On Calculations in Conformal Invariant Field Theories.

K. SYMANZIK

Deutsches Elektronen-Synchrotron DESY - Hamburg

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Recently, MIGDAL [1,2] has proposed a very interesting technique to deal with the large-momenta limit of strong-interaction field theories, namely, by the use of conformal invariance [3] in conjunction with Wilson's anomalous dimensions [4]. Since it appears that the Gell-Mann–Low limit [5] of renormalizable theories indeed is conformal invariant [6,7] and Migdal's bootstrap equations [2] have been shown [8] to be ultraviolet- and infra-red–divergence free, it is challenging to analyse Migdal's algebraic vertex bootstrap condition [2] and similar algebraic conditions derivable from analogous propagator bootstraps [8,9]. Equivalent algebraic bootstrap equations for critical exponents have been proposed by PARISI and PELITI [10].

We show here how to obtain the general term in these bootstrap relations by way of a simple manifestly conformal invariant calculus, based on a generalization of a formula of D'ERAMO, PELITI and PARISI [11] from three to n points. We use Euclidean metric and comment on the relation to Minkovski metric only at the end. Consider the integral over D dimensions [*]

$$(1) \qquad I(x_1\delta_1, \ldots, x_n\delta_n) = \pi^{-\frac{1}{2}D}\int \mathrm{d}^D u \prod_{i=1}^{n} \left((u-x_i)^2\right)^{-\delta_i}\Gamma(\delta_i)\,, \qquad \sum \delta_i = D$$

(1) A. A. MIGDAL: *Phys. Lett.*, **37** B, 98 (1971).
(2) A. A. MIGDAL: *Phys. Lett.*, **37** B, 386 (1971).
(3) A. M. POLYAKOV: *Žurn. Èksp. Teor. Fiz. Pis. Red.*, **12**, 538 (1970) (English translation: *Sov. Phys. JETP Lett.*, **12**, 381 (1970)).
(4) K. G. WILSON: *Phys. Rev.*, **179**, 1499 (1969).
(5) K. G. WILSON: *Phys. Rev. D*, **3**, 1818 (1971).
(6) B. SCHROER: *Lett. Nuovo Cimento*, **2**, 867 (1971).
(7) M. HORTACSU, R. SEILER and B. SCHROER: NYO-3829-80, University of Pittsburgh, Sept. 1971.
(8) G. MACK and I. T. TODOROV: IC/71/139, Trieste, Oct. 1971.
(9) G. MACK and K. SYMANZIK: in preparation.
(10) G. PARISI and L. PELITI: *Lett. Nuovo Cimento*, **2**, 627 (1971).
(11) M. D'ERAMO, L. PELITI and G. PARISI: *Lett. Nuovo Cimento*, **2**, 878 (1971).
(*) All formulae are interpretable, and valid, also if D is not an integer, provided the points $x_1 \ldots x_n$ lie in a subspace of integer dimension. Also in the present context, the use of noninteger dimension is occasionally a convenient technical device, cf. ref. (12).
(12) K. G. WILSON and M. E. FISHER: CLNS-173, Cornel University, Oct. 1971.

with $\operatorname{Re}\delta_i > 0$, $\forall i$. Familiar procedures give

$$(2)\qquad I = \int_0^\infty \!\!\dots\! \int_0^\infty \prod_i (\mathrm{d}\alpha_i \alpha_i^{\delta_i - 1}) \Big(\sum \varkappa_i \alpha_i\Big)^{-\frac{1}{2}D} \exp\Big[-\Big(\sum \varkappa_i \alpha_i\Big)^{-1} \sum_{i<j} \alpha_i \alpha_j (x_i - x_j)^2\Big]\,.$$

with $\varkappa_i = 1$, $\forall i$, however, the independence of (2) of the $\varkappa_i$, provided $\varkappa_i \geqslant 0$, $\sum \varkappa_i^2 > 0$, follows from the easily proved equality of (2) with

