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The Computation of Dynamic Cournot-Nash Traffic Network Equilibria in Discrete Time

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Abstract

The competition among a finite number of firms who must transport the fixed volume of traffic over a prescribed planning horizon is considered on a congested transportation network with one origin-destination pair connected by parallel routes. It is assumed that each firm attempts to minimize individual transportation cost by making a sequence of simultaneous decisions of departure time, route, and departure flow rate based on the trade-off between arc traversal time and schedule delay penalty. The model is formulated as an N-person nonzero-sum discrete-time dynamic game. A Cournot-Nash network equilibrium is defined under the open-loop information structure. Optimality conditions are derived using the Kuhn-Tucker theorem and given economic interpretation as a dynamic game theoretic generalization of Wardrop's second principle which requires equilibration of certain marginal costs. A computational algorithm based on the augmented Lagrangian method and the gradient method is proposed and a numerical example is provided. Future extensions of the model and the algorithm are also discussed.

Keywords: cournot-nash game, traffic equilibria, traffic asssignment, lagrangian method

1. Introduction

The aim of this paper is to apply dynamic noncooperative game theory (Basar and Olsder, 1982) to the prediction of Cournot-Nash equilibria on a congested transportation network. The dynamic game considered here is a simple dynamic extension of the noncooperative game in Haurie and Marcotte (1985). The extension is simple in that the network has only one origin-destination pair connected by a finite number of parallel routes. The extension is dynamic in that the players have an additional dimension of choice, that is, departure time choice.

Our dynamic game is also closely related to dynamic system optimal traffic assignment models in the work of Merchant and Nemhauser (1978), Wie (1988), and Friesz, Luque, Tobin, and Wie (1989). However, there are two major differences: [1] those models assume that a single player (e.g., a central traffic controller) wants to minimize the total network transportation cost, and [2] they assume that network users have no departure time choice. The model presented in this paper is a discrete time version of the open-loop differential game in the work of Wie (1993). This paper focuses more on development of a solution algorithm and analysis of numerical results.

The dynamic game considered in this paper corresponds to several different situations. For example, automated highways will be built in the 21st century and in-vehicle dynamic route guidance systems will revolutionize ancient tasks of navigation by providing drivers with optimal routing information. Suppose a finite number of private communication firms sell route guidance services to customers who are equipped with in-vehicle navigation systems. In such a situation, these firms compete with each other to minimize their customers' travel times and an equilibrium will be reached when no firm can do better by unilaterally changing their route guidances. Another corresponding situation may be found at a congested airport. Suppose a finite number of airlines serve a common airport and each airline has the limited number of landing slots. In the case where no price competition prevails, airlines may compete with each other to maximize their profits by changing departure times. An equilibrium will be reached when no airline can do better by unilaterally changing departure times.

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This paper is organized as follows. In Section 2, the model is formulated as an N-person nonzero-sum discrete-time dynamic game under the open-loop information structure. In Section 3, the necessary and sufficient conditions for an equilibrium solution are derived using the Kuhn-Tucker theorem. In Section 4, economic interpretation is provided for the optimality conditions as a dynamic game theoretic generalization of Wardrop's second principle (Wardrop, 1952) which requires equilibration of marginal costs. In Section 5, a computation algorithm is proposed and a numerical example is presented. The paper concludes with a discussion on future extensions of the model and the algorithm.

2. Model Formulation

Consider a transportation network with one origin-destination pair connected by parallel arcs. We assume that a finite number of transportation firms compete on this network for sending flows (e.g., automobiles, trucks, aircrafts, petroleum products, etc.) and attempt to minimize their individual transportation costs over a prescribed time interval $[0,T]$. We also assume that each firm must transport the fixed volume of traffic to the destination by the terminal time T. Each firm has a most desired time of arrival at the destination and wishes to reduce the total penalty associated with schedule delays (i.e., differences between desired arrival time and actual arrival time).

The entire time interval is divided into (K+1) discrete time periods of uniformly small length and the index k will be used to denote a time period. To ensure dimensional consistencies throughout the paper, the length of each time period is assumed to be unity. Let A denote the set of arcs which will be interchangeably called routes or paths hereinafter. Also, let $I=\{1,...., N\}$ denote the set of firms or players and let (I/i) denote the set of players except a player indexed by i. It should be noted that flow rate is defined as traffic volume per unit time (e.g., vehicles per minute) and all flow variables are taken to be continuous in the present analysis. We shall employ the following notation:

$u_a^i(k)$ = the decision of player i on arc a in period k denoting the rate of flow that player i dispatches onto arc a in period k

$u_a(k) = [u_a^i(k)\colon i \in I]$

$u^i = [u_a^i(k)\colon a \in A, k = 0,\dots,K-1]$

$u^{-i} = [u_a^j(k)\colon a \in A, k = 0,\dots,K-1, j \in I]$

$x_a(k)$ = the volume of aggregate traffic on arc a in period k

$x = [x_a(k)\colon a \in A, k = 0,\dots,K]$

$g_a[x_a(k)]$ = the rate of aggregate flow exiting from arc a in period k

$c_a^i[x_a(k), u_a(k)]$ = the unit arc traversal cost perceived by player i if entrance to arc a occurs in period k

$\phi_a^i[x_a(k), u_a(k), k]$ = the unit schedule delay penalty perceived by player i entrance to arc a occurs in period k

Q^{iK} = the cumulative volume of traffic that player i would have dispatched from the origin during the time interval $[0,T]$

