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Tensor Decomposition of the Transition Operator (*).

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Summary. — The properties of the tensor components of the transition operator with respect to an arbitrary compact group are investigated. The principal result is that as a consequence of unitarity the transition operator always has an invariant component whose imaginary part is nonnegative. The connection of this result with the consistency of phenomenological calculations in broken-symmetry models is discussed.

The consequences of higher (broken) symmetry schemes in scattering processes have been the subject of numerous investigations (1). However, the subject has always been plagued with serious conceptual difficulties particularly in connection with the phenomenological perturbation-theoretic approach which was so successful in obtaining mass formulae. We have no proposals in mind for a more well-founded treatment of broken symmetries in such problems. Nevertheless, the transition operator T is virtually unique among physically interesting operators in that a general analysis of its tensor decomposition with respect to any compact group is possible without further assumptions. From this fact we will be able to demonstrate at least some measure of consistency within the present rather ill-defined framework.

The tensor decomposition of T will be defined in terms of the following

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(1) See H. HARARI: *High-Energy Physics and Elementary Particles* (I.A.E.A., Vienna, 1965), p. 353, for a review of the topic in the case of SU_3. Some recent work on the completely symmetric limit of SU_3 has been carried out by S. MESHKOV and G. B. YODH: *Phys. Rev. Lett.*, **19**, 603 (1967).

particular characterization of the broken-symmetry problem (2). Let $\mathscr{H}$ denote the physical (hadronic) space of states upon which is defined a representation of a compact group G by unitary operators, $U(g)$, where $g \in G$ (3). Since T is a bounded operator on $\mathscr{H}$ it possesses a decomposition into a sum of (tensor) operators which transform irreducibly with respect to G (see Appendix).

Now T is related to the unitary scattering operator S by

$$S = 1 + iT\,,$$

so that

$$2\,\mathrm{Im}[T] \equiv i(T^\dagger - T) = TT^\dagger = T^\dagger T\,.$$

Therefore, by the Lemma of the Appendix the invariant part, $\mathrm{Im}[T^{(0)}]$, of $\mathrm{Im}[T]$ is not identically zero and is, moreover, nonnegative irrespective of the (compact) group G involved or, equivalently, irrespective of the magnitude of the symmetry breaking. Since

$$\mathrm{Im}[T^{(0)}] \not\equiv 0\,,$$

it necessarily follows that $T^{(0)}$ is not identically zero as well (4). As a consequence of this, some aspect of respectability can be assigned to the approximation

$$T \simeq T^{(0)};$$

namely, the amplitudes in question exist and, secondly, they satisfy those positivity requirements necessary for meaningful statements concerning total cross-sections and dominance theorems (5) within this approximation. Except under additional very strong assumptions no analogous theorems obtain for

(2) This characterization appears to be implicitly assumed in most phenomenological investigations of symmetry breaking.

(3) The $U(g)$ could be introduced, for example, by defining their transformations on the «in» or «out» states. Of course, all the $U(g)$ do not commute with all the generators of the Poincaré group, in general, and this in turn implies that all the $U(g)$ do not commute with T and therein lie the basic conceptual problems of this entire approach.

(4) One can imagine this scalar part occurring even if the Hamiltonian had no scalar part. For example, if $H = H^{(8)}$ in the unitary-symmetry model we would still get a scalar part of T through the reduction of the various products of $H^{(8)}$ which would appear if we thought of S as being formally generated by something like the Dyson prescription. Thus, the existence of a scalar part really provides no insight in the structure of the interaction as we might expect from the very generality of the result.

(5) L. L. FOLDY and R. F. PEIERLS: *Phys. Rev.*, **130**, 1585 (1963); D. AMATI, L. L. FOLDY, A. STANGHELLINI and L. VAN HOVE: *Nuovo Cimento*, **32**, 1685 (1964).

the nonnegative but unbounded Hamiltonian (H) or mass (M) operators; we shall comment upon this in more detail below.