$$(3)\qquad I = \Gamma(\tfrac{1}{2}D) \int_0^\infty \!\!\dots\! \int_0^\infty \prod_i (\mathrm{d}\alpha_i \alpha_i^{\delta_i - 1})\, \delta\Big(1 - \sum \lambda_i \alpha_i\Big) \Big(\sum_{i<j} \alpha_i \alpha_j (x_i - x_j)^2\Big)^{-\frac{1}{2}D},$$

independently of the λ_i, $\lambda_i \geqslant 0$, $\sum \lambda_i^2 > 0$. In formulae (2) and (3), conformal covariance in accord with (1) expresses itself in the arbitrariness of the $\varkappa_i$ and the λ_i, respectively. Choosing in (2) $\varkappa_i = \delta_{i1}$ and using for the $2(n-3) + \frac{1}{2}(n-3)(n-4) = \frac{1}{2}n(n-3)$ terms in the exponential with $ij = 2k, 3k$ $(4 \leqslant k \leqslant n)$ and $4 \leqslant i < j \leqslant n$, the representation ([13])

$$(4)\qquad \exp[-z] = (2\pi i)^{-1} \int_{c - i\infty}^{c + i\infty} \mathrm{d}s\, \Gamma(-s) z^s \qquad (c < 0,\ |\arg z| < \tfrac{1}{2}\pi),$$

one can perform the α-integrations with the result (for typographical simplicity, we set $(x_i - x_j)^2 = r_{ij}$ and restrict in (5) and (6) the values of i, j to $4 \dots n$)

$$\begin{aligned}(5)\quad I = {} & r_{12}^{-\delta_1 - \delta_2 + \frac{1}{2}D} r_{13}^{-\delta_1 - \delta_3 + \frac{1}{2}D} r_{23}^{\delta_1 - \frac{1}{2}D} \prod_i r_{1i}^{-\delta_i} (2\pi i)^{-\frac{1}{2}n(n-3)} \cdot \\ & \cdot \int_{\uparrow} \!\dots\! \int_{\uparrow} \prod_i \Big(\mathrm{d}s_{2i}\, \Gamma\Big(-s_{2i} + \sum_{j>i} s_{ij}\Big) \mathrm{d}s_{3i}\, \Gamma\Big(-s_{3i} + \sum_{j<i} s_{ji}\Big) \cdot \Gamma(\delta_i + s_{2i} + s_{3i})\Big) \prod_{i<j} (\mathrm{d}s_{ij}\, \Gamma(-s_{ij})) \cdot \\ & \cdot \Gamma\Big(\tfrac{1}{2}D - \delta_1 + \sum s_{2i} + \sum s_{3i} - \sum_{i<j} s_{ij}\Big) \Gamma\Big(\delta_1 + \delta_2 - \tfrac{1}{2}D - \sum s_{3i}\Big) \cdot \\ & \qquad \cdot \Gamma\Big(\delta_1 + \delta_3 - \tfrac{1}{2}D - \sum s_{2i}\Big) \prod_i (h_{2i}^{s_{2i}} h_{3i}^{s_{3i}}) \prod_{i<j} h_{ij}^{s_{ij}}\,,\end{aligned}$$

where we have introduced the (squares of the) independent (*) harmonic ratios

$$(6)\qquad h_{2i} = r_{1i}^{-1} r_{23}^{-1} r_{2i} r_{13}\,, \qquad h_{3i} = r_{1i}^{-1} r_{23}^{-1} r_{3i} r_{12}\,, \qquad h_{ij} = r_{2i}^{-1} r_{3j}^{-1} r_{ij} r_{23} \qquad (i < j)\,,$$

in terms of which all others can be expressed, and where, to validate the interchanges of integrations that lead to (5), the paths of the s-integrations have to be choosen parallel to the imaginary axis with real parts such that the real parts of the arguments of all Γ-functions are positive. The necessary and sufficient condition (**) for this to be

([13]) *Higher Transcendental Functions*, Vol. 1, edited by A. ERDELYI (New York, 1955).

(*) We disregard the relations, for large n, between the harmonic ratios stemming from the vanishing of the Gram determinants, see the later remark.