The problem of determining Cournot-Nash equilibria on a congested transportation network is formulated as an N-person nonzero-sum discrete-time dynamic game as follows: for each $i \in I$,

$$\min_{u^i} J^i(x, u^i, u^{-i}) = \sum_{k=0}^{K-1}\sum_{a\in A} u_a^i(k)\ \{[c_a^i(k), u_a(k)] + \phi_a^i[x_a(k), u_a(k), k]\} \tag{1}$$

subject to

$$x_a(k+1) = x_a(k) + \sum_{j=i}^{N} u_a^j(k) - g_a[x_a(k)] \tag{2}$$

$$\forall a \in A, k = 0,\dots,K-1$$

$$Q^{iK} = \sum_{k=0}^{K-1}\sum_{a\in A} u_a^i(k) \tag{3}$$

$$u_a^i(k) \geq 0 \quad \forall a \in A, k = 0,\dots,K-1 \tag{4}$$

$$x_a(k) \geq 0 \quad \forall a \in A, k = 0,\dots,K \tag{5}$$

$$x_a(0) = x_a^o \geq 0 \quad \forall a \in A \tag{6}$$

where u^{-i} is treated as fixed.

The sum of $c_a^i[x_a(k), u_a(k)]$ and $\phi_a^i[x_a(k), u_a(k), k]$ represents the unit generalized transportation cost perceived by player i if entrance to arc a occurs in period k. The value of the objective function $J^i(x, u^i, u^{-i})$ thus represents the total generalized transportation cost of player i who sends flows from the origin to the destination over the time interval $[0,T]$. It implies that simultaneous decisions of departure time, route, and departure flow rate of player i are based on the trade-off between arc traversal time and schedule delay penalty. We assume that the arc traversal cost functions $c_a^i[x_a(k), u_a(k)]$ are convex, continuously differentiable, nondecreasing, and positive for all $x_a(k) \geq 0$ and $u_a(k) \geq 0$. We also assume that the schedule delay penalty functions $\phi_a^i[x_a(k), u_a(k), k]$ are convex, continuously differentiable, and nonnegative for all $x_a(k) \geq 0$ and $u_a(k) \geq 0$.

The evolution of the state of each arc is described by nonlinear difference equation (2) with initial condition (6). In order to represent the congestion phenomenon, the arc exit flow functions $g_a[x_a(k)]$ are assumed to be concave, continuously differentiable, nondecreasing, and nonnegative for all $x_a(k) \geq 0$ with the restriction that $g_a(0) = 0$ for all a $a \in A$. Because of this assumption, we do not have to subsequently consider the state nonnegativity constraints (5) in analyzing the dynamic game formulated above. In fact, the analysis is considerably simplified by this assumption. The isoperimetric constraint (3) ensures that each player must dispatch the fixed volume of traffic from the origin by the terminal time T. We assume that T is large enough for all players to complete their transportation tasks during the time interval $[0,T]$. It should be noted that the value of Q^{iK} is predetermined for each player and the interest rate is neglected. In the remainder of the paper, a N-person nonzero-sum discrete-time dynamic game formulated in equations (1)-(6) will be refered to as the dynamic game G.

We now specify the information structure of the dynamic game G. In this paper, we consider only one type of information structure which simplifies the analysis considerably.

Definition 1

The set $\eta = \{\eta_a^i(k): a \in A, i \in I, k = 0, \ldots, K\}$ is called the information structure of the dynamic game g. The information structure is said to be open loop if $\eta_a^i(k) = [k, x_a(0)]$ for all $a \in A, i \in I$, and $k = 0, \ldots, K$.

Under the open-loop information structure of Definition 1, we also introduce the following notation:

$\gamma_a^i[\eta_a^i(k)]$ =the strategy of player i on arc a in period k

$\gamma^i = [\gamma_a^i[\eta_a^i(k)]: a \in A, k = 0, \ldots, K-1]$

Γ^i =the set of admissible strategies for player i

A strategy of player i is here referred to as a rule for determining the decision variable $u_a^i(k)$ when the information available to player i in period k is $[k, x_a(0)]$.

When no binding agreement can be made among the players, the noncooperative Cournot-Nash equilibrium solution concept appears to be a reasonable mood of play for the dynamic game G.

Definition 2

An N-tuple of strategies $(\gamma^{1*}, \ldots \ldots, \gamma^{N*})$, with $\gamma^{i*} \in \Gamma^i$ for each $i \in I$, is said to constitute an open-loop Cournot-Nash equilibrium solution for the dynamic game G with the information structure of Definition 1, if the inequalities

$$J^i(\gamma^{1*}, \ldots\ldots, \gamma^{N*}) \leq J^i(\gamma^{1*}, \ldots, \gamma^{(i-1)*}, \gamma^i, \gamma^{(i+1)*}, \ldots, \gamma^N) \quad (7)$$

are satisfied for all $\gamma^i \in \Gamma^i$ and $i \in I$.

The Cournot-Nash strategy has the property that if all but one player use their Cournot-Nash strategies, the deviating player cannot reduce the value of $J^i(x, u^i, u^{-i})$. In other words, the Cournot-Nash equilibrium strategy is the optimal strategy for each of the players provided that all of the other players continue to use their Cournot-Nash strategies.

3. Derivation of Optimality Conditions

In this section, we derive the Kuhn-Tucker necessary conditions which a Cournot-Nash equilibrium solution of Definition 2 must satisfy. We then derive the sufficient conditions to determine whether an N-tuple of strategies characterized by the necessary conditions is indeed a global open-loop Cournot-Nash equilibrium solution.