It is always tempting to think of $T^{(0)}$ as the sum of a « symmetric limit » transition operator plus a scalar perturbation. Therefore, we seek any other similarities between $T^{(0)}$ and T other than those just proven. One finds by direct computation that [6]

$$2\,\mathrm{Im}[T^{(0)}] = T^{(0)\dagger}T^{(0)} + \sum_{\mu\neq 0} d(\mu)^{-1}\sum_{i,k}^{d(\mu)} T_k^{(\mu)}(i)^\dagger T_k^{(\mu)}(i) =$$

$$= T^{(0)}T^{(0)\dagger} + \sum_{\mu\neq 0} d(\mu)^{-1}\sum_{i,k}^{d(\mu)} T_k^{(\mu)}(i)\, T_k^{(\mu)}(i)^\dagger\,,$$

which explicitly demonstrates our previous statements, since $\mathrm{Im}[T^{(0)}]$ is obviously equal to a sum of nonnegative operators, and also shows that

$$S^{(0)} \equiv 1 + iT^{(0)}$$

will not be a unitary operator except in the limit of complete symmetry. So, except under the latter circumstance, there will always be a scalar « perturbation » present in S or T [4]. Finally, we observe that $T^{(0)}$ will not even be a normal operator, in general.

Equations for $\mathrm{Im}[T^{(\mu)}]$, $\mu \neq 0$, similar to the preceding reveal very little concerning the structure of the higher tensor components of T. However, from the Lemma in the Appendix we see that neither $\mathrm{Im}[T^{(\mu)}]$ for any $\mu\neq 0$ nor

$$\mathrm{Im}\,[T - T^{(0)}] = \sum_{\mu\neq 0}\mathrm{Im}\,[T^{(\mu)}]$$

can be nonnegative operators. Also, it is clear that the assumption that the decomposition of T consists of a finite sum over μ will be, in general, in conflict with unitarity unless $T = T^{(0)}$ or if G is a finite group.

Obviously, a similar analysis applies to any unitary operator on $\mathscr{H}$, such as the Poincaré transformations $U(a, \Lambda)$, although nothing very useful appears to result. For example, the $U^{(0)}(a, \Lambda)$ do not satisfy any group properties nor can the fact that $U^{(0)}(a, \Lambda)$ is not equal to the identity for all a and Λ be used used to infer that any of the generators of the Poincaré group necessarily have invariant parts with respect to G or even tensor decompositions with respect to G.

In connection with this last observation we would like to discuss very briefly the question of tensor decompositions for unbounded operators like H

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[6] For this calculation we take the matrices $D^{(\mu)}(g)$ introduced in the Appendix to be unitary.

or M. Clearly, without prior knowledge concerning the common domains and ranges of the $U(g)$ and those of the operator in question, say H, nothing can be said concerning tensor decompositions, and the use of such an expansion for H in broken-symmetry models constitutes a *physical assumption* concerning the structure of both H and the groups on $\mathscr{H}$ (7). Thus, the customary usage of these expansions already implies an extremely intimate interconnection between the group and the observables in question. In a certain (somewhat vague) sense such an assumption is roughly equivalent to the intuitive physical expectation that the « symmetry » is not arbitrarily broken.

We point out, finally, that within the context of the usual implicit and explicit assumptions introduced in conjuction with the derivation of mass formulae, it is easy to convince oneself that the conclusions of the Lemma in the Appendix obtain for H and M.

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We would like to thank Profs. L. L. FOLDY and P. KANTOR for several very stimulating conversations. I am especially indebted to the latter for pointing out the difficulties in the case of unbounded operators.

APPENDIX

We outline here those aspects of tensor decompositions which are employed in the body of the paper. Let $\chi^{(\mu)}(g)$ denote the character of $g \in G$ with respect to the irreducible representation (IR) μ of G (8). The characters corresponding to the entire set of inequivalent, irreducible representations of G, form a complete orthonormal set on the space of class functions defined on the group manifold $\mathscr{M}$. If $F(g)$ is any bounded continuous class function on $\mathscr{M}$ then

$$F(g) = \sum_{\mu} \chi^{(\mu)}(g) F^{(\mu)}$$

and in particular

$$F(e) = \sum_{\mu} d(\mu) F^{(\mu)},$$

(7) It is interesting to compare these questions of domains, etc., with those encountered in the specification of the Poincaré transformation properties of fields. See R. F. STREATER and A. S. WIGHTMAN: *PCT, Spin and Statistics, and All That* (New York, 1964), p. 98, 99.