(**) This can be proven by giving an explicit construction, but is made plausible simplest by considering a mechanical analogy. The condition also implies all propagators to possess ordinary Fourier transforms, and is equivalent to the UV convergence of (1).

possible is that $0 < \operatorname{Re}\delta_i < \frac{1}{2}D, \forall i$. (5) expresses I as a function of the harmonic ratios in the form of a multiple Mellin-Barnes integral [13], which converges absolutely for all h positive.

Inserting (6) in (5) and allowing linear transformations of the s-variables, (5) can be written more symmetrically as

$$(7a)\qquad I = (2\pi i)^{-\frac{1}{2}n(n-3)}\int\limits_{-i\infty}^{+i\infty}\dots\int\limits_{-i\infty}^{+i\infty} \mathrm{d}s_i \,..\, \mathrm{d}s_{\frac{1}{2}n(n-3)} \prod_{1\leqslant i<j\leqslant n} \Gamma\Big(\delta_{ij}^0 + \sum_k c_{ij,k}s_k\Big)\, r_{ij}^{-\delta_{ij}^0-\sum_k c_{ij,k}s_k}\,,$$

where δ_{ij}^0 is a particular solution, with positive real parts, of

$$(7b)\qquad \sum_{j\neq i}\delta_{ij}^0 = \delta_i\,,$$

and the real $c_{ij,k} = c_{ji,k}$ obey

$$(7c)\qquad c_{ii,k} = 0\,,\qquad \sum_{j\neq i} c_{ij,k} = 0\,.$$

Here $k = 1 \dots \frac{1}{2}n(n-3)$, and the $(\frac{1}{2}n(n-3))^2 c_{ij\,k}$ with $2 \leqslant i < j \leqslant n$, excepting $c_{23,k}$, which may be taken as the independent ones, must satisfy

$$(7d)\qquad |\det c_{ij,k}| = 1\,.$$

The simple and symmetric form of (5) or (7) makes it unadvisable to attempt to use the dependences among the harmonic ratios for large n to reduce the number of complex integrations. Also the triangle inequalities play no role in the uses, to be described below, of these formulae.

For propagators to nonscalar fields, the integral over a conformal invariant vertex (*i.e.* one that is Lorentz invariant and at which $\sum\delta_i = D$ holds) can be reduced to (1). We will consider only the propagator $((\gamma\cdot x) = x^\mu\gamma_\mu)$

$$(8)\qquad \Gamma(\delta+\tfrac{1}{2})\,\gamma\cdot x(x^2)^{-\delta-\frac{1}{2}} = -\tfrac{1}{2}\Gamma(\delta-\tfrac{1}{2})\,\not\partial(x^2)^{-\delta+\frac{1}{2}}\,.$$

If lines $1 \dots 2k$ have this propagator, we need consider, with again $\sum\delta_i = D$, instead of (2)

$$(9)\qquad I_{1\dots 2k} = 2^{-2k}\int\limits_0^\infty\dots\int\limits_0^\infty\prod_i(\mathrm{d}\alpha_i\,\alpha_i^{\delta_i-1})(\textstyle\sum\alpha)^{-\frac{1}{2}D}\cdot$$
$$\cdot(\alpha_1\dots\alpha_{2k})^{-\frac{1}{2}}(\not\partial_1)\dots(\not\partial_{2k})\exp\Big[-(\textstyle\sum\alpha)^{-1}\displaystyle\sum_{i<j}\alpha_i\alpha_j(x_i-x_j)^2\Big]\,.$$

For $k = 1$ and scalar (and similarly pseudoscalar) coupling we find, using $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ and $\gamma_\mu\gamma^\mu = D$

$$(10)\qquad \not\partial_1\not\partial_2\exp[\dots] = 4(\textstyle\sum\alpha)^{-1}\alpha_1\alpha_2\cdot$$
$$\cdot\Big[\sum_{i=3}^n\alpha_i\big((\gamma\cdot x)_1-(\gamma\cdot x)_i\big)\big((\gamma\cdot x)_2-(\gamma\cdot x)_i\big)+\tfrac{1}{2}D+\sum_{i=1}^n\alpha_i(\partial/\partial\alpha_i)\Big]\exp[\dots]\,,$$

where the homogeneity in α of the exponent has been used. The two last terms in the square bracket in (10) cancel in (9) upon partial integration, such that