As a first step to apply the Kuhn-Tucker theorem, let us define the Lagrangian for player i as follows:

$$\begin{aligned} L^i(x, u, \lambda^i, \mu^i) = &\sum_{k=0}^{K-1} \sum_{a \in A} u_a^i(k) \{ c_a^i[x_a(k), u_a(k) \\ &+ \phi_a^i x_a(k), u_a(k) k] \} + \sum_{k=0}^{K-1} \sum_{a \in A} \lambda_a^i(k+1) \\ &\left[x_a(k) + \sum_{j=1}^{N} u_a^j(k) - g_a[x_a(k)] - x_a(k+1) \right] \\ &+ \mu^i \left[Q^{iK} - \sum_{k=0}^{K-1} \sum_{a \in A} u_a^i(k) \right] \end{aligned} \quad (8)$$

where $\lambda_a^i(k+1)$ is the adjoint variable of player i associated with the state difference equation (2); $\lambda^i = [\lambda_a^i(k):, a \in A, k = 0, \ldots, K]$; and μ^i is the Lagrange multiplier of player i associated with the isoperimetric constraint (3).

Using the Kuhn-Tucker theorem (see e.g., Bazararaa and Shetty, 1979), the necessary conditions can be written as follows:

$$\begin{aligned} \frac{\partial L^i}{\partial x_a(k)} = &\, u_a^i(k) \left[\frac{\partial c_a^i[x_a(k), u_a(k)]}{\partial x_a(k)} + \frac{\partial \phi_a^i[x_a(k), u_a(k), k]}{\partial x_a(k)} \right] \\ &+ \left[1 - \frac{dg_a[x_a(k)]}{dx_a(k)} \right] \lambda_a^i(k+1) - \lambda_a^i(k) = 0 \\ &\forall a \in A, i \in I, k = 0, \ldots, K-1 \end{aligned} \quad (9)$$

$$\lambda_a^i(K) = 0 \quad \forall a \in A, i \in I \quad (10)$$

$$\begin{aligned} \frac{\partial L^i}{\partial u_a^i(k)} = &\, c_a^i[x_a(k), u_a(k)] + \phi_a^i[x_a(k), u_a(k), k] \\ &+ u_a^i(k) \left[\frac{\partial c_a^i[x_a(k), u_a(k)]}{\partial u_a^i(k)} + \frac{\partial \phi_a^i[x_a(k), u_a(k), k]}{\partial u_a^i(k)} \right] \\ &+ \lambda_a^i(k+1) - \mu^i \geq 0 \quad \forall a \in A, i \in I, k = 0, \ldots, K-1 \end{aligned} \quad (11)$$

$$u_a^i(k) \frac{\partial L^i}{\partial u_a^i(k)} = 0 \quad \forall a \in A, i \in I, k = 0, \ldots, K-1 \quad (12)$$

$$u_a^i(k) \geq 0 \quad \forall a \in A, i \in I, k = 0, ..., K-1 \tag{13}$$

$$\frac{\partial L^i}{\partial \lambda_a^i(k+1)} = x_a(k) + \sum_{j=1}^{N} u_a^j(k)$$

$$-g_a[x_a(k)] - x_a(k+1) = 0$$

$$\forall a \in A, i \in I, k = 0, ..., K-1 \tag{14}$$

$$x_a(0) = x_a^o \geq 0 \tag{15}$$

$$\frac{\partial L^i}{\partial \mu^i} = Q^{iK} - \sum_{k=0}^{K-1} \sum_{a \in A} u_a(k) = 0 \quad \forall a \in I \tag{16}$$

where all variables, functions, and derivatives are evaluated at a Cournot-Nash equilibrium solution $(x^*, u^{i*}, u^{-i*}, \lambda^*, \mu^*)$.

The necessary conditions (9)-(16) can propose a certain number of admissible open-loop Cournot-Nash equilibrium solutions, while assuring that there will be no other candidate that solves the dynamic game G. However, we need to establish the sufficient conditions to ensure whether $(x^*, u^{i*}, u^{-i*}, \lambda^*, \mu^*)$ characterized by the necessary conditions (9)-(16) is indeed a global open-loop Cournot-Nash equilibrium solution to the dynamic game g. To simplify our presentation, we make the following definition of regularity.

Definition 3

The dynamic game G is said to be regular if it satisfies the following conditions:

1. the arc traversal cost functions $c_a^i[x_a(k), u_a(k)]$ are convex, continuously differentiable, nondecreasing, and positive for all $x_a(k) \geq 0$ and $u_a(k) \geq 0$
2. the schedule delay penalty functions $\phi_a^i[x_a(k), u_a(k), k]$ are convex, continuously differentiable, and nonnegative for all $x_a(k) \geq 0$ and $u_a(k) \geq 0$;
3. the arc exit flow functions $g_a[x_a(k)]$ are concave, continuously differentiable, nondecreasing, and nonnegative for all $x_a(k) \geq 0$;
4. $\partial c_a^i[x_a(k), u_a(k)]/\partial u_a^i(k) \geq \partial \phi_a^i[x_a(k), u_a(k), k]/\partial u_a^i(k)$ $\forall a \in A, i \in I$, and $k = 0, ..., K-1$

Note that the regularity condition 4 is the behavioral assumption such that each player prefers one minute of schedule delay to one minute of travel time.

Essentially, the sufficient conditions are the Kuhn-Tucker necessary conditions (9)-(16) plus the convexity of $J^i(x, u^i, u^{-i})$ in x, u^i and the constraint qualification. The convexity of $J^i(x, u^i, u^{-i})$ in x, u^i is apparent from the regularity conditions of Definition 3. The constraint qualification is satisfied as long as the arc exit flow functions $g_a[x_a(k)]$ are concave for all $x_a(k) \geq 0$ and the following conditions are satisfied:

$$\mu^i \geq 0 \quad \forall i \in I \tag{17}$$

$$\lambda_a^i(k+1) \geq 0 \quad \forall a \in A, i \in I, k = 0, ..., K-1 \tag{18}$$

Nonnegativity of the Lagrange multipliers can be ensured by its well-known interpretation as the sensitivity of the optimal value of the objective function $J^i(x^*, u^{i*}, u^{-i*})$ to variations in the constraint constant Q^{iK} as follows:

$$\mu^{i*} = \frac{\partial J^i(x^*, u^{i*}, u^{-i*})}{\partial Q^{iK}} \quad \forall a \in I \tag{19}$$

Since we require both arc traversal costs and schedule delay penalties to be nonnegative, nonnegativity of the Lagrange multipliers is ensured. As will also be demonstrated in Theorem 1, the Lagrange multiplier μ^i can be interpreted as the marginal cost of an additional unit of traffic that player i has to transport from the origin to the destination during the time interval $[0,T]$. Nonnegativity of the adjoint variables can be ensured by the complementary slackness condition (11) and the regularity condition 4. Hence, the optimality conditions (9)-(16) are both necessary and sufficient for an open-loop Cournot-Nash equilibrium solution of the dynamic game g when it is regular in the sense of Definition 3.

