl., IV, **166** (1994a), pp. 335-342.

CAPUTO M, *A unified model for the dispersion in anelastic and dielectric media: consequences, abstract, Proceedings of the 14th IMACS* (Int. Ass. for Mathem. and Computer in Simulation), World Congress, Atlanta, USA, 1994b.

CAPUTO M. - MAINARDI F, *Linear models of dissipation in anelastic solids*, Riv. Nuovo Cimento, II, 1 (1971), pp. 161-198.

CISOTTI V, *L'ereditarietà lineare ed i fenomeni dispersivi*, Nuovo Cimento, **6**, 2 (1911), pp. 234-244.

COLE K. S. - COLE R. H, *Dispersion and absorption in dielectrics*, J. Chem. Phys., **9** (1941), pp. 341-349.

HEAVISIDE O, *Electromagnetic Theory*, vol. II, The Electrician Printing and Publishing Co. (1889).

JACQUELIN J, *A number of models for CPA impedences of conductors and for relaxation in non Debye dielectrics*, J. Non-Cryst. Solids, **131** (1991), pp. 1080-1083.

KÖRNIG H. - MÜLLER G, *Rheological models and interpretation of postglacial uplift*, Geophys. J. R. Astron. Soc., **98** (1989), pp. 245-253.

MAINARDI F, *On the initial value problem for the fractional diffusione wave equation*, Rionero S. and Ruggeri T. Eds., *Waves and Stability in Continuous Media*, World Scientific, Singapore (1994), pp. 246-251.

PODLUBNY IGOR, *Analytical solution of linear differential equations of the fractional order, Proceedings of the 12th IMACS* (Intern. Assoc. for Mathematics and Comp. in Simulation), World Congress, Atlanta, USA (1994a).

PODLUBNY IGOR, *Numerical solution of initial value problems for ordinary fractional-order differential equations, Proceedings IMACS* (Intern. Assoc. for Mathematics and Comp. in Simulation), World Congress, Atlanta, USA (1994b).

Pervenuto in Redazione l'11 maggio 1995.

In forma definitiva il 10 marzo 1996.