$$(11)\qquad I_{1\cdot 2}(x_1\delta_1, \ldots, x_n\delta_n) = \sum_{i=3}^{n}\big((\gamma\cdot x)_1 - (\gamma\cdot x)_i\big)\big((\gamma\cdot x)_2 - (\gamma\cdot x)_i\big)\cdot$$
$$\cdot I\big(x_1(\delta_1+\tfrac{1}{2}), x_2(\delta_2+\tfrac{1}{2}), x_3\delta_3, \ldots, x_{i-1}\delta_{i-1}, x_i(\delta_i+1), x_{i+1}\delta_{i+1}, \ldots, x_n\delta_n\big)\,,$$

i.e. using (7), we obtain a sum of $n-2$ terms where in the $(i-2)$-th term the propagators $1i$ and $2i$ have been replaced by the propagator (8), with corresponding changes in the dimensions of the scalar parts by $\frac{1}{2}$ for each, and corresponding changes in the arguments of the Γ functions.

For $k>1$, more complicated formulae hold. We give here for $k=2$ only the formula for scalar-scalar (and similarly, for scalar-pseudoscalar etc.) coupling:

$$(12)\qquad I_{1\cdot 2\,3\cdot 4}(x_1\delta_1, \ldots, x_n\delta_n) =$$
$$= \sum_{i\neq 1,2}\sum_{j\neq 3,4}\big((\gamma\cdot x)_1 - (\gamma\cdot x)_i\big)\big((\gamma\cdot x)_2 - (\gamma\cdot x)_i\big)\times\big((\gamma\cdot x)_3 - (\gamma\cdot x)_j\big)\big((\gamma\cdot x)_4 - (\gamma\cdot x)_j\big)\cdot$$
$$\cdot I\big(x_1(\delta_1+\tfrac{1}{2}), \ldots, x_4(\delta_4+\tfrac{1}{2}), x_5\delta_5, \ldots, x_i(\delta_i+1), \ldots, x_j(\delta_j+1), \ldots, x_n\delta_n\big) +$$
$$+\big[\tfrac{1}{2}\big((\gamma\cdot x)_1\gamma^\mu + \gamma^\mu(\gamma\cdot x)_2\big)\times\big((\gamma\cdot x)_3\gamma_\mu + \gamma_\mu(\gamma\cdot x)_4\big) - (\gamma\cdot x)_1(\gamma\cdot x_2)\times 1 - 1\times(\gamma\cdot x)_3(\gamma\cdot x)_4\big]\cdot$$
$$\cdot I\big(x_1(\delta_1+\tfrac{1}{2}), \ldots, x_4(\delta_4+\tfrac{1}{2}), x_5\delta_5, \ldots, x_n\delta_n\big)\,,$$

whereby the I-arguments in the first term, if $i=j$ or $i=3,4$ and/or $j=1,2$ are obvious, the dimensional excesses summing up.

Using formulae (7), (11), (12) and higher ones in the computation of skeleton expansion graphs as appear in Migdal's theory ([1,2]), the $n\geqslant 4$-point functions are ultimately obtained as $\frac{1}{2}n(n-3)$ fold Mellin-Barnes integral similar to (7) with, however, a kernel $K(s_1 \ldots s_{\frac{1}{2}n(n-3)})$ in the integrand, which in turn is a sum of multiple Mellin-Barnes integrals, whose integrands are products of Beta functions arising through use of

$$(13)\qquad \Gamma(\delta_1)(x^2)^{-\delta_1}\Gamma(\delta_2)(x^2)^{-\delta_2} = B(\delta_1, \delta_2)\,\Gamma(\delta_1+\delta_2)(x^2)^{-\delta_1-\delta_2}\,,$$

since in all formulae (1), (7), (11), (12) only (*) the combinations $\Gamma(\delta)(x^2)^{-\delta}$ occur, which obviously is the convenient propagator normalization. In these calculations, one can always avoid the appearence of dimensions δ violating $0<\operatorname{Re}\delta<\frac{1}{2}D$. This is due to the facts that in a graph that is UV-convergent by the criteria of MACK and TODOROV ([8]), carrying out the integration over one vertex cannot bring about a UV-divergence, and that (« catastrophic » ([8]), *i.e.* apart from those at exceptional momenta) UR-divergences never arise in a UV-convergent Migdal graph.