4. Economic Interpretation

In this section, we wish to ascertain that an N-tuple of strategies characterized by the optimality conditions (9)-(16) corresponds to the notion of a Cournot-Nash equilibrium of Definition 2. To this end, let us define the marginal generalized transportation cost function for player $i \in I$ as

$$\begin{aligned}
&\Psi_a^i[x_a(k), u_a(k), \lambda_a^i(k+1), \lambda_a^i(k), k] \\
&\equiv c_a^i[x_a(k), u_a(k)] + \phi_a^i[x_a(k), u_a(k), k] \\
&+ u_a^i(k)\left[\frac{\partial c_a^i}{\partial u_a(k)} + \frac{\partial \phi_a^i}{\partial u_a(k)}\right] \\
&+ \left\{u_a^i(k)\left[\frac{\partial c_a^i}{\partial x_a(k)} + \frac{\partial \phi_a^i}{\partial x_a(k)}\right] + \lambda_a^i(k+1) - \lambda_a^i(k)\right\} \\
&\times \frac{1}{g_a'[x_a(k)]} \qquad \forall a \in A, k = 0, ..., K-1
\end{aligned} \tag{20}$$

where $g_a'[x_a(k)]$ denotes $dg_a[x_a(k)]/dx_a(k)$. We observe that the marginal cost function defined in equation (20) consists of the *static* and *dynamic* cost components. The term $[\lambda_a^i(k+1) - \lambda_a^i(k)]$ is considered to be *dynamic* in that it represents the intertemporal change of the impact of an additional unit of traffic on the optimal value of the objective function $J^i(x^*, u^{i*}, u^{-i*})$. This *dynamic* component is scaled by $1/g_a'[x_a(k)]$ which represents the physical effect of congestion. In fact, $1/g_a'[x_a(k)]$ is positive, increasing, continuously differentiable, and convex for all $x_a(k) \geq 0$. The *static* component is the sum of three terms:

- $c_a^i[x_a(k), u_a(k)] + \phi_a^i[x_a(k), u_a(k), k]$ =the unit cost perceived by player *i* on arc a in period *k*
- $u_a^i(k)[\partial c_a^i/\partial u_a(k) + \partial \phi_a^i/\partial u_a(k)]$ =the additional travel-cost burden for player *i* inflicted by an infinitesimal increment of traffic entering arc a in period *k*
- $u_a^i(k)[\partial c_a^i/\partial x_a(k) + \partial \phi_a^i/\partial x_a(k)]$ =the additional travel-cost burden inflicted by an infinitesimal increment of traffic already traveling on arc a in period *k*

We are now ready to state and prove the following theorem.

Theorem 1

If, for each player $i \in I, u_a^i(k) > 0$ in some time period $\Psi_a^i[x_a(k), u_a(k), \lambda_a^i(k+1), \lambda_a^i(k), k] = \inf\{\Psi_b^i[x_b(k), u_b(k)\ \lambda_b^i(k+1), \lambda_b^i(k), k]\} : \forall b \in A\}$ for an open-loop Cournot-Nash equilibrium solution of the dynamic game *G* when it is regular in the sense of Definition 3.

Proof

Manipulating the adjoint difference equation (9) yields

$$\lambda_a^i(k+1) = \left\{ u_a^i(k) \left[\frac{\partial c_a^i}{\partial x_a(k)} + \frac{\partial \phi_a^i}{\partial x_a(k)} \right] + \lambda x_a(k-1) - \lambda x_a(k) \right\} \times \frac{1}{g_a'[x_a(k)]} \qquad \forall a \in A, i \in I, k = 0, \ldots, K-1 \quad (21)$$

However, we know from equation (11) that

$$c_a^i[x_a(k), u_a(k)] + \phi_a^i[x_a(k), u_a(k), k] + u_a^i(k) + \left[\frac{\partial c_a^i}{\partial u_a(k)} + \frac{\partial \phi_a^i}{\partial u_a(k)} \right] + \lambda_a(k+1) \geq \mu^i \qquad \forall a \in A, i \in I, k = 0, \ldots, K-1 \quad (22)$$

It follows at once from equations (21) and (22) that

$$\Psi_a^i[x_a(k), u_a(k), \lambda_a^i(k+1), \lambda_a^i(k), k] \geq \mu^i \qquad \forall a \in A, i \in I, k = 0, \ldots, K-1 \quad (23)$$

If $u_a^i(k) > 0$ in some time period k, then equation (20) holds as an equality because of the complementary slackness condition (12). Hence, the theorem is proved. Q.E.D.

Theorem 1 states that the marginal costs on the arcs that are being used (i.e., $u_a^i(k) > 0$) are the same and equal to the minimum marginal cost. In other words, player *i* discharges flow onto arc a in period *k* only if $\Psi_a^i[x_a(k), u_a(k), \lambda_a^i(k+1), \lambda_a^i(k), k]$ is equal to μ^i. Clearly, the optimality conditions (9)-(16) can be interpreted as a dynamic game theoretic generalization of Wardrop's second principle which requires equilibration of marginal costs for all the arcs that are being used in each time period. Thus, an *N*-tuple of strategies characterized by those optimality conditions is consistent with the notion of an open-loop Cournot-Nash equilibrium of Definition 2.

