ON BOUNDED SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

JEROME A. GOLDSTEIN (*)

§ 1. The following result was recently proved by S. ZAIDMAN in a slightly modified form.

THEOREM (ZAIDMAN [2]). *Let A be the infinitesimal generator of a* (C_0) *semigroup* $\{T_t\,;\ t \geq 0\}$ *of bounded linear operators on a Banach space X. Suppose that either*

(i) $\lim_{t \to \infty} \| T_t f \| = 0$ *for each* $f \in X$, *and* $(\lambda I - A)^{-1}$ *is compact for some complex number* λ, *or*

(ii) $\lim_{t \to \infty} \| T^*_t \varphi \| = 0$ *for each* $\varphi \in X^*$.

Let $u : (-\infty, \infty) \to \mathfrak{D}(A)$ (= *the domain of* A) *be a strongly continuously differentiable solution of*

$$du(t)/dt = Au(t) \qquad (-\infty < t < \infty). \tag{*}$$

Then either $u \equiv 0$ *or else* $\lim_{t \to -\infty} \| u(t) \| = \infty$.

This result is useful in the study of almost periodic solutions of parabolic systems (cf. the references cited in [2]). It seems natural to ask if the compactness condition on the resolvent of A can be removed in (i), since no analogous condition occurs in (ii). The answer is no, and the purpose of this note is to prove this be means of an example.

§ 2. We use the notation and terminology of [1]. We shall construct a (C_0) semigroup $\{T_t\,;\ t \geq 0\}$ of bounded linear operators on a Hilbert space X satisfying:

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(I) $\lim_{t \to \infty} \| T_t f \| = 0$ *for each* $f \in X$,

(II) *if* A *denotes the infinitesimal generator of* $\{T_t\,;\ t \geq 0\}$, *then there exists a nontrivial bounded strongly continuously differentiable solution of* $(*)$.

Let $X = L^2(-\infty, 0]$. Define T_t on X to be translation to the right followed by restriction to $(-\infty, 0]$, i.e.

$$T_t f(x) = f(x-t) \qquad (-\infty < x \leq 0,\ t \geq 0).$$

Then $\{T_t\,;\ t \geq 0\}$ is a (C_0) contraction semigroup acting on the Hilbert space X. Note that if $f \in X$,

$$\| T_t f \|^2 = \int_{-\infty}^{-t} | f(y) |^2 dy \to 0 \text{ as } t \to \infty,$$

so that (I) holds. Before going further, it will be convenient to obtain another expression for T_t.

Let $Y = L^2(-\infty, \infty)$. Let $J : X \to Y$ be the canonical injection:

$$Jf(x) = f(x) \text{ if } x \leq 0, \qquad Jf(x) = 0 \text{ if } x > 0.$$

Define the restriction map $P : Y \to X$ by $Pf = f\,|_{(-\infty, 0]}$. Clearly $J^* = P$, $P^* = J$. Define S_t on Y to be translation to the right by t units, i.e.

$$S_t f(x) = f(x-t) \qquad (-\infty < x,\ t < \infty).$$

Then $\{S_t\,;\ -\infty < t < \infty\}$ is a (C_0) unitary group on Y with infinitesimal generator $G = -d/dx$, the domain of G being

$\mathfrak{D}(G) = \{f \in Y : f$ absolutely continuous on $(-\infty, \infty)$, $f' \in Y\}$. Then $\{T_t = P \cdot S_t \cdot J;\ t \geq 0\}$ is the semigroup acting on X defined in the above paragraph.

Let

$\mathfrak{D} = \{f \in X : f$ absolutely continuous on $(-\infty, 0]$, $f' \in X$, $f(0) = 0\}$. Then if $f \in \mathfrak{D}$, it follows that $Jf \in \mathfrak{D}(G)$ and

$$Af = P(G(Jf)) = -f'.$$

Hence $\mathfrak{D} \subset \mathfrak{D}(A)$ and $A\,|_{\mathfrak{D}} = -d/dx\,|_{\mathfrak{D}}$.

Next,

$$T^*_t = (P \cdot S_t \cdot J)^* = P \cdot S_{-t} \cdot J \qquad (t \geq 0)$$

is translation to the left by t units, i.e.

$$T^*_t f(x)=f(x+t) \text{ if } x\leq -t,\ T^*_t f(x)=0 \text{ if } -t<x\leq 0.$$

Clearly $\{T^*_t\ ;\ t\geq 0\}$ is a (C_0) semigroup of isometries on X.

Now let $0\neq f\in\mathfrak{D}$ ($\mathfrak{D}$ is dense in X). Define

$$u:(-\infty,\ \infty)\to\mathfrak{D}(A)$$

by

$$u(t)=T_t f \text{ if } t\geq 0, \qquad u(t)=T^*_{-t}f \text{ if } t\leq 0.$$

We claim that (*) holds. This is clear for $t\geq 0$ since $f\in\mathfrak{D}\subset\mathfrak{D}(A)$. On the other hand, if $t\geq 0$, $T^*_{-t}(\mathfrak{D})\subset\mathfrak{D}$ and so

$$\frac{d}{dt}T^*_{-t}f=\frac{d}{dt}P(S_t(Jf))=AT^*_{-t}f;$$

thus (*) also holds for $t\leq 0$. Finally,

$$0<\|f\|=\sup_{-\infty<t<\infty}\|u(t)\|,$$

and so (II) holds.