The results of integrating over the vertices in a Migdal graph in different order differ from each other only by linear substitutions on the complex integration variables, which is due to the fact that in principle the final formula can be obtained also directly from the completely symmetric configuration space formula ([14]) for Feynman integrals by copious use of (4) and interchange of s- with α-integrations.

(*) When $(\gamma\cdot x)(\gamma\cdot x) = x^2$ is used to combine parallel scalar-coupled spinor lines, an obvious modification occurs.

([14]) C. S. LAM and J. P. LEBRUN: *Nuovo Cimento*, **59** A, 397 (1969); N. NAKANISHI: *Progr. Theor. Phys.*, **42**, 966 (1969).

For $n = 3$, as needed in Migdal's vertex bootstrap equations [2] and in those of PARISI and PELITI [10], the co-ordinate dependence factors out, and the coefficients in the algebraic relation

$$g = g^3 f_1(\delta_M, \delta_B) + g^5 f_2(\delta_M, \delta_B) + \dots, \tag{14}$$

are directly obtained in the form of multiple Mellin-Barnes integrals. *E.g.*, in the computation of f_1 in ps-ps theory, only formulae (7) and (11) are needed, and a sum of fourfold integrals is obtained. A more elegant way, using complex dimension as remarked at the beginning, yields one threefold integral, which can be converted, by insertion of the well-known integral representations of beta- and gamma-functions, interchange of integration, and use of (4), into a sixfold real integral in suitable form for numerical estimate but apparently not reducible to named functions (*).

In Migdal's theory, the vertex bootstrap eq. (14) must be complemented by propagator bootstrap equations [8]. For these, rather than using the unitarity condition, it is more convenient [9] to use the equations for momentum-differentiated inverse propagators [15]. This avoids the difficulties inherent in the use of conformal invariance in Minkowski space [7].

The results presented here suggest how one should proceed in a theory with fundamental quadrilinear coupling such as φ^4 or four-fermion coupling theory. One would have to insert a Mellin-Barnes integral representation (which implies, *e.g.*, growth properties of the function represented, for large and for small values of the harmonic ratios) of the four-point vertex function (and, conveniently, also for its totally two-particle irreducible part) into the appropriate bootstrap skeleton equations; unfortunately, hereby (14) becomes replaced by a nonlinear integral equation for kernels $K(s_1 s_2)$ (subjected to crossing symmetry). Here, and also in analysing the representations of conformal covariant amplitudes in terms of multiple Mellin-Barnes integrals described earlier, one should insert for the $K(\dots)$ their representations by Bergmann-Weil formulae [16] and interchange integrations. This will be undertaken elsewhere.

A Migdal approach, with tools as described here, appears possible also for perturbation-theoretically unrenormalizable theories such as four-fermion coupling in $D \geqslant 3$ dimensions (**) since the UV-convergence condition $\frac{1}{4}D < \dim \psi < \frac{1}{2}D$ is compatible with the positive-definiteness condition $\dim \psi > \frac{1}{2}(D-1)$.

Finally, concerning Minkowski metric, we note that the convergence of some of the Mellin-Barnes integrals for amplitudes may be conditional (or worse), such that the analytic continuation [17] from Euclidean to Minkowskian arguments should, as is always recommendable, be performed only in final formulae, invoking hereby the analytic properties [18] deriving from causality and spectrum conditions.

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(*) The expression for $f(n, \nu)$ given in ref. [11] is incorrect.

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(16) F. SOMMER: *Math. Ann.*, **125**, 172 (1952).

(**) This emerged in a discussion between G. MACK, K. WILSON, and the author.

(17) J. SCHWINGER: *Proc. Nat. Acad. Sci.*, **44**, 956 (1958); K. SYMANZIK: *Journ. Math. Phys.*, **7**, 510 (1966).

(18) R. F. STREATER and A. S. WIGHTMAN: *PCT, Spin and Statistics and All That* (New York, 1964); R. JOST: *The General Theory of Quantized Fields* (Providence, 1965).