5. Solution Algorithm

In this section, we propose an iterative algorithm for solving the dynamic game . The proposed algorithm makes use of the augmented Lagrangian method in conjunction with the gradient method. The augmented Lagrangian method, also known as the method of multipliers, was initially proposed by Hestenes (1969) and Powell (1969) and has been applied to the solution of optimal control problems (see e.g., Rupp 1972 and Glad 1979). The augmented Lagrangian method can be regarded as a combination of the penalty method and the primal-dual method (Bertsekas 1982). The gradient method (see e.g., Bryson and Ho 1975) has been widely used to solve optimal control problems.

The motivation for choosing the augmented Lagrangian method is related to the presence of the the isoperimetric constraints (5). Although the gradient method can be modified and applied to solve the dynamic game *g*. by means of the penalty functions, the penalty functions usually become ill-conditioned when the penalty parameter is increased to infinity (see e.g., Bryson and Ho 1975). In order to improve slow convergence and numerical instabilities associated with ill-conditioned penalty functions, the augmented Lagrangian method is selected in the proposed algorithm.

The role of the gradient method in the proposed algorithm is to solve two-point value problems with fixed values of the Lagrange multipliers associated with the isoperimetric constrains (3). For the sake of computational simplicity, the first-order gradient method with a fixed step-size is chosen in the proposed algorithm. The rate of convergence might be improved by employing other gradient methods such as steepest-descent, conjugate gradient, and second-order gradient methods.

Let us define the augmented Lagrangian for player i as

$$L^i(x, u, u^{-i}, \lambda^i, \mu^i) = L^i(x, u^i, u^{-i}, \lambda^i, \mu^i) + \frac{\rho}{2}\left[Q^{iK} - \sum_{k=0}^{K-1} \sum_{a \in A} u_a^i(k) \right]^2 \qquad (24)$$

where $L^i(x, u^i, u^{-i}, \lambda^i, \mu^i)$ is the lagrangian defined in equation (8) and ρ is a suitably large positive constant. The minimization of the augmented Lagrangian is equivalent to the minimization of the Lagrangian because the constraints adjoined as the penalty functions do not affect the minimum value of the Lagrangian. In fact, at the optimum $[x^*, u^{i^*}, u^{-i^*}, \lambda^{i^*}, \mu^{i^*}]$, the gradient of the augmented Lagrangian is zero for any value of ρ and the constraints (3) are binding.

A double set of iterations constitutes the solution algorithm. The objective of the main iterations is to determine the new values of the Lagrange multipliers until a Cournot-Nash equilibrium solution is obtained. On the other hand, the objective of the subiterations is to solve for each player i a two-point boundary value problem which is transformed from the augmented Lagrangian minimization problem when the value of the Lagrange multiplier is fixed.

The solution algorithm is structed as follows:

Step 1: *Initialization*

choose the initial values of the Lagrange multipliers:

$\mu^r = [\mu^{ir}: i \in I]$

Set the main iteration index r=1 and go to Step 2.

Step 2: *Solving the Two-Point Boundary Value Problem with Fixed Lagrange Multipliers*

Substep 1: Choose the initial values of the decision variables:

$u^{rs} = [u_a^{irs}(k): a \in A, i \in I, k = 0, \ldots, K-1]$

Set the subiteration index s=1 and go to Substep 2.

Substep 2: Solve the state difference equations forward in time:

$$x_a^{rs}(k+1) = x_a^{rs}(k) + \sum_{j=1}^{N} u_a^{jrs}(k) - g_a[x_a^{rs}(k)]$$

$$\forall a \in A, k = 0, \ldots, K-1$$

$$x_a(0) = x_a^o \geq 0 \quad \forall a \in A$$

Substep 3: Solve the adjoint difference equations backward in time:

$$\lambda_a^{irs}(k) = \left[1 - \frac{dg_a[x_a^{rs}(k)]}{dx_a^{rs}(k)}\right]\lambda_a^{irs}(k+1) + u_a^{irs}(k)$$
$$\times\left[\frac{\partial c_a^i[x_a^{rs}(k), u_a^{rs}]}{\partial x_a^{rs}(k)} + \frac{\partial \phi_a^i[x_a^{rs}(k), u_a^{rs}, k]}{\partial x_a^{rs}(k)}\right]$$

$$\forall a \in A, i \in I, k = 0, \ldots, K-1$$

$$\lambda_a^{irs}(K) = 0 \quad \forall a \in A, i \in I$$

Substep 4: Determine the gradient of the augmented Lagrangian with respect to the decision variables:

$$\frac{\partial L^{irs}}{\partial u_a^{irs}(k)} = c_a^i[x_a^{rs}(k), u^{rs}(k)]$$
$$+ \phi_a^i[x_a^{rs}(k), u_a^{rs}(k), k] + u_a^{irs}(k)$$
$$+ u_a^{irs}(k)\left[\frac{\partial c_a^i[x_a^{rs}(k), u_a^{rs}(k)]}{\partial u_a^{irs}(k)}\right.$$
$$\left.+ \frac{\partial \phi_a^i[x_a^{rs}(k), u_a^{rs}(k), k]}{\partial u_a^{irs}(k)}\right] + \lambda_a^{irs}(k+1) - \mu^{ir}$$
$$-\rho\left[Q^{iK} - \sum_{k=0}^{K-1}\sum_{a \in A} u_a^{irs}(k)\right]$$

$$\forall a \in A, i \in I, k = 0, \ldots, K-1$$

Substep 5: Compute the new values of the decision variables:

$$u_a^{ir,s+1}(k) = u_a^{irs}(k) - \theta \frac{\partial L^{irs}}{\partial u_a^{irs}(k)}$$

$$\forall a \in A, i \in I, k = 0, \ldots, K-1$$

where θ is the fixed stepsize. Set s=s+1 and repeat the computation starting at Substep 2 until the following stopping criterion is satisfied:

$$\sum_{i=1}^{N}\sum_{k=0}^{K-1}\sum_{a \in A}\left[\frac{\partial L^{irs}}{\partial u_a^{irs}(k)}\right]^2 \leq \pi$$