§ 3. Note in the above example that since A^* is the infinitesimal generator of $\{T^*_t\ ;\ t\geq 0\}$, we have shown

$$A^*|_{\mathfrak{D}}=-A|_{\mathfrak{D}}.$$

However, $A^*\neq -A$ since $\{T_t\ ;\ t\geq 0\}$ is not a unitary semigroup. Since $\{T^*_t\ ;\ t\geq 0\}$ is a (C_0) isometric (non-unitary) semigroup, A^* is maximal skew-symmetric (but not skew-adjoint).

§ 4. It seems natural to ask that if condition (i) of Zaidman's theorem holds, does it follow that there exist constants $M>0$, $\beta>0$ such that

$$\|T_t\|\leq Me^{-\beta t} \qquad (t\geq 0).$$

If this were the case then (i) would imply (ii) and so Theorem 1 of [2] would be a trivial consequence of Theorem 2 of [2]. To see that this is not the case we construct a (C_0) semigroup $\{T_t\ ;\ t\geq 0\}$ of normal operatos on Hilbert space X satisfying

(1) $\lim_{t\to\infty}\|T_t f\|=0$ *for each* $f\in X$,

(2) $\|T_t\|=1$ *for each* $t\geq 0$,

(3) *If A denotes the infinitesimal generator of* $\{T_t\,;\, t \geq 0\}$, *then* $(I-A)^{-1}$ *is compact.*

This example settles a question posed recently by Professor ZAIDMAN [personal communication].

Let $\{x_n : n=1, 2, \dots\}$ be an orthonormal basis for a separable Hilbert space X. Define T_t by

$$T_t x_n = \exp\{-tn^{-1}+itn\}x_n \qquad (n=1, 2, \dots),$$

and extend the domain of T_t to all of X by linearity and continuity. Clearly $\{T_t\,;\, t \geq 0\}$ is a (C_0) contraction semigroup of normal operators on X.

We have

$$\lim_{t\to\infty} \| T_t f \| = 0$$

for each f which is a finite linear combination of $x_1, x_2, \dots$; since such f's are dense and since each T_t is a contraction, (1) follows. Next,

$$1 \geq \| T_t \| \geq \| T_t x_n \| = \exp(-t/n) \to 1 \text{ as } n \to \infty;$$

thus (2) holds. Now let A denote the infinitesimal generator of $\{T_t\,;\, t \geq 0\}$. Then $(I-A)^{-1}$ is a bounded operator on X and

$$(I-A)^{-1}x_n = n(n+1-in^2)^{-1}x_n \qquad (n=1, 2, \dots).$$

We use this to show the compactness of $(I-A)^{-1}$. Let $\{y^n = \sum_{m=1}^{\infty} y_m{}^n x_m : n = 1, 2, \dots\}$ be a bounded sequence in X. Then $\sup_{n,m} | y_m{}^n | < \infty$. By the Bolzano - Weierstrass theorem, there is a convergent subsequence $\{y^1{}_{n,1} : n=1, 2, \dots\}$ of $\{y_n{}^1 : n=1, 2, \dots\}$. Similarly, there is a convergent subsequence $\{y^2{}_{n,2} : n=1, 2, \dots\}$ of $\{y^2{}_{n,1} : n=1, 2, \dots\}$. Continue this process in the obvious way. Consider the diagonal sequence $\{y_{n,n} : n=1, 2, \dots\}$. $y^m = \lim_{n\to\infty} y^m{}_{n,n}$ exists for $m=1, 2, \dots$ and $\{y^m : m=1, 2, \dots\}$ is a bounded sequence. Moreover, as $n \to \infty$,

$$(I-A)^{-1}y_{n,n} = \sum_{m=1}^{\infty} m(m+1-im^2)^{-1}y^m{}_{n,n}x_m$$

$$\to \sum_{m=1}^{\infty} m(m+1-im^2)^{-1}y^m x_m \in X,$$

the converge being in X. Hence $(I-A)^{-1}$ is compact. Thus $(I-A)^{-1}$ is a compact normal operator with eigenvalues $\{n(n+1-in^2)^{-1} : n=1, 2, \dots\}$; in particular, (3) holds.

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RIASSUNTO

Si costruiscono soluzioni limitate non triviali di una equazione differenziale astratte

$$u'(t) = Au(t) \qquad (-\infty < t < \infty)$$

dove A è il generatore di un semigruppo di operatori $\{T_t\,;\ t \geq 0\}$ tale che T_t converge fortemente verso 0 per $t \to \infty$.

SUMMARY

We construct nontrivial bounded solutions of an abstract evolution equation

$$u'(t) = Au(t) \qquad (-\infty < t < \infty)$$

where A generates a (C_0) semigroup of operators $\{T_t\,;\ t \geq 0\}$ such that T_t converges strongly to zero as $t \to \infty$.

REFERENCES

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