(8) As a matter of convention we let $\mu = 0$ correspond to the identity representation of G.

where e is the identity element of G, $d(\mu)$ is the dimensionality of the IR μ,

$$F^{(\mu)} = \sum_g \chi^{(\mu)}(g)^* F(g) ,$$

and the summation over g represents the (normalized) invariant integration over $\mathscr{M}$.

Next, let A be a bounded operator on $\mathscr{H}$. Then for any $|x\rangle$, $|y\rangle \in \mathscr{H}$, $\langle x| U(g) A U(g)^\dagger |y\rangle$ is a bounded continuous function on $\mathscr{M}$. If we construct the class function

$$A(g') = \sum_g \langle x| U(\bar{g}) A U(\bar{g})^\dagger |y\rangle ,$$

where $\bar{g} = g g' g^{-1}$ we see that $A(g')$ is bounded and continuous on $\mathscr{M}$ and

$$A(e) = \langle x| A |y\rangle .$$

Using the results of the preceding paragraph we infer the tensor decomposition

$$A = \sum_\mu A^{(\mu)},$$

where

$$A^{(\mu)} = d(\mu) \sum_g \chi^{(\mu)}(g)^* U(g) A U(g)^\dagger .$$

Given some choice of matrix realizations, $D^{(\mu)}(g)$, of the IR μ, we find that

$$A^{(\mu)} = \sum_i A_i^{(\mu)}(i) .$$

The various tensor components are defined by

$$A_k^{(\mu)}(i) \equiv d(\mu) \sum_g D_{ik}^{(\mu)}(g^{-1}) U(g) A U(g)^\dagger$$

and transform according to

$$U(g) A_k^{(\mu)}(i) U(g)^\dagger = \sum_j D_{jk}^{(\mu)}(g) A_j^{(\mu)}(i) ,$$

so that $A_i^{(\mu)}(i)$ belongs to the tensorial set $[A_k^{(\mu)}(i);\ k = 1, \ldots, d(\mu)]$. The operators $A_k^{(\mu)}(i)$ and $A_l^{(\nu)}(j)$ are linearly independent if $\mu \neq \nu$ or if $k \neq l$.

In the case that A is both bounded and nonnegative, namely,

$$\langle x| A |x\rangle \geqslant 0 , \qquad \text{for all } |x\rangle ,$$

we see that $\langle x| U(g) A U(g)^\dagger |x\rangle$ for any $|x\rangle$ is a bounded, continuous, nonnegative function on $\mathcal{M}$. Thus,

Lemma: If A is a bounded, nonnegative operator, then $A^{(0)} \not\equiv 0$ and A^0 is also nonnegative.

As a matter of terminology we point out that $A^{(0)}$ is called the scalar or invariant component of A. We also mention the fact that the preceding results can also be derived using the completeness of the components $D_{ij}^{(\mu)}(g)$ of the inequivalent IR representation matrices by considering the bounded, continuous function $\langle x| U(g) A U(g)^\dagger |y\rangle$ directly. Questions concerning the consequences of the possible Hermiticity of A or the existence of symmetries of A with respect to subgroups of G can be answered in a straightforward manner using the preceding formalism.

RIASSUNTO (*)

Si studiano le proprietà delle componenti tensoriali dell'operatore di transizione rispetto ad un arbitrario gruppo compatto. Il risultato principale è che, in conseguenza dell'unitarietà, l'operatore di transizione ha sempre una componente invariante la cui parte immaginaria è non negativa. Si discute come questo risultato sia connesso con la consistenza dei calcoli fenomenologici nei modelli di simmetria infranta.

(*) *Traduzione a cura della Redazione.*

Тензорное разложение оператора перехода.

Резюме (*). — Исследуются свойства тензорных компонент оператора перехода по отношению к произвольной компактной группе. Как следствие унитарности, основной результат представляет, что оператор перехода всегда имеет инвариантную компоненту, чья мнимая часть не является отрицательной. Обсуждается связь этого результата с последовательностью феноменологических вычислений в моделях с нарушением симметрии.

(*) *Переведено редакцией.*