Step 3: *Updating and Convergence Test* Update the values of the Lagrange multipliers:

$$\mu^{i,r+1} = \mu^{ir} + \rho\left[Q^{ik} - \sum_{k=0}^{K-1}\sum_{a \in A} u_a^{ir}(k)\right] \quad \forall i \in I$$

Set r=r+1 and repeat the computation starting at Step 2 until the following stopping criterion is satisfied:

$$\sum_{i=1}^{N} \frac{\mu^{i,r+1} - \mu^{ir}}{\mu^{ir}} \leq \xi$$

6. Numerical Example

The solution algorithm is tested for a small numerical example. The test network is comprised of one origin-destination pair connected by two parallel arcs as follows:

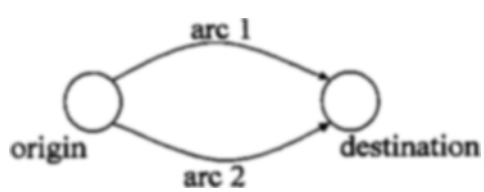

Fig. 1 Two Arcs Network.

We assume that two Cournot-Nash players share the network. The time interval is discretized into 200 periods of one minute length. The cumulative volume of traffic that each player would have sent from the origin during the time interval is given as follows: $Q^{1=1000}$ and $Q^{2=1000}$. The arc exit flow functions are expressed in the following functional forms:

$$g_a[x_a(k)] = g_a^{\max}\left[1-e^{-\frac{x_a(k)}{\gamma_a}}\right] \qquad \forall a=1,2,k=0,\ldots,200 \tag{25}$$

where $g_a^{\max}$ is the maximum exit flow rate on arc a in each time period, and γ_a is the parameter that is determined by arc characteristics such as length, speed limit, traffic signal setting, etc. The values of g_a^{max} and γ_a assumed are: $g_a^{\max} = 10$, $\gamma_1 = 10$, and $\gamma_2 = 10$.

The average arc traversal time functions are expressed as

$$\begin{aligned} c_a^i[x_a(k),u_a(k)] &= \xi^i D_a[x_a(k),u_a(k)] \\ &= \xi^i \frac{x_a(k)+0.5u_a(k)}{g_a[x_a(t)]} \\ &\forall a=1,2,k=0,\ldots,200 \end{aligned} \tag{26}$$

where $D_a[x_a(k),u_a(k)]$ is the anticipated arc traversal time if entrance to arc a occurs in time period k and ξ^i is the transportation cost per minute incurred by one unit of traffic of player k. In this numerical example, the values of ξ^i are assumed to be identical for each player and given as unity. The initial values of the state variables are also given as $x_1(0)=3$ and $x_2(0)=3$.

After de Palma *et al.* (1983), the average schedule delay penalty functions are expressed in the following piecewise linear functional form:

$$\phi_a^i[x_a(k),u_a(k),k]=$$

$$\begin{Bmatrix} \alpha^i\{k+D_a[x_a(k)]-(\tau^i-\Delta^i)\} \text{ if } k+D_a[x_a(k)]<\tau^i-\Delta^i \\ 0, \qquad \text{if } \tau^i=\Delta^i \le k+D_a[x_a(k)] \le \tau^i+\Delta^i \\ \beta^i\{k+D_a[x_a(k)]-(\tau^i+\Delta^i)\} \text{ if } (\tau^i+\Delta^i<k+D_a(x_a(k))) \end{Bmatrix} \tag{27}$$

where α^i is the penalty coefficient of early arrival of player i, τ^i is the penalty coefficient of late arrival of player i, τ^i is the most desired arrival time period of player i, and Δ^i is the number of time periods measuring the flexibility of arrival time of player i. The values of these parameters are given as

$$\alpha^1=-0.5 \quad \beta^1=0.8 \quad \tau^1=100 \quad \Delta^1=5$$

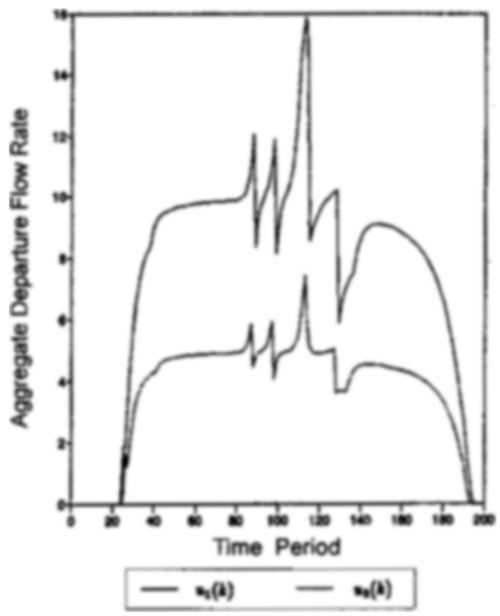

Fig. 2 AggregatE Departure Flow Rates.

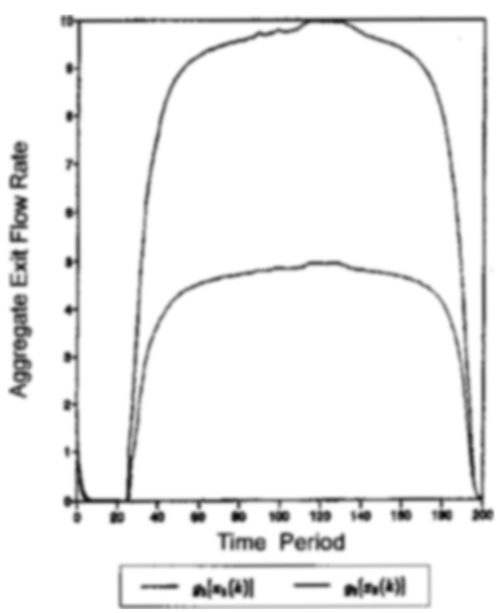

Fig. 3 Aggregate Exit Flow Rates.

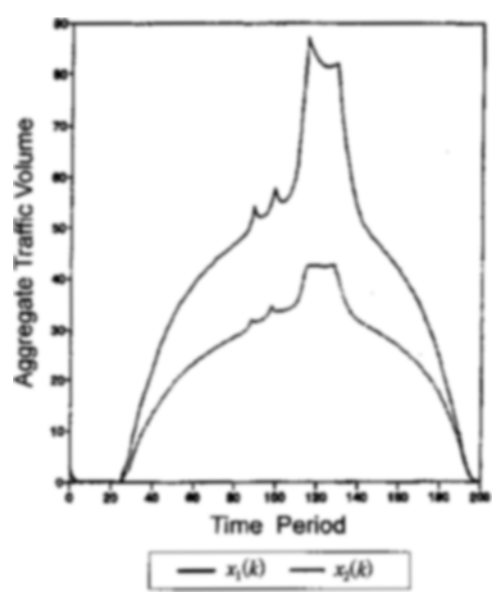

Fig. 4 Aggregate Traffic Volumes.

$\alpha^1 = -0.5 \quad \beta^1 = 0.8 \quad \tau^1 = 100 \quad \Delta^1 = 5$

Figure 2 presents time trajectories of aggregate departure flow rates on arcs 1 and 2. Figure 3 presents time trajectories of aggregate exit flow rates on arcs 1 and 2.

It can be observed that each arc is capacitated and traffic congestion develops as inflow rate exceeds maximum exit flow rate on each arc. Figure 4 shows that time trajectories of aggregate traffic volumes on arcs 1 and 2. Since all two players wish to arrive at the destination within specified time windows which are close each other, the network tends to be severely congested during peak periods. Figure 5 plots time trajectories of average traversal costs on arcs 1 and 2. Figures 6 and 7 present time trajectories of average schedule delay penalties for both players on arc 1 and arc 2, respectively.

We now wish to ensure as to whether the solution of this numerical example satisfies the optimality conditions (9)-(16) and is indeed an open-loop Cournot-Nash equilibrium of Definition 2. To this end, we plot time trajectories of marginal costs as defined in equation (20) and compare them with time trajectories of departure flow rates. Figure 8 illustrates that departure flow rates of player 1 on arc 1 are positive only when marginal costs are equal to the minimum marginal cost. The same observations can be made in Figure 9-11. Hence, we can conclude that proposed solution algorithm converges to an open-loop Cournot-Nash equilibrium solution.

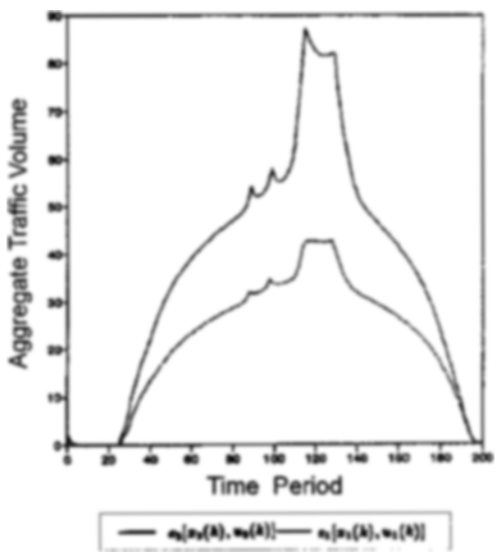

Fig. 5 Aggregate Traffic Volumes.

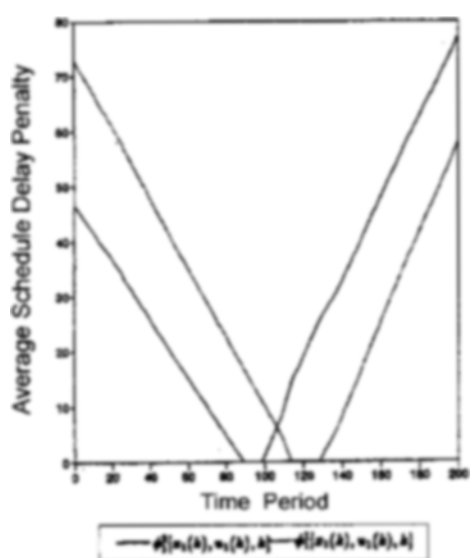

Fig. 6 Average Schedule Delay Penalties on Arc 1.

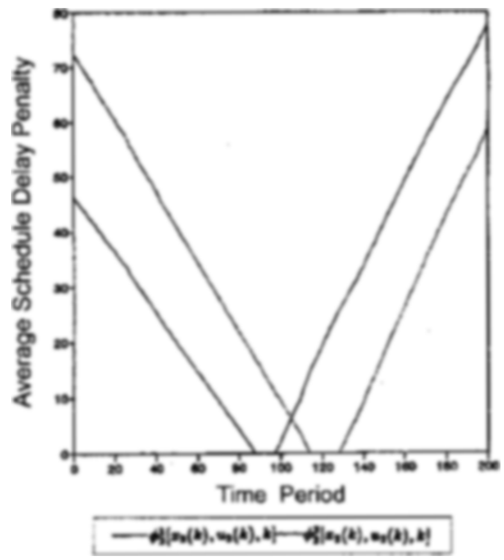

Fig. 7 Average Schedule Delay Penalties on Arc2.

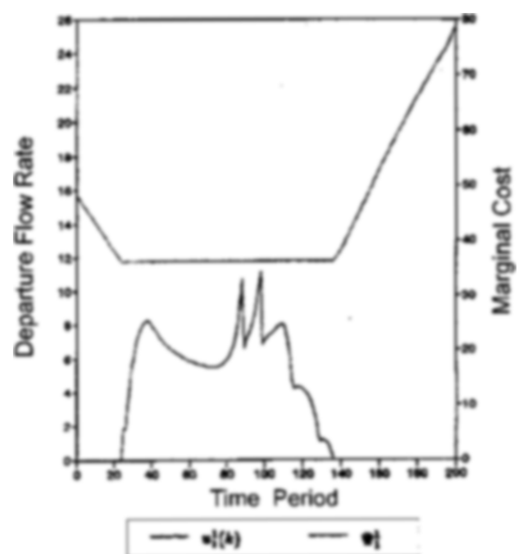

Fig. 8 Comparison of Departure Flow Rate and Marginal Cost for Player 1 on Arc 1.

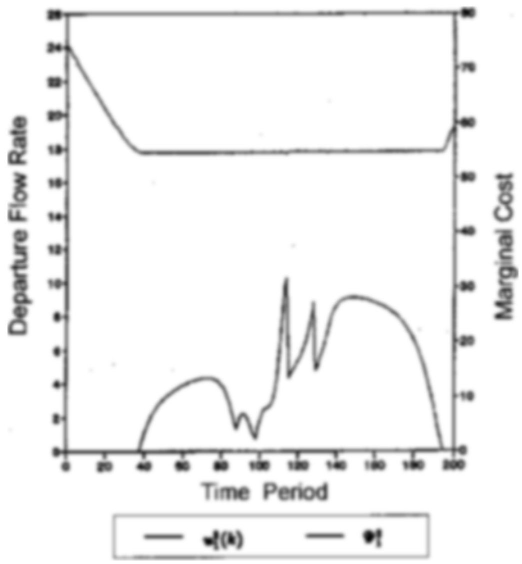

Fig. 9 Comparison of Department Flow Rate and Marginal Cost foR Player 2 on Arc 1.

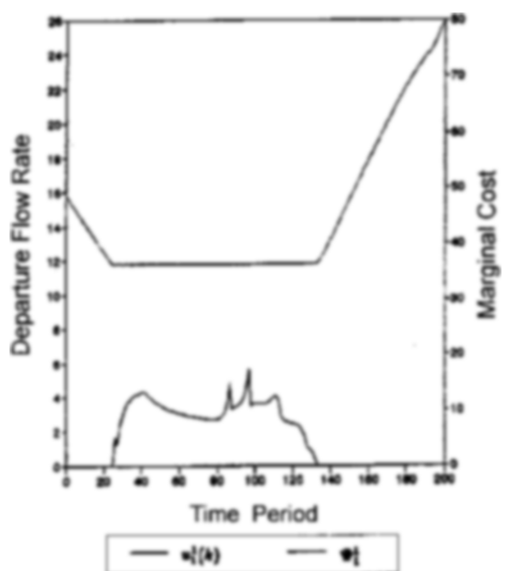

Fig. 10 Comparison of Departure Flow Rate and Marginal Cost for Player 1 on Arc 1.

7. Conclusion

We have shown that a discrete-time dynamic game model can be formulated to determine open-loop Cournot-Nash equilibria on a simple congested traffic network. We have also developed an iterative algorithm to solve the model. Computation results of a numerical example illustrate that the algorithm converges to a discrete-time dynamic Cournot-Nash network equilibrium solution.

Future extensions of the model and the algorithm include the following issues. First, the model needs to be extented to a general network with many origins and many destinations. This extension appears to be very difficult because path traversal times should be predicted while preserving the first-in-first-out queue discipline. Second, the model can be reformulated under different information structures. Recent advances in computerized traffic information systems convince that a feedback information structure may give rise to more realistic modeling of traffic flows than an open-loop information structure. However, analyses and solutions of dynamic game models are generally very difficult under the feedback information structure. Third, the proposed solution algorithm needs to be modified to accelerate the rate of convergence. Convergence properties of the algorithm should also be investigated theoretically.

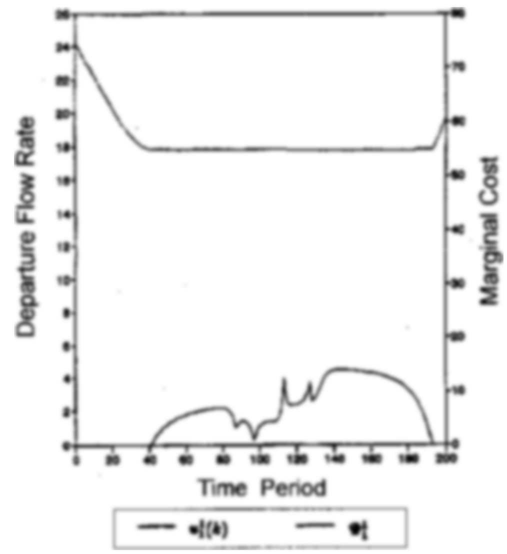

Fig. 11 Comparison of Departure Flow Rate and Marginal Cost For Player 2 on Arc 